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# PERMANENCE OF METRIC FRACTALS

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*Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor*

Abstract. The paper studies energy functionals on quasimetric spaces, defined by quadratic measure-valued Lagrangeans. This general model of medium, known as metric fractals, includes nested fractals and sub-Riemannian manifolds. In particular, the quadratic form of the Lagrangean satisfies Sobolev inequalities with the critical exponent determined by the (quasimetric) homogeneous dimension, which is also involved in the asymptotic distribution of the form's eigenvalues. This paper verifies that the axioms of the metric fractal are preserved by space products, leading thus to examples of non-differentiable media of arbitrary intrinsic dimension.

## 1. Introduction

Many models of continuous medium can be put into a general framework of Dirichlet forms (cf. [2, 7]) on topological measure spaces that are not necessarily differentiable (or piecewise differentiable) manifolds, or are manifolds whose natural metric structure is no longer Riemannian. Sobolev inequalities formalize a basic consistency of such medium by subordinating a characteristic of displacement ($L^p$-norm) to the value of the energy, and they can be derived from the scaled Poincaré inequality. Theory of the abstract Sobolev spaces for Dirichlet forms on metric spaces (cf.[8, 9] and references therein), when applied to fractals, requires one substantial reconsideration: in the case of fractal media the scaling factor $R^s$ in the Poincaré inequality on metric balls $B_R(x)$ has an exponent $s$ whose values vary with the fractal. To extend the abstract Sobolev theory to fractals one needs to replace the metric $d$ with a quasimetric $d^q$ with a $q > 0$ that returns the standard value of the exponent in the scaling factor of the Poincaré inequality. Once this is done, the critical Sobolev exponent and the spectral asymptotics attain the classical magnitudes, $\frac{2\nu}{\nu-2}$ and $n(\lambda) = O(\lambda^{\frac{\nu}{2}})$ respectively, where $\nu$ is the homogeneous dimension derived from the doubling property of measure with respect to the chosen quasimetric, which allows to call it the intrinsic quasimetric. Sobolev inequalities in the quasimetric framework may admit minimizers (ground states), similarly to

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the Euclidean case. Existence of minimizers is known for compact spaces due to compactness in Sobolev imbeddings ([4], cf. [9] for the metric case), for compact problems on non-compact spaces, [4], and for non-compact problems in [3]. A quasi-metric space with Dirichlet form satisfying a scaled Poincaré inequality is called a metric fractal and the dimension $\nu$ is called intrinsic or spectral dimension.

This paper considers the set of axioms for a metric fractal from [16], stemming from the notion of measure-valued Lagrangeans from [12]. This axiomatic system sets a framework that, on one side, describes a wide range of media, and on the other, inherits many essential properties of energy functionals associated with elliptic operators on Euclidean space, but it also covers subelliptic operators on manifolds and most common fractals (Koch and Sierpinski curves and snowflakes, bi-dimensional carpets) and more general elastic fractal media, such as the variational fractals of [14], endowed with its intrinsic Lagrangean metric ([15]). This paper addresses a more general case than the paper of R. Strichartz [19] that generalizes Kigami's construction to products of p.c.f. fractals, a general class of fractals where the energy functionals have been constructed, but which does not include, for instance, Sierpinski carpet. This note, instead of constructed energies, uses common properties of the latter in an axiomatic definition of the energy functional.

Our main result (Theorem 3.3) establishes the permanence property of these axioms, namely that the product of two metric fractals $X_1, X_2$ of spectral dimensions $\nu_1, \nu_2$ is a metric fractal of spectral dimension $\nu_1+\nu_2$. This result implies, for example, a Sobolev inequality on such spaces as a product of the Sierpinski gasket (with the usual self-similar measure and energy, and the quasidistance $d(x,y) = |x-y|^s$ with $s$ chosen so that the homogeneous dimension is equalized with the spectral dimension) and the realization of the Heisenberg group on $\mathbb{R}^{2m+1}$, endowed with the left Haar measure (which is the Lebesgue measure), the homogeneous quasi-distance and the quadratic form of the Heisenberg-Kohn Laplacian.

## 2. Definition of metric fractal

**Definition 2.1.** A metric fractal is a quintuple $(X, d, \mu, \mathcal{L}, \mathcal{C})$, where

(i) $(X, d)$ is a complete connected quasimetric space with a quasidistance $d : X \times X \to [0,\infty)$ (a symmetric nonnegative function vanishing only on the diagonal and satisfying $d(x,y) \le k(d(x,z) + d(z,y))$ with some $k \ge 1$);

(ii) $\mu$ is a doubling measure (a positive Borel measure supported on $X$ and satisfying the inequality

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \le C\big(\frac{R}{r}\big)^{\nu} \tag{1}$$

for all $x \in X$ and all $r, R$ satisfying $0 < r \le R$ with some $R_0 \in (0, +\infty]$);

(iii) $\mathcal{C}$ is a dense subalgebra of $C_c(X)$ (the space of continuous functions with compact support on $X$), – and $\mathcal{L}$ is a (signed) Radon measure-valued, positive symmetric bilinear form on the set $\mathcal{D}_{\mathcal{L}} := \{\varphi(u) : \varphi \in C^1(\mathbb{R}), u \in \mathcal{C}\}$ – to be called a Lagrangean - with $\mathcal{L}(u,u)$ of finite mass on $X$ for every $u \in \mathcal{D}_{\mathcal{L}}$ and

$$\mathcal{L}(\varphi(u), v) = \varphi'(u)\mathcal{L}(u, v) \tag{2}$$

for any $u, v \in \mathcal{D}_{\mathcal{L}}$ and any $\varphi \in C^1(\mathbb{R})$.

(iv) $d$, $\mu$ and $\mathcal{L}$ are related, with some $c > 0$ and $\lambda \geq 1$, by the inequality

$$\frac{1}{\mu(B_R(x))} \int_{B_R(x)} |u - u_{B_R(x)}| \, d\mu \, \leq \, cR \left( \frac{1}{\mu(B_{\lambda R}(x))} \int_{B_{\lambda R}(x)} d\mathcal{L}(u,u) \right)^{1/2} \tag{3}$$

for $u \in \mathcal{D}_{\mathcal{L}}$, $x \in X$, $0 < \lambda R < R_0$.

In (iv) above, the notation $u_A$ is used for the average value of $u$ on the set $A$ with respect to the given measure. In what follows an abbreviated notation $B_R$ is used for quasimetric balls $B_R(x)$ in $X$ when the statements concern all $x \in X$.

The term *metric fractal*, borrowed from [16], is used here in a broader sense: the definition omits the capacity conditions, required in [16] with the purpose to obtain fractal Harnack inequalities, and follows the set of conditions from [12] that suffice to verify Sobolev and Morrey inequalities.

The doubling property together with completeness of $(X, d)$ assures that the quasi-metric balls $B_R(x)$ are compact. Therefore $X$ is locally compact and the measure space $(X, \mu)$ is $\sigma-$finite. We consider the Hilbert space $L^2(X, \mu)$ with inner product

$$(u, v) = \int_X uv \, d\mu.$$

The space $C_c(X)$ is dense in $L^2(X, \mu)$. By our assumption in (2.1), $\mathcal{C}$ is dense in $C_c(X)$ (for the uniform convergence of sequences supported on compact sets), therefore $\mathcal{C}$ is dense in $L^2(X, \mu)$. Since $\mathcal{D}_{\mathcal{L}} \cap L^2(X, \mu) \supset \mathcal{D}_{\mathcal{L}} \cap C_c(X) \supset \mathcal{C}$, then $\mathcal{D}_{\mathcal{L}} \cap L^2(X, \mu)$ is dense in $L^2(X, \mu)$.

Under assumptions of Definition 2.1 the following Sobolev inequality is established in [12] (cf. [8, 5] that use similar but less general conditions): If $p \in [1, \frac{2\nu}{\nu - 2})$ when $\nu > 2$ or $p \geq 1$ when $\nu \leq 2$, there exist $C > 0$ and $\sigma \geq 1$ for every $u \in \mathcal{D}_{\mathcal{L}}$ and every quasimetric ball $B_R$ :

$$\Big( \frac{1}{\mu(B_R)} \int_{B_R} |u - u_{B_R}|^p \Big)^{1/p} \leq cR \Big( \frac{1}{\mu(B_{\sigma R})} \int_{B_{\sigma R}} d\mathcal{L}(u,u) \Big)^{1/2}. \tag{4}$$

¿From the local inequality (4) follows the global inequality, with additional requirement $p \geq 2$ when $X$ is not compact:

$$\Big( \int_X |u|^p \Big)^{2/p} \leq c \int_X d\mathcal{L}(u,u) + \int_X |u|^2 d\mu. \tag{5}$$

By Cauchy inequality one has $|u_{B_R}| \leq C(\int |u^2|)^{1/2}$, so that from (4) follows

$$\Big( \int_{B_R} |u|^p \Big)^{2/p} \leq c \int_{B_{\sigma R}} d\mathcal{L}(u,u) + \int_{B_R} |u|^2 d\mu, \tag{6}$$

which easily extends to (5) if $X$ is compact. If $X$ is not compact, one considers a covering of $X$ by a collection of $B_R(x_i)$ such that the multiplicity of the covering of $X$ with corresponding $B_{\sigma R}$ is finite (existence of such coverings is a well-known consequence of the doubling property) and adds (6) over the covering. Condition $p \geq 2$ is required for the superadditivity in the left hand side.

We consider a Sobolev space $H_0^1(X)$ defined as the completion of $\mathcal{D}_L$ in the *energy norm*

$$\Big( \int_X d\mathcal{L}(u,u) + \int_X |u|^2 d\mu \Big)^{1/2}.$$

By the definition of the energy norm, $H_0^1(X)$ is continuously imbedded into $L^2(X,\mu)$ and so may be regarded as the space of measurable functions.

**Proposition 2.2.** *The Lagrangean $\mathcal{L}$ admits a continuous extension to a Radon measure-valued positive symmetric bilinear form on $H_0^1(X)$.*

*Proof.* Let $u \in H^1(X)$ be given by a Cauchy sequence $u_k \in \mathcal{D}_{\mathcal{L}}$. Then $\mathcal{L}(u_k,u_k)A$ will be a Cauchy sequence for any Borel set $A$ and by Theorem 30.2, [1], $\mathcal{L}(u_k,u_k)$ converges weakly to some Radon measure $m_u$. The measure $m_u$ inherits from $\mathcal{L}(u_k,u_k)$ bilinearity and the parallelogram identity with respect to $u$. By setting $\mathcal{L}(u,u)=m_u$, we define the extension of $\mathcal{L}$ as a positive symmetric measure-valued quadratic form to the whole $H^1(X)$. Continuity of $(u,v)\mapsto \int_A d\mathcal{L}(u,v)$ is then immediate. $\square$

## 3. Permanence of metric fractals under space products

Let $(X_i,d_i,\mu_i,\mathcal{L}_i,\mathcal{C}_i)$, $i=1,2$, be two metric fractals. We define the product metric fractal as the quasimetric space $X=X_1\times X_2$ equipped with the quasidistance $d(x,y)=\max\{d_1(x_1,y_1),d(x_2,y_2)\}$ and the standard product measure $\mu=\mu_1\times\mu_2$. We will denote balls in respective spaces as $B_R^i\subset X_i$, $i=1,2$, and omit the notation for the center of the ball). The quasidistance for the product space is chosen so that $B_R=B_R^1\times B_R^2$

Let now $\mathcal{C}$ be the set of finite linear combinations of functions of the form $u_1(x_1)u_2(x_2)$, $u_i\in\mathcal{C}_i$. It is obviously an algebra and it is dense in $C_c(X)$ due to the following argument. For every $i=1,2$, and $R>0$, the function $\chi_R^i(x_i)=\frac{d_i(x_i,X_i\setminus B_R^i)}{d_i(x_i,X_i\setminus B_R^i)+d_i(x_i,B_{R/2}^i)}$ is in $C_c(X_i)$ and so it can be approximated by some sequence $\chi_{R,n}^i\in\mathcal{C}_i$. Then the function $\chi_R(x_1,x_2):=\chi_R^1(x_1)\chi_R^2(x_2)$ can be approximated by $\chi_{R,n}^1(x_1)\chi_{R,n}^2(x_2)\in\mathcal{C}$. Given a $w\in C_c(X)$ and an $\varepsilon>0$, let $R>0$ be such that the modulus of continuity of $w$ on any ball of radius $R$ does not exceed $\varepsilon$ and consider a locally finite cover of $X$ with $B_R(x_j)$, $j\in\mathbb{N}$. Then the functions $\varphi_j=\frac{\chi_{R;x_j}}{\sum_k\chi_{R;x_k}}$ form a partition of unity on $X$ and $|w-\sum_j w(x_j)\varphi_j|\le\varepsilon$. Since the sum above is finite and every $\varphi_j$ can be approximated by functions from $\mathcal{C}$, we conclude that $\mathcal{C}$ is dense in $C_c(X)$.

We define the product Lagrangean on products of functions $u_i,v_i\in\mathcal{C}_i$:

$$\begin{aligned}\mathcal{L}(u_1u_2,v_1v_2)&=u_2v_2\mathcal{L}_1(u_1,v_1)\times\mu_2+u_1v_1\mathcal{L}_2(u_2,v_2)\times\mu_1\\&=\mathcal{L}_1(u_1u_2,v_1v_2)\times\mu_2+\mathcal{L}_2(u_1u_2,v_1v_2)\times\mu_1.\end{aligned}\tag{7}$$

and extend it by bilinearity to $\mathcal{C}$.

**Lemma 3.1.** *The product Lagrangean $\mathcal{L}$ admits a continuous extension (in the energy norm) to $\mathcal{D}_{\mathcal{L}}$ (defined as $\{\varphi(\mathcal{C}),\varphi\in C^1(\mathbb{R})\}$). Moreover, if $u,v\in\mathcal{C}$ and $\varphi\in C^1(\mathbb{R})$,*

$$\mathcal{L}(\varphi(u),v)=\varphi'(u)\mathcal{L}(u,v).\tag{8}$$

*Proof.* First consider $u=u_1(x_1)u_2(x_2)$. Then

$$\begin{aligned}\mathcal{L}(\varphi(u_1u_2),v)&=\mathcal{L}_1(\varphi(u_1u_2),v)\times\mu_2+\mathcal{L}_2(\varphi(u_1u_2),v)\times\mu_1\\&=u_2\varphi'(u_1u_2)\mathcal{L}_1(u_1,v)\times\mu_2+u_1\varphi'(u_1u_2)\mathcal{L}_2(u_2,v)\times\mu_1\\&=\varphi'(u_1u_2)\mathcal{L}(u_1u_2,v).\end{aligned}\tag{9}$$

¿From here the chain rule extends by bilinearity to all functions of the form $\varphi(u)$, $u \in \mathcal{C}$ where $\varphi$ is a polynomial.

Assume now that $\varphi \in C^1(\mathbb{R})$. By the Weierstrass approximation theorem for functions of real variable, applied to $\varphi'$, we get a sequence of polynomials $\varphi_n$ that approximates $\varphi$ in $C^1(\mathbb{R})$, uniformly on compact subsets. We claim that the sequence $\varphi_n(u)$ is a Cauchy sequence in the energy norm over any compact set $K$, that is

$$\sup_{m,n\geq N}\Big(\int_K |\varphi_n(u)-\varphi_m(u)|^2 d\mu + \int_K d\mathcal{L}(\varphi_n(u)-\varphi_m(u), \varphi_n(u)-\varphi_m(u))\Big) \to 0.$$

By uniform convergence,

$$\sup_{m,n\geq N}\int_K |\varphi_n(u)-\varphi_m(u)|^2 d\mu \leq \sup_{m,n\geq N}\sup_K |\varphi_n(u)-\varphi_m(u)|^2 \mu(K) \to 0.$$

In particular, we have $\varphi_n(u)_K \to \varphi(u)_K$.

Applying the chain rule for each of the polynomials $\varphi_n$, we have:

$$\begin{aligned}
&\sup_{m,n\geq N}\int_K d\mathcal{L}(\varphi_n(u)-\varphi_m(u), \varphi_n(u)-\varphi_m(u))\\
&\leq \sup_{m,n\geq N}\int_K (\varphi_n'(u)-\varphi_m'(u))^2 d\mathcal{L}(u,u)\\
&\leq \sup_{m,n\geq N}\sup_K |\varphi_n(u)-\varphi_m(u)|^2 \int_K d\mathcal{L}(u,u) \to 0.
\end{aligned}$$

Let $w = \lim \varphi_n(u)$. Since $\varphi_n$ converges pointwise, with necessity $w = \varphi(u)$.

Due to [1, Theorem 30.2], there exists a Radon measure on $X$, which we denote here by $\mu(w,w)$, such that $\int_K d\mathcal{L}(\varphi_n(u), \varphi_n(u)) \to \int_K d\mu(w,w)$. We now prove that $\mu(w,w)$ is a Lagrangean. The measure $\mu(w,w)$ inherits homogeneity and parallelogram identity from $\mathcal{L}(\varphi_n(u), \varphi_n(u))$, therefore it is a quadratic measure-valued functional of $w$, $\mu(w,w) = \mathcal{L}(w,w)$ associated with a (measure-valued) positive symmetric bilinear form $\mathcal{L}(u,v)$ defined now for all $u, v \in \mathcal{D}_{\mathcal{L}}$.

Since $\int_K \varphi_n'(u) d\mathcal{L}(u,v) \to \int_K \varphi'(u) d\mathcal{L}(u,v)$ by the uniform convergence theorem for integrals, and since $\int_K d\mathcal{L}(\varphi_n(u), v) \to \int_K d\mathcal{L}(\varphi(u), v)$ by the definition of the Lagrangean on $\mathcal{D}_{\mathcal{L}}$, we have the chain rule on $\mathcal{D}_{\mathcal{L}}$. □

**Lemma 3.2.** *There is a $q \geq 1$, $c > 0$ such that for every $R > 0$ and $u \in \mathcal{D}_{\mathcal{L}}$, we have*

$$\frac{1}{\mu(B_R(x))}\int_{B_R(x)} |u - u_{B_R(x)}| \leq cR\Big(\frac{1}{\mu(B_{qR}(x))}\int_{B_{qR}(x)} d\mathcal{L}(u,u)\Big)^{1/2}. \tag{10}$$

*Proof.* In the calculations below we will denote $u_{B_R^2}(x_1)$ as $v$. We consider first $u \in \mathcal{C}$. We have:

$$\begin{aligned}
&\frac{1}{\mu_1(B_R^1)\mu_2(B_R^2)}\int_{B_R} |u(x_1,x_2) - u_{B_R}| d\mu_1 d\mu_2\\
&\leq \frac{1}{\mu_1(B_R^1)}\frac{1}{\mu_2(B_R^2)}\int_{B_R} |u(x_1,x_2) - u_{B_R^2}(x_1)| d\mu_2 d\mu_1\\
&\quad + \frac{1}{\mu_2(B_R^2)}\frac{1}{\mu_1(B_R^1)}\int_{B_R} |v(x_1) - v_{B_R^1}| d\mu_1 d\mu_2
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{cR}{\mu_1(B^1_R)}\int_{B^1_R}\Big(\frac{1}{\mu_2(B^2_{q_2R})}\int_{B^2_{q_2R}} d\mathcal{L}_2(u(x_1,\cdot),u(x_1,\cdot))\Big)^{1/2} d\mu_1\\
&\quad + cR\Big(\frac{1}{\mu_1(B^1_{q_1R})}\int_{B^1_{q_1R}} d\mathcal{L}_1(v,v)\Big)^{1/2}\\
&\le \frac{cR}{\mu_1(B^1_R)^{1/2}\mu_2(B^2_{q_2R})^{1/2}}\Big(\int_{B^1_R\times B^2_{q_2R}} d\mathcal{L}_2(u(x_1,\cdot),u(x_1,\cdot))d\mu_1(x_1)\Big)^{1/2}\\
&\quad + \frac{cR}{\mu_1(B^1_{q_1R})^{1/2}\mu_2(B^2_R)^{1/2}}\Big(\int_{B^1_{q_1R}\times B^2_R} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2)\Big)^{1/2}. \qquad (11)
\end{aligned}
$$

In the calculation above we use (3) for $\mathcal{L}_1,\mathcal{L}_2$, and, several times, the Cauchy inequality, including its following variation:

$$
\begin{aligned}
&\frac{1}{\mu_2(B^2_R)^2}\int_{B^1_{q_1R}} d\mathcal{L}_1\Big(\int_{B^2_R} u(\cdot,x_2)d\mu_2(x_2),\int_{B^2_R} u(\cdot,x_2)d\mu_2(x_2)\Big)\\
&\le \frac{1}{\mu_2(B^2_R)}\int_{B^1_{q_1R}\times B^2_R} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2).
\end{aligned} \qquad (12)
$$

To verify the above inequality, we change first the integral limits (which for $u\in\mathcal{C}$ needs nothing but a trivial use of bilinearity). After that the Cauchy inequality is applied to the bilinear form $\mathcal{L}_1(\cdot,\cdot)B^1_{q_1R}$, and once again, to the integral over $B^2_R$:

$$
\begin{aligned}
&\frac{1}{\mu_2(B^2_R)^2}\int_{B^1_{q_1R}} d\mathcal{L}_1\Big(\int_{B^2_R} u(\cdot,x_2)d\mu_2(x_2),\int_{B^2_R} u(\cdot,x_2')d\mu_2(x_2')\Big)\\
&= \frac{1}{\mu_2(B^2_R)^2}\int_{B^2_R}\int_{B^2_R}\int_{B^1_{q_1R}} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2'))d\mu_2(x_2)d\mu_2(x_2')\\
&\le \frac{1}{\mu_2(B^2_R)^2}\Big(\int_{B^2_R}\Big(\int_{B^1_{q_1R}} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))\Big)^{1/2} d\mu_2(x_2)\Big)\\
&\quad\times\Big(\int_{B^2_R}\Big(\int_{B^1_{q_1R}} d\mathcal{L}_1(u(\cdot,x_2'),u(\cdot,x_2'))\Big)^{1/2} d\mu_2(x_2')\Big)\\
&= \frac{1}{\mu_2(B^2_R)^2}\Big(\int_{B^2_R}\Big(\int_{B^1_{q_1R}} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))\Big)^{1/2} d\mu_2(x_2)\Big)^2\\
&\le \frac{1}{\mu_2(B^2_R)}\int_{B^1_{q_1R}\times B^2_R} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2).
\end{aligned}
$$

From (11) it follows that

$$
\begin{aligned}
&\frac{1}{\mu(B_R)}\int_{B_R}|u(x_1,x_2)-u_{B_R}|d\mu\\
&\le \frac{1}{\mu_1(B^1_R)\mu_2(B^2_R)}\int_{B_R}|u(x_1,x_2)-u_{B_R}|d\mu_1 d\mu_2\\
&\le \frac{cR}{\mu_1(B^1_R)^{1/2}\mu_2(B^2_{q_2R})^{1/2}}\Big(\int_{B^1_R\times B^2_{q_2R}} d\mathcal{L}_2(u(x_1,\cdot),u(x_1,\cdot))d\mu_1(x_1)\Big)^{1/2}\\
&\quad + \frac{cR}{\mu_1(B^1_{q_1R})^{1/2}\mu_2(B^2_R)^{1/2}}\Big(\int_{B^1_{q_1R}\times B^2_R} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2)\Big)^{1/2}
\end{aligned}
$$

$$\le \frac{cR}{\mu(B_{qR})^{1/2}}\Big(\int_{BqR} d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2)$$
$$+ d\mathcal{L}_2(u(x_1,\cdot),u(x_1,\cdot))d\mu_1(x_1)\Big)^{1/2}.$$

This completes the proof of (10) when $u \in \mathcal{C}$.

Let us now replace $u$ in (10) by $\varphi(u)$ with a polynomial $\varphi$ and use the chain rule for $\mathcal{L}_1$,$\mathcal{L}_2$.

$$\begin{aligned}
&\frac{1}{\mu_1(B_R^1)\mu_2(B_R^2)}\int_{B_R}|\varphi\circ u(x_1,x_2)-(\varphi\circ u)_{B_R}|d\mu_1 d\mu_2\\
&\le \frac{cR}{\mu_1(B_R^1)^{1/2}\mu_2(B_{q_2R}^2)^{1/2}}\\
&\quad\times\Big(\int_{B_R^1\times B_{q_2R}^2}(\varphi'\circ u(x_1,\cdot))^2 d\mathcal{L}_2(u(x_1,\cdot),u(x_1,\cdot))d\mu_1(x_1)\Big)^{1/2}\\
&\quad+\frac{cR}{\mu_1(B_{q_1R}^1)^{1/2}\mu_2(B_R^2)^{1/2}}\\
&\quad\times\Big(\int_{B_{q_1R}^1\times B_R^2}(\varphi'\circ u(\cdot,x_2))^2 d\mathcal{L}_1(u(\cdot,x_2),u(\cdot,x_2))d\mu_2(x_2)\Big)^{1/2}.
\end{aligned}$$

Let $\varphi_n$ be as in the previous lemma so that $\varphi_n(u)\to\varphi(u)$ locally in the energy norm and in $L^1_{loc}$. Then, repeating the argument of Lemma 3.1 on extension of the Lagrangean to the $\mathcal{D}_{\mathcal{L}}$, we have $\int_{B_R} d\mathcal{L}(\varphi_n(u),\varphi_n(u))\to\int_{B_R} d\mathcal{L}(\varphi(u),\varphi(u))$. The assertion of the lemma follows. $\square$

**Theorem 3.3.** *The quintuple $(X,d,\mu,\mathcal{L},\mathcal{C})$ defined above is a metric fractal in accordance to the Definition 2.1 with $X=X_1\times X_2$, $d(x,y)=\max\{d(x_1,y_1),d(x_2,y_2)\}$ $\nu=\nu_1+\nu_2$, $\mu=\mu_1\times\mu_2$,$\mathcal{C}$ defined as an algebra of finite sums of the form $\sum_i u_i(x_1)v_i(x_2)$, $u_i\in\mathcal{C}_1$, $v_i\in\mathcal{C}_2$ and $\mathcal{L}$ given by* (7) *and extended by continuity to $\mathcal{D}_{\mathcal{L}}$ by Lemma 3.1.*

*Proof.* Property (i) is immediate. To prove (ii) we verify that
$$\frac{\mu(B_R^1\times B_R^2)}{\mu(B_r^1\times B_r^2)}=\frac{\mu_1(B_R^1)}{\mu_1(B_r^1)}\frac{\mu_2(B_R^2)}{\mu_2(B_r^2)}\le C(\frac{R}{r})^{\nu_1+\nu_2},$$
so the relation (1) holds for the product space with $\nu=\nu_1+\nu_2$. The chain rule (8) extends trivially to all $u,v\in\mathcal{D}_{\mathcal{L}}$. The Poincaré inequality is proved in Lemma 3.2. $\square$

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