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A MINIMAX FORMULA FOR THE PRINCIPAL EIGENVALUES OF DIRICHLET PROBLEMS AND ITS APPLICATIONS

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Dedicated to Jacqueline Fleckinger on the occasion of
an international conference in her honor

Abstract. A minimax formula for the principal eigenvalue of a nonselfadjoint Dirichlet problem was established in [8, 18]. In this paper we generalize this formula to the case where an indefinite weight is present. Our proof requires less regularity and, unlike that in [8, 18], does not rely on semigroups theory nor on stochastic differential equations. It makes use of weighted Sobolev spaces. An application is given to the study of the uniformity of the antimaximum principle.

1. Introduction

The main purpose of this paper is to establish a variational formula of minimax type for the principal eigenvalues of the (generally nonselfadjoint) Dirichlet problem

$$\begin{aligned} Lu &= \lambda m(x)u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

eq1.1

Here Ω is a bounded domain in $\mathbb{R}^N$, L is a second order elliptic operator of the form

$$Lu := -\operatorname{div}(A(x)\nabla u) + \langle a(x), \nabla u\rangle + a_0(x)u \tag{1.2}$$

eq1.2

with $\langle , \rangle$ the scalar product in $\mathbb{R}^N$, and $m(x)$ is a possibly indefinite weight.

Calculating the principal eigenvalues of a selfadjoint operator via minimization of the Rayleigh quotient is a classical matter. Problem (1.1) above is generally nonselfadjoint and this Euler-Lagrange technique does not apply anymore. Other approaches were introduced in [8], [18]. In [8] $m \equiv 1$ and a minimax formula was derived through the consideration of an associated semigroup of positive operators. In [18] the weight is definite (i.e. $m(x) \geq \varepsilon > 0$ in Ω) and a similar minimax formula was derived by using results on stochastic differential equations. Both [8],

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[18] assume C^∞ smoothness for the coefficients of L, and [18] assume m of class C^2.

The formula we obtain (cf. Theorem 3.1 and Theorem 3.5) is rather similar to that in [8], [18]. Our contribution is triple. First we deal with the general case where the weight m may vanish or change sign in Ω. Secondly much less regularity on the coefficients and on the weight is required. Finally our proof does not rely on semigroups theory nor on stochastic differential equations.

Our proof follows the general approach initially introduced in [18] and further developed in [13] in the case of the Neumann-Robin problem. The main difficulty in adapting this approach to the case of the Dirichlet problem comes from the fact that several auxiliary equations which in the Neumann-Robin case are uniformly elliptic now degenerate on $\partial\Omega$ (cf. equations (3.2) and (3.3)). A large part of the present paper is devoted to the study of these degenerate equations, to which the classical results of [25] do not apply. Our study is carried out in the context of weighted Sobolev spaces, and Moser's iteration technique is in particular used in that context to derive a crucial L^∞ bound (cf. Lemma 4.7).

The second part of this paper briefly deals with an application to the antimaximum principle (in short AMP). This principle concerns the problem

$$Lu = \lambda m(x)u + h(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

and says roughly the following : if λ^* denotes the largest principal eigenvalue of (1.1) , then for any $h \geq 0$, $h \not\equiv 0$, there exists $\delta > 0$ such that for $\lambda \in]\lambda^*, \lambda^* + \delta[$, the solution u of (1.3) is < 0. This AMP was first established in [4] in the case where there is no weight, i.e. $m(x) \equiv 1$ in Ω. It was later extended in [15] to the case of an indefinite weight $m \in C(\bar{\Omega})$. In this paper we extend it further to the case of an indefinite weight $m \in L^\infty(\Omega)$ (cf. Theorem 5.1). Our method of proof differs from that in [4], [15] and is more in the line of the approach introduced in [9] to deal with nonlinear operators. It was also proved in [4] that for $L = -\Delta$ and $m(x) \equiv 1$, the AMP is nonuniform, in the sense that a $\delta > 0$ cannot be found which would be valid for all h. This was derived in [4] from considerations involving the associated Green function. In this paper we use our minimax formula to prove that this nonuniformity still holds in the general case of (1.3). For further recent results involving the uniformity of the AMP, see [5], [13]. See also [12] in the selfadjoint case.

The plan of the paper is the following. In section 2, which has a preliminary character, we collect some known results on the existence of principal eigenvalues for (1.1) in the presence of an indefinite weight. Section 3 deals with the minimax formula itself, while the study of the auxiliary degenerate equations is postponed to section 4. Section 5 deals with the AMP and section 6 with its nonuniformity.

2. Principal eigenvalues

Let us start by stating the assumptions to be imposed on the operator L and the domain Ω in (1.1). Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 1$, and the coefficients of L satisfy: A is a symmetric uniformly positive definite $N \times N$ matrix, with $A \in C^{0,1}(\bar{\Omega})$, a and $a_0 \in L^\infty(\Omega)$. The weight m in (1.1) belongs to $L^\infty(\Omega)$, with $m \not\equiv 0$. These conditions will be assumed throughout the paper. More restrictions on Ω and a will be imposed later.

Our purpose in this preliminary section is to collect some known results on the existence of principal eigenvalues of (1.1), with some indications of proofs in order to allow later use. Standard references include [17], [16], [21], [7], [10].

By a principal eigenvalue we mean $\lambda \in \mathbb{R}$ such that (1.1) admits a solution $u \not\equiv 0$ with $u \geq 0$. Unless otherwise stated, solutions are understood in the strong sense, i.e. $u \in W^{2,p}(\Omega)$ for some $1 < p < \infty$, the equation is satisfied a.e. in Ω and the boundary condition is satisfied in the sense of traces. We will denote by $W(\Omega)$ the intersection of all $W^{2,p}(\Omega)$ spaces for $1 < p < \infty$.

A fundamental tool is the following form of the maximum principle, which can be derived from [11, Theorem 9.6 and Lemma 3.4].

prop2.1 **Proposition 2.1.** *Assume $a_0 \geq 0$. Let $u \in W^{2,p}(\Omega)$ satisfy*

$$Lu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

where $p > N$, $f \geq 0$, $g \geq 0$ and f or $g \not\equiv 0$. Then $u > 0$ in Ω. Moreover, if $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then $\partial u/\partial\eta(x_0) < 0$ for any exterior direction η at x_0.

Another tool is the following existence, unicity and regularity result, which follows e.g. from [14, Theorem 2.4, 2.5].

prop2.2 **Proposition 2.2.** *Let $1 < p < \infty$. If $l \in \mathbb{R}$ is sufficiently large, then the problem*

$$(L + l)u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.1}$$ eq2.1

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in L^p(\Omega)$. Moreover, the solution operator $S_l : f \to u$ is continuous from $L^p(\Omega)$ into $W^{2,p}(\Omega)$. In addition, the above holds with $l = 0$ if the problem $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$ has only the trivial solution $u \equiv 0$. This is the case in particular if $a_0 \geq 0$.

The solution operator S_l provided by Proposition 2.2 will be mainly looked at as an operator from $C_0^1(\bar{\Omega})$ into itself (and then denoted by S_{lC}). Here $C_0^1(\bar{\Omega})$ denotes the space of the C^1 functions on $\bar{\Omega}$ which vanish on $\partial\Omega$; it is endowed with its natural ordering and norm. Note that the interior of the positive cone P in $C_0^1(\bar{\Omega})$ is nonempty and made of those $u \in C_0^1(\bar{\Omega})$ such that $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$, where ν denotes the unit exterior normal.

Combining the above two propositions with the Krein-Rutman theorem for strongly positive operators (cf. e.g. [1]), one easily gets the following

lem2.3 **Lemma 2.3.** *Assume l sufficiently large. Then: (i) S_{lC} is compact and strongly positive (i.e. $f \geq 0$ with $f \not\equiv 0$ implies $u \in$ int P). (ii) The spectral radius ρ_l of S_{lC} is > 0 and ρ_l is an algebraically simple eigenvalue of S_{lC}, having an eigenfunction u in int P; in addition, there is no other eigenvalue having a nonnegative eigenfunction. (iii) For every $f \in C_0^1(\bar{\Omega})$ such that $f \geq 0$, $f \not\equiv 0$, the equation $\rho u - S_{lC}u = f$ has exactly one solution u, which belongs to int P, if $\rho > \rho_l$, and has no solution $u \geq 0$ if $\rho \leq \rho_l$.*

The above considerations apply in particular to the operator $L - \lambda m$. It follows that for each $\lambda \in \mathbb{R}$ there is a unique $\mu = \mu(\lambda) \in \mathbb{R}$ such that

$$Lu - \lambda mu = \mu u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.2}$$ eq2.2

has a solution $u = u_\lambda$ with $u \geq 0$, $u \not\equiv 0$. Moreover this solution u belongs to $W(\Omega)\cap$ int$P$, and the space of solutions of (2.2) is one dimensional.

This function $\mu : \mathbb{R} \to \mathbb{R}$ is directly related with the principal eigenvalues of (1.1) since $\lambda \in \mathbb{R}$ is a principal eigenvalue of (1.1) if and only if $\mu(\lambda) = 0$. Various properties of this function are collected in the following lemma, whose proof can for instance be adapted from that [13, Lemma 2.5]. (Note that under further assumptions on L and m, the concavity of $\mu(\lambda)$ could also be derived from Holland's formula of [18] or from Kato's result of [19] on the concavity of the spectral radius).

lem2.4 **Lemma 2.4.** *(i) If $a_0 \geq 0$, then $\mu(0) > 0$. (ii) If $m^+ \not\equiv 0$, then $\mu(\lambda) \to -\infty$ as $\lambda \to +\infty$; if $m^- \not\equiv 0$, then $\mu(\lambda) \to -\infty$ as $\lambda \to -\infty$. (iii) $\lambda \to \mu(\lambda)$ is concave and real analytic.*

We are now in a position to state the main result of this section, whose proof easily follows from Lemma 2.4 and Proposition 2.2.

prop2.5 **Proposition 2.5.** *Assume $a_0 \geq 0$. (i) If m changes sign, then (1.1) admits exactly two principal eigenvalues, one is > 0, the other is < 0. (ii) If $m \geq 0$, $m \not\equiv 0$, then (1.1) admits exactly one principal eigenvalue, which is > 0. (iii) If $m \leq 0$, $m \not\equiv 0$, then (1.1) admits exactly one principal eigenvalue, which is < 0.*

We now turn to the case where the condition $a_0 \geq 0$ of Proposition 2.5 does not hold.

prop2.6 **Proposition 2.6.** *If m changes sign, then (1.1) may have zero, one or two principal eigenvalues. If m does not change sign, then (1.1) may have zero or one principal eigenvalue.*

Proof. If m changes sign, then there exists l_0 such that the problem

$$Lu + lu = \lambda m(x)u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{2.3}$$ eq2.3

has two (resp. one, zero) principal eigenvalues for $l > l_0$ (resp. $l = l_0, l < l_0$). This is easily deduced from Lemma 2.4 since the function $\mu_l(\lambda)$ associated to (2.3) is given by $\mu_0(\lambda) + l$.

Suppose now that m does not change sign, say $m \geq 0$ in Ω. Then (1.1) has at most one principal eigenvalue. Indeed if it had two, then by Lemma 2.4, $\mu_0(\lambda) \to -\infty$ not only as $\lambda \to +\infty$ but also as $\lambda \to -\infty$. This implies that $\mu_l(\lambda)$ has two distinct zeros for $l \geq 0$; taking l such that $a_0(x) + l \geq 0$ in Ω, one gets a contradiction with part (ii) of Proposition 2.5.

We finally give a simple example showing that (1.1) with $m \geq 0$ may have no principal eigenvalue. (More refined results in this direction can be found in [21], [7], [10]). We will show that if $m \in L^\infty(\Omega)$, $m \not\equiv 0$ vanishes on a ball $B \subset \Omega$, then

$$-\Delta u - lu = \lambda m(x)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.4}$$ eq2.4

has no principal eigenvalue for $l > \lambda_1^B$, where λ_1^B is the principal eigenvalue of $-\Delta$ on $H_0^1(B)$. Indeed, for such a value of l, there exists $v \in H_0^1(B)$ such that $\int_B (|\nabla v|^2 - lv^2) < 0$. Using the fact that if $u \in H_0^1(\Omega)$ satisfies $\int_\Omega mu^2 = 1$, then $\int_\Omega m(u + r\widetilde{v})^2 = 1$ for any $r \in \mathbb{R}$ ($\widetilde{v}$ denotes v extended by 0 on $\Omega \backslash B$), one deduces that

$$\inf\{\int_\Omega (|\nabla u|^2 - lu^2) : u \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega mu^2 = 1\} = -\infty. \tag{2.5}$$ eq2.5

Suppose now by contradiction that (2.4) admits a principal eigenvalue λ^*. Applying the classical Rayleigh formula to $-\triangle u - lu + ku = (\lambda^* m(x) + k)u$ with k taken $> l$, one gets

$$1 = \inf\{\frac{\int_\Omega(|\nabla u|^2 - lu^2 + ku^2)}{\int_\Omega(k+\lambda^* m)u^2} : u \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega (k+\lambda^* m)u^2 > 0\}. \tag{2.6}$$ eq2.6

Choosing k larger if necessary so that $k > \lambda^*\|m\|_\infty$, one observes that any nonzero $u \in H_0^1(\Omega)$ satisfies the constraint in (2.6). Consequently, by (2.6),

$$\lambda^* \leq \int_\Omega (|\nabla u|^2 - lu^2)$$

for all $u \in H_0^1(\Omega)$ with $\int_\Omega mu^2 = 1$. But this contradicts (2.5). □

3. Minimax Formula

The operator L and the weight m in this section are assumed to satisfy the conditions indicated at the beginning of section 2, with in addition $a \in C^{0,1}(\bar{\Omega})$ and Ω of class C^2. Our purpose is to give a formula of minimax type for the principal eigenvalues of (1.1).

Let us define the distance function to the boundary $d(x) := \text{dist}\,(x, \partial\Omega)$ and call

$$D(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable } : \exists c_i = c_i(u) > 0 \\ \text{such that } c_1 d \leq u \leq c_2 d \text{ a.e. in } \Omega\}.$$

Note that the positive eigenfunctions associated to the principal eigenvalues to $D(\Omega)$. Let us also define, for $\sigma \in \mathbb{R}$, the weighted Sobolev space

$$H^1(\Omega, d^\sigma) := \{u \in H^1_{\text{loc}}(\Omega) : \int_\Omega d^\sigma(u^2 + |\nabla u|^2) < \infty\},$$

which is endowed with the norm given by the square root of the above integral.

theo3.1 **Theorem 3.1.** *Suppose $a_0 \geq 0, m^+ \not\equiv 0$ and let λ^* be the largest principal eigenvalue of* (1.1) *(cf. Proposition 2.5). Then*

$$\lambda^* = \inf_{u\in U} \sup_{v \in H^1(\Omega, d^2)} \frac{\Lambda(u) - Q_u(v)}{\int_\Omega mu^2} \tag{3.1}$$ eq3.1

where

$$U := \{u \in H^1(\Omega) \cap D(\Omega) : \int_\Omega mu^2 > 0\},$$

$$\Lambda(u) := \int_\Omega (\langle A\nabla u, \nabla u\rangle + \langle a, \nabla u\rangle u + a_0 u^2),$$

$$Q_u(v) := \int_\Omega u^2(\langle A\nabla v, \nabla v\rangle - \langle a, \nabla v\rangle).$$

Moreover, the infimum and the supremum in (3.1) *are achieved.*

Note that the smallest principal eigenvalue can be handled via Theorem 3.1, after changing m into $-m$.

The following two lemmas will be used in the proof of Theorem 3.1. They concern auxiliary equations which degenerate on $\partial\Omega$ and which will be considered in a suitable weak sense. The proof of these two lemmas will be given in section 4. The first one deals with Q_u. The second one introduces a function G whose role is the following : in the selfadjoint case, the minimum of the Rayleigh quotient is achieved at an eigenfunction; it will turn out that in the present nonselfadjoint situation, the infimum in (3.1) is achieved for u equal to an eigenfunction multiplied by $\sqrt{G}$.

lem3.2 **Lemma 3.2.** *For any $u \in D(\Omega)$, the infimum of Q_u on $H^1(\Omega, d^2)$ is achieved at some W_u. This W_u is unique up to an additive constant and can be characterized as the solution of*

$$\begin{gathered} W_u \in H^1(\Omega, d^2), \\ \int_\Omega u^2 \langle 2A\nabla W_u - a, \nabla\varphi\rangle = 0 \quad \forall \varphi \in H^1(\Omega, d^2). \end{gathered} \tag{3.2}$$

eq3.2

Moreover

$$Q_u(W_u) = -\int_\Omega u^2 \langle A\nabla W_u, \nabla W_u\rangle = -\frac{1}{2}\int_\Omega u^2\langle a, \nabla W_u\rangle.$$

lem3.3 **Lemma 3.3.** *Let $u \in D(\Omega) \cap C^1(\bar{\Omega})$. Then the problem*

$$\begin{gathered} G \in H^1(\Omega, d^2), \\ \int_\Omega u^2 \langle A\nabla G + aG, \nabla\varphi\rangle = 0 \quad \forall \varphi \in H^1(\Omega, d^2) \end{gathered} \tag{3.3}$$

eq3.3

has a non trivial solution G, which is unique up to a multiplicative constant and satisfies

$$c_1 \le G \le c_2 \quad a.e.\ in\ \Omega \tag{3.4}$$

eq3.4

for some constants $c_i > 0$.

Note that by Lemma 3.2, formula (3.1) can be stated equivalently as

$$\lambda^* = \inf_{u\in U} \frac{\Lambda(u) - Q_u(W_u)}{\int_\Omega mu^2}. \tag{3.5}$$

eq3.5

Once these two lemmas are accepted, the proof of (3.1) can be carried out by following the same general lines as in [13], and we will only indicate below the main differences. In this adaptation of [13], special care must be taken to the boundary behaviour of the functions involved, and the introduction of $D(\Omega)$, $H^1(\Omega, d^2)$ plays in this respect a central role.

Proof of Theorem 3.1. Let u^* be an eigenfunction associated to λ^* and satisfying $u^* \in W(\Omega) \cap \quad \text{int}\, P$. We will first prove that inequality $\le$ holds in (3.5), i.e.

$$\lambda^* \int_\Omega mu^2 \le \Lambda(u) - Q_u(W_u) \tag{3.6}$$

eq3.6

for all $u \in U$. Call $v^* := -\log u^*$. Then $v^* \in W^{2,p}_{\rm loc}(\Omega)$ for all $1 < p < \infty$ and satisfies

$$-\operatorname{div}(A\nabla v^*) = -\langle A\nabla v^*, \nabla v^*\rangle - \langle a, \nabla v^*\rangle + a_0 - \lambda^* m \quad \text{in } \Omega. \tag{3.7}$$

eq3.7

Note that, unlike (3.10) from [13], no boundary condition appears here since $v^* = +\infty$ on $\partial\Omega$. Now one takes $u \in U$, multiply both sides of equation (3.7) by u^2, integrate and use as in formula (3.11) of [13] an argument based on the idea of completing a square to obtain

$$\int_\Omega \langle A\nabla u^2 - u^2 w_u, \nabla v^*\rangle + \lambda^* \int_\Omega mu^2 \le \frac{1}{4}\int_\Omega u^2\langle a + w_u, A^{-1}(a + w_u)\rangle + \int_\Omega a_0 u^2 \tag{3.8}$$

eq3.8

where $w_u := -a + 2A((\nabla u/u) + \nabla W_u)$. In this process one should verify that all the integrals involved do make sense in the usual $L^1(\Omega)$ sense, which is easy by using the regularity of u^* and the fact that $u^* \in D(\Omega)$, $u \in D(\Omega) \cap H^1(\Omega)$ and $W_u \in H^1(\Omega, d^2)$. One should also justify the use of the divergence theorem to write

$$\int_\Omega [\operatorname{div}(A\nabla v^*)]u^2 = -\int_\Omega \langle A\nabla v^*, \nabla u^2\rangle. \tag{3.9}$$ eq3.9

This latter formula follows by applying Lemma 3.4 below to the vector field $V = A(\nabla v^*)u^2$.

Once (3.8) is obtained, the calculation on page 96 from [13] can be pursued without any change to derive (3.6) above. The only point to be observed at this stage is the (easily verified) fact that $\log u \in H^1(\Omega, d^2)$, which allows the use of equation (3.2) for W_u with $\varphi = \log u$ as testing function.

We will now show that if we put $\tilde{u} := u^*\sqrt{G^*}$, where G^* is a function provided by Lemma 3.3 for $u = u^*$, then $\tilde{u} \in U$ and equality holds in (3.6). This will conclude the proof of Theorem 3.1.

One first observes that $\tilde{u} \in H^1(\Omega) \cap D(\Omega)$ and then argue as on page 97 from [13], multiplying both sides of equation (3.7) by $\tilde{u}^2$ and integrating to reach now

$$\int_\Omega \langle A\nabla\tilde{u}^2 - \tilde{u}^2\eta, \nabla v^*\rangle + \lambda^* \int_\Omega m\tilde{u}^2 = \frac{1}{4}\int_\Omega \tilde{u}^2[\langle a+\eta, A^{-1}(a+\eta) + a_0] \tag{3.10}$$ eq3.10

where $\eta := -a - 2A\nabla v^*$. The rest of the calculation on page 97 from [13] can be pursued without any change. It uses in particular the fact that $W_{\tilde{u}} = -(\log G^*)/2$ up to an additive constant, which follows from (3.2) and (3.3) above. Proceeding in this way, one reaches equality in (3.6) for $u = \tilde{u}$.

It remains to see that $\tilde{u} \in U$, i.e. that $\int_\Omega m\tilde{u}^2 > 0$. For this purpose one deduces as in formula (3.16) from [13] that

$$\lambda^* \int_\Omega m\tilde{u}^2 = \int_\Omega \tilde{u}^2[\langle A\nabla v^*, \nabla v^*\rangle + a_0],$$

and the conclusion follows since $\lambda^* > 0$, v^* is not a constant and $a_0 \geq 0$. □

lem3.4 **Lemma 3.4.** *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$. Let $V : \Omega \to \mathbb{R}^N$ be a vector field in $L^\infty(\Omega)$ such that* $\operatorname{div} V \in L^1(\Omega)$ *and $\|V\|_{L^\infty(\Gamma_\epsilon)} \to 0$ as $\epsilon \to 0$, where*

$$\Gamma_\epsilon := \{x \in \bar{\Omega} : d(x) < \epsilon\}.$$

Then $\int_\Omega \operatorname{div} V = 0$.

The proof of the above lemma is an easy adaptation of the proof in [6, Lemma A.1]. We now turn to the case where the condition $a_0 \geq 0$ of Theorem 3.1 does not hold.

theo3.5 **Theorem 3.5.** *Assume $m^+ \not\equiv 0$. Assume also the existence of a principal eigenvalue for* (1.1) *and let λ^* be the largest of these principal eigenvalues (cf. Proposition 2.6). Then formula* (3.1) *holds for λ^* .*

Note that the smallest principal eigenvalue can be handled by Theorem 3.5, after changing m into $-m$.

Proof of Theorem 3.5. Applying formula (3.1) to $Lu - \lambda mu + lu = (\mu(\lambda) + l)u$ with l sufficiently large, one deduces that

$$\mu(\lambda) = \inf_{u \in H^1(\Omega) \bigcap D(\Omega)} \frac{\Lambda(u) - \lambda\int_\Omega mu^2 - \inf Q_u}{\int_\Omega u^2}, \tag{3.11}$$ eq3.11

where here and below inf Q_u denotes inf $\{Q_u(v) : v \in H^1(\Omega, d^2)\}$. Consequently $\mu(\lambda)$ is ≥ 0 at a given λ if and only if the following three conditions hold:

$$\lambda \leq \Lambda(u) - \inf Q_u \quad \text{for all} \quad u \in D(\Omega) \cap H^1(\Omega) \quad \text{with} \int_\Omega mu^2 = 1, \tag{3.12} \qquad \boxed{\texttt{eq3.12}}$$

$$\lambda \geq -\Lambda(u) + \inf Q_u \quad \text{for all} \quad u \in D(\Omega) \cap H^1(\Omega) \quad \text{with} \int_\Omega mu^2 = -1, \tag{3.13} \qquad \boxed{\texttt{eq3.13}}$$

$$0 \leq \Lambda(u) - \inf Q_u \quad \text{for all} \quad u \in D(\Omega) \cap H^1(\Omega) \quad \text{with} \int_\Omega mu^2 = 0. \tag{3.14} \qquad \boxed{\texttt{eq3.14}}$$

Note that the class of u's in (3.13) and (3.14) may be empty.
Claim. If (3.12) holds for some λ, then (3.14) also holds.

Proof of the claim. Let $u \in D(\Omega) \cap H^1(\Omega)$ with $\int_\Omega mu^2 = 0$. Take $\psi \in C_c^\infty(\Omega)$ such that $\int_\Omega m(u + \varepsilon\psi)^2 > 0$ for $\varepsilon > 0$ sufficiently small and call $u_\varepsilon = u + \varepsilon\psi$. Condition (3.12) gives

$$\lambda \int_\Omega mu_\varepsilon^2 \leq \Lambda(u_\varepsilon) - \inf Q_{u\varepsilon}$$

and so, since $u_\varepsilon \to u$ in $H^1(\Omega)$ as $\varepsilon \to 0$, the conclusion (3.14) will follow if we show that

$$\inf Q_{u_\varepsilon} \to \inf Q_u \quad \text{as } \varepsilon \to 0. \tag{3.15} \qquad \boxed{\texttt{eq3.15}}$$

To prove (3.15) fix a ball $\bar{B} \subset \Omega$ and recall that by Lemma 3.2,

$$\inf Q_{u_\varepsilon} = Q_{u_\varepsilon}(W_{u_\varepsilon}) = -\int_\Omega u_\varepsilon^2 \langle A\nabla W_{u_\varepsilon}, \nabla W_{u_\varepsilon}\rangle = -\frac{1}{2}\int_\Omega u_\varepsilon^2 \langle a, \nabla W_{u_\varepsilon}\rangle \tag{3.16} \qquad \boxed{\texttt{eq3.16}}$$

for a unique $W_{u_\varepsilon} \in H_B^1(\Omega, d^2)$, where this later space is defined below in Lemma 4.2. Using (3.16), the ellipticity of L and the fact that $c_1 d \leq u_\varepsilon \leq c_2 d$ for some positive constants c_1, c_2 and all $\varepsilon > 0$ sufficiently small, one gets

$$\int_\Omega d^2 |\nabla W_{u_\varepsilon}|^2 \leq c_3 \int_\Omega d^2 \langle a, \nabla W_{u_\varepsilon}\rangle \leq c_4 (\int_\Omega d^2 |\nabla W_{u_\varepsilon}|^2)^{\frac{1}{2}}$$

for some other constants c_3, c_4. This implies that ∇W_{u_ε} remains bounded in $L^2(\Omega, d^2)$, and consequently, by Lemma 4.2 below, W_{u_ε} remains bounded in the space $H_B^1(\Omega, d^2)$. It follows that for some subsequence, $W_{u_\varepsilon} \to W$ weakly in $H_B^1(\Omega, d^2)$. Going to the limit in equation (3.2) for W_{u_ε} and using the fact that $(\frac{u_\varepsilon}{d})^2 \to (\frac{u}{d})^2$ in $L^2(\Omega, d^2)$, one then sees that $W = W_u$. Finally one deduces (3.15) from the last equality in (3.16). This completes the proof of the claim. □

Recall that by Lemma 2.4, the existence of a principal eigenvalue is equivalent to the existence of λ with $\mu(\lambda) \geq 0$. It then follows from (3.12), (3.13) and (3.14), using the above claim and Lemma 2.4, that $\{\lambda \in \mathbb{R} : \mu(\lambda) \geq 0\}$ is a nonempty closed interval with left and right extremities given respectively by

$$\sup\{-\Lambda(u) + \inf Q_u : u \in D(\Omega) \cap H^1(\Omega) \quad \text{with} \int_\Omega mu^2 = -1\},$$

$$\inf\{\Lambda(u) - \inf Q_u : u \in D(\Omega) \cap H^1(\Omega) \quad \text{with} \int_\Omega mu^2 = 1\},$$

where the above supremum is $-\infty$ in case $m \geq 0$ in Ω; moreover the largest principal eigenvalue λ^* is the right extremity of this interval, i.e. the infimum above. This is exactly saying that formula (3.1) holds for λ^*. □

rem3.6 **Remark 3.6.** In the context of Theorem 3.5, it is not clear whether the infimum in (3.1) is achieved. This is however so when $m(x) \geq \varepsilon > 0$ since then, by writing (1.1) as $Lu + lmu = (\lambda + l)mu$, one can reduce to Theorem 3.1.

rem3.7 **Remark 3.7.** The proof of Theorem 3.5 shows that by using Lemmas 2.4 and 3.2, formula (3.1) for a problem with weight can be deduced from formula (3.1) for a problem without weight.

rem3.8 **Remark 3.8.** Formula (3.1) in the presence of an indefinite weight was considered recently in [2] in the particular case where $a_0 = div\ a$. Beside C^∞ smoothness of the coefficients and of the weight, [2] requires an extra hypothesis on the principal eigenvalue λ^*, namely $\int_\Omega m(u^*)^2 G^* > 0$. Theorem 3.5 shows that this extra hypothesis is not needed for formula (3.1) to hold. The proof in [2] relies as in [18] on stochastic differential equations.

rem3.9 **Remark 3.9.** When $A^{-1}a$ in (1.2) is a gradient, then (3.1) reduces to a formula of Rayleigh quotient type. Indeed, if $-A^{-1}a = \nabla\alpha$, then (1.1) can be rewritten as

$$\tilde{L}u := -\operatorname{div}(\tilde{A}(x)\nabla u) + \tilde{a}_0(x)u = \lambda\tilde{m}(x)u \quad \text{in } \Omega, \ \ u = 0 \quad \text{on } \partial\Omega,$$

where $\tilde{A} = e^\alpha A$, $\tilde{a}_0 = e^\alpha a_0$ and $\tilde{m} = e^\alpha m$. So by the usual Rayleigh formula,

$$\lambda^* = \min\{\int_\Omega (\langle \tilde{A}\nabla u, \nabla u\rangle + \tilde{a}_0 u^2) : u \in H_0^1(\Omega) \text{ and } \int_\Omega \tilde{m}u^2 = 1\}. \tag{3.17}$$ eq3.17

Since the minimum in (3.17) is achieved at one u which belongs to $D(\Omega)$, one can limit oneself in (3.17) to taking u in $H_0^1(\Omega) \cap D(\Omega)$; moreover, writing u as $e^{-\alpha/2}w$, (3.17) becomes

$$\begin{aligned}\lambda^* = \min\{ &\int_\Omega (\langle A\nabla w, \nabla w\rangle + \langle a, \nabla w\rangle w + a_0 w^2 + \frac{1}{4}\langle a, A^{-1}a\rangle w^2) : \\ &w \in H_0^1(\Omega) \cap D(\Omega) \text{ and } \int_\Omega mw^2 = 1\}.\end{aligned} \tag{3.18}$$ eq3.18

But, by completing a square,

$$\begin{aligned}&\max\big\{ -Q_w(v) : v \in H^1(\Omega, d^2)\big\} \\ &= \max\{-\int_\Omega w^2 \langle A(\nabla v - \frac{1}{2}A^{-1}a), \nabla v - \frac{1}{2}A^{-1}a\rangle + \frac{1}{4}\int_\Omega w^2\langle a, A^{-1}a\rangle \\ &\quad : v \in H^1(\Omega, d^2)\big\} \\ &= \frac{1}{4}\int_\Omega w^2 \langle a, A^{-1}a\rangle,\end{aligned}$$

which implies that (3.18) reduces to the minimax formula (3.1).

rem3.10 **Remark 3.10.** Minimax formulas of a different nature, in the line of the classical formula of Barta, can be found in [3].

4. Two degenerate elliptic equations

In this section we give a proof of Lemmas 3.2 and 3.3. The assumptions on L and Ω are the same as in section 3. Beside the weighted Sobolev space $H^1(\Omega, d^\sigma)$, we will use for $\sigma \in \mathbb{R}$ the weighted Lebesgue space

$$L^p(\Omega, d^\sigma) := \{u \text{ measurable on } \Omega : \int_\Omega d^\sigma |u|^p < \infty\},$$

which is endowed with the norm given by the p-th root of the above integral.

The following three lemmas concern these spaces. Lemma 4.1 is a particular case of imbedding results in [22, Theorems 2.4 and 2.5]. See also [20, Theorem 8.2] and [23, Theorem 19.5]. Lemma 4.2 is a Poincaré type inequality, which follows easily from Lemma 4.1. Lemma 4.3 is a particular case of another imbedding result in [23, Theorem 19.9].

lem4.1 **Lemma 4.1.** *$H^1(\Omega, d^2)$ is continuously imbedded into $L^2(\Omega)$, and compactly imbedded into $L^2(\Omega, d^\epsilon)$ for any $\epsilon > 0$.*

lem4.2 **Lemma 4.2.** *Fix a ball B such that $\bar{B} \subset \Omega$ and let $\epsilon > 0$. Then there exists $c = c(\Omega, B, \epsilon)$ such that*

$$\|u\|_{L^2(\Omega,d^\epsilon)} \le c\|\nabla u\|_{L^2(\Omega,d^2)} \quad \forall u \in H^1_B(\Omega, d^2),$$

where $H^1_B(\Omega, d^2)$ denotes the subspace of $H^1(\Omega, d^2)$ made of those u such that $\int_B u = 0$.

Proof. It clearly suffices to consider the case where $\epsilon \le 2$. Assume by contradiction that for each $k = 1, 2, \dots$ there exists $u_k \in H^1_B(\Omega, d^2)$ such that

$$\|u_k\|_{L^2(\Omega,d^\epsilon)} > k\|\nabla u_k\|_{L^2(\Omega,d^2)}.$$

One can assume $\|u_k\|_{H^1(\Omega,d^2)} = 1$ and so, for a subsequence, u_k converges weakly to some u in $H^1_B(\Omega, d^2)$. By Lemma 4.1, $u_k \to u$ in $L^2(\Omega, d^\epsilon)$, and by the inequality above, $\nabla u_k \to 0$ in $L^2(\Omega, d^2)$. So $u_k \to u$ in $H^1_B(\Omega, d^2)$, $\|u\|_{H^1_B(\Omega,d^2)} = 1$, and $u \equiv$ constant. But this is impossible, since $\int_B u = 0$. □

lem4.3 **Lemma 4.3.** *$H^1(\Omega, d^2)$ is continuously imbedded into $L^p(\Omega, d^2)$ for $p \le 2 + 4/N$.*

Proof of Lemma 3.2. Let $u \in D(\Omega)$ and fix $B \subset \Omega$ as in Lemma 4.2. It is clear that Q_u is continuous on $H^1(\Omega, d^2)$; moreover, by ellipticity, one has

$$Q_u(v) \ge c_1\|\nabla v\|^2_{L^2(\Omega,d^2)} - c_2\|\nabla v\|_{L^2(\Omega,d^2)}$$

for some constants $c_1 > 0$ and $c_2 \ge 0$ and all $v \in H^1(\Omega, d^2)$. Combining with Lemma 4.2 yields that Q_u is coercive on $H^1_B(\Omega, d^2)$. It follows that the strictly convex functional Q_u achieves its minimum on $H^1_B(\Omega, d^2)$ at a unique $W_u \in H^1_B(\Omega, d^2)$. Moreover, this W_u is characterized by

$$\int_\Omega u^2\langle 2A\nabla W_u - a, \nabla\varphi\rangle = 0 \quad \forall \varphi \in H^1_B(\Omega, d^2). \tag{4.1}$$ eq4.1

Clearly the minimum of Q_u on $H^1_B(\Omega, d^2)$ coincides with its minimum on $H^1(\Omega, d^2)$; moreover (4.1) holds for all $\varphi \in H^1_B(\Omega, d^2)$ if and only if it holds for all $\varphi \in H^1(\Omega, d^2)$. It follows that W_u is characterized up to an additive constant as the solution of (3.2). Finally taking $\varphi = W_u$ in (3.2), one deduces the formulas for $Q_u(W_u)$. □

The proof of Lemma 3.3 will be more involved. Writing the equation in (3.3) as

$$\mathcal{L}G := -\operatorname{div}(u^2(A\nabla G + aG)) = 0, \tag{4.2}$$ eq4.2

we will first show that for l sufficiently large, some sort of inverse of $(\mathcal{L} + lu^2)$ is well-defined and compact (cf. Lemma 4.4), and enjoys a rather strong positivity property (cf. Lemma 4.5). This allows the application of a version of the Krein-Rutman theorem for irreducible operators, which yields a positive solution G of (3.3) (cf. Lemma 4.6). The remaining parts of the proof of Lemma 3.3 consist in proving that G belongs to $L^\infty(\Omega)$ (cf. Lemma 4.7) and is bounded away from zero (cf. Lemma 4.8).

lem4.4 **Lemma 4.4.** *Let $u \in D(\Omega)$. Then for l sufficiently large, the problem*

$$\begin{cases} g \in H^1(\Omega, d^2), \\ \int_\Omega u^2 \langle A\nabla g + ag, \nabla\varphi\rangle + \int_\Omega lu^2 g\varphi = \int_\Omega u^2 f\varphi \quad \forall \varphi \in H^1(\Omega, d^2) \end{cases} \tag{4.3}$$ eq4.3

has for each $f \in L^2(\Omega, d^2)$ a unique solution g. Moreover the solution operator $T_l : f \to g$ is continuous from $L^2(\Omega, d^2)$ into $H^1(\Omega, d^2)$, and compact from $L^2(\Omega, d^2)$ into itself.

Proof. The left-hand side of (4.3) defines a bilinear form $B_l(g, \varphi)$ which is clearly continuous on $H^1(\Omega, d^2)$. It is also coercive for l sufficiently large. Indeed, using the inequality $2rs \le (\varepsilon r)^2 + (s/\varepsilon)^2$, one easily obtains, for l sufficiently large,

$$B_l(\varphi, \varphi) \ge c\|\varphi\|^2_{H^1(\Omega, d^2)} \tag{4.4}$$ eq4.4

for some constant $c > 0$ and all $\varphi \in H^1(\Omega, d^2)$. The right-hand side of (4.3) defines for $f \in L^2(\Omega, d^2)$ a continuous linear form on $H^1(\Omega, d^2)$. It thus follows from the Lax-Milgram lemma that (4.3) has a unique solution g, with moreover the continuous dependance of $g \in H^1(\Omega, d^2)$ with respect to $f \in L^2(\Omega, d^2)$. Finally the compactness of the solution operator T_l in $L^2(\Omega, d^2)$ follows from Lemma 4.1. $\square$

lem4.5 **Lemma 4.5.** *Let $u \in D(\Omega) \cap C^1(\bar{\Omega})$. Then for l sufficiently large, the solution operator T_l of Lemma 4.4 enjoys the following positivity property : if $f \in L^2(\Omega, d^2)$ is ≥ 0 and $\not\equiv 0$, then for any $\Omega' \subset\subset \Omega$,*

$$\operatorname*{ess\,inf}_{x \in \Omega'} (T_l f)(x) > 0. \tag{4.5}$$ eq4.5

Proof. Let f be as in the statement of the lemma and call $g = T_l f$. Taking $-g^-$ as testing function in (4.3), one obtains $B_l(g^-, g^-) \le 0$ and consequently, by (4.4), g is ≥ 0, with clearly $g \not\equiv 0$. It remains to prove (4.5).

To do so, we will first consider the particular case where the vector field a in (1.2) satisfies

$$\langle a, \nu\rangle > 0 \quad \text{on } \partial\Omega. \tag{4.6}$$ eq4.6

Since $u \in D(\Omega) \cap C^1(\bar{\Omega})$, ∇u on $\partial\Omega$ is a strictly negative multiple of ν and consequently, by continuity, (4.6) implies

$$\langle a, \nabla u\rangle < 0 \quad \text{on } \Gamma_\varepsilon \tag{4.7}$$ eq4.7

for some $\varepsilon = \varepsilon(a, u) > 0$, where Γ_ε was defined in Lemma 3.4. Consider now the zero order coefficient of $\mathcal{L} + lu^2$, where $\mathcal{L}$ is defined in (4.2). It is equal to $u(-2\langle a, \nabla u\rangle - (\operatorname{div} a)u + lu)$ and so, using (4.7) and the fact that $a \in C^{0,1}(\bar{\Omega})$, one easily sees that taking l larger if necessary (depending on u and a), this coefficient can be made ≥ 0 on Ω. Since the solution g of (4.3) belongs to $H^1_{\rm loc}(\Omega)$ and is a weak solution of $(\mathcal{L} + lu^2)g = fu^2$ in Ω, the strong maximum principle can be applied on any $\Omega'' \subset\subset \Omega$ (cf. [11, Theorem 8.19]), which yields the conclusion (4.5).

Let us now consider the general case where (4.6) possibly does not hold. Let us write g as hw, where w is a (fixed) function with the following properties :

$$w \in C^{1,1}(\bar{\Omega}),\ w > 0 \quad \text{on } \bar{\Omega}, \quad \langle A\frac{\nabla w}{w} + a, \nu\rangle > 0 \quad \text{on } \partial\Omega. \tag{4.8}$$ eq4.8

The existence of such a function w will be shown later. Clearly $h = g/w \in H^1_{\text{loc}}(\Omega)$ and is a weak solution of

$$-\operatorname{div}\left[(u^2w)(A\nabla h + (A\frac{\nabla w}{w} + a)h)\right] + l(u^2w)h = \frac{f}{w}(u^2w) \quad \text{in } \Omega. \tag{4.9}$$ eq4.9

Equation (4.9) is of the same type as $(\mathcal{L} + lu^2)g = fu^2$: u is replaced by $u\sqrt{w}$ (which still belongs to $D(\Omega) \cap C^1(\bar{\Omega})$), a is replaced by $A(\nabla w/w) + a$ (which still belongs to $C^{0,1}(\bar{\Omega})$ but now satisfies (4.6)), and f is replaced by f/w (which still belongs to $L^2(\Omega, d^2)$). It follows that the preceding argument can be repeated for (4.9), which yields that

$$\operatorname*{ess\,inf}_{x\in\Omega'} h(x) > 0 \text{ for any } \Omega' \subset\subset \Omega.$$

The conclusion (4.5) for $g = hw$ then follows.

It remains to show the existence of a function w satisfying (4.8). Putting $w = e^v$, it suffices to construct $v \in C^{1,1}(\bar{\Omega})$ such that

$$\langle A\nabla v + a, \nu\rangle > 0 \quad \text{on } \partial\Omega. \tag{4.10}$$ eq4.10

By the regularity of Ω, any point in $\partial\Omega$ belongs to an open set U such that there exists a $C^{1,1}$ diffeomorphism X from U onto the unit ball $B \subset \mathbb{R}^N$ with the properties that $B \cap \{x_N > 0\}$ corresponds to $\Omega \cap U$ and $B \cap \{x_N = 0\}$ corresponds to $\partial\Omega \cap U$. We take an open covering $\{V^j : j = 1, \dots, m\}$ of $\partial\Omega$ such that $V^j \subset\subset U^j$ with (U^j, X^j) as (U, X) above. We also take functions $\Psi^j \in C^{1,1}(\mathbb{R}^N)$ such that $\operatorname{supp} \psi^j \subset U^j, \psi^j \equiv 1$ on V^j and $0 \le \Psi^j \le 1$. Define for $P \in \bar{\Omega}$

$$v(P) = r \sum_j \Psi^j(P) X^j_N(P)$$

where X^j_N is the N^{th} component of X^j and r is a constant to be chosen later. Clearly $v \in C^{1,1}(\mathbb{R}^N)$, and for $P \in \partial\Omega$,

$$\nabla v(P) = r \sum_j \Psi^j(P) c^j(P) \nu(P) \tag{4.11}$$ eq4.11

since $\nabla X^j_N(P) = c^j(P)\nu(P)$ where $\nu(P)$ is the exterior normal at P and $c^j(P) < 0$. Calling $rf(P)$ the coefficient of $\nu(P)$ in the right-hand side of (4.11), one has $f \in C^{0,1}(\partial\Omega)$ and $f < 0$ on $\partial\Omega$ (since the V^j's cover $\partial\Omega$). One also has

$$\langle A\nabla v + a, \nu\rangle = rf\langle A\nu, \nu\rangle + \langle a, \nu\rangle,$$

which is > 0 on $\partial\Omega$ if the constant r is chosen sufficiently large (< 0). Inequality (4.10) thus follows. $\square$

lem4.6 **Lemma 4.6.** *Let $u \in D(\Omega) \cap C^1(\bar{\Omega})$. Then problem (3.3) has a solution G, which is unique up to a multiplicative constant and which satisfies*

$$\operatorname*{ess\,inf}_{x\in\Omega'} G(x) > 0 \tag{4.12}$$ eq4.12

for any $\Omega' \subset\subset \Omega$.

Proof. We recall that in the context of a Lebesgue space $L^p(E, d\mu)$ with $1 \le p < \infty$, the irreducibility of a positive operator T can be characterized by the property that E does not admit any nontrivial subset F which is invariant for T (cf. [26], [24]); invariant here means that $f = 0$ a.e. on F implies $Tf = 0$ a.e. on F. Lemmas 4.4 and 4.5 thus imply that the Krein-Rutman theory for compact positive irreducible operators (cf. e.g. [26], [24]) can be applied to T_l in $L^2(\Omega,d^2)$ for l sufficiently large. This yields that the spectral radius $\bar\rho_l$ of T_l is >0 and is a simple eigenvalue of T_l having an eigenfunction $G\geq 0$, $G\not\equiv 0$. Now $T_lG=\bar\rho_lG$ implies that G satisfies (4.12) and that

$$\int_\Omega u^2\langle A\nabla(\bar\rho_lG)+a(\bar\rho_lG),\nabla\varphi\rangle+\int_\Omega lu^2(\bar\rho_lG)\varphi=\int_\Omega u^2G\varphi \tag{4.13}$$ eq4.13

for all $\varphi\in H^1(\Omega,d^2)$. Taking $\varphi\equiv 1$ yields $\bar\rho_l=1/l$, which shows that (4.13) reduces to (3.3). So G solves (3.3). Finally the statement about unicity in Lemma 4.6 follows from the fact that (3.3) can now be rewritten as $T_lG=\bar\rho_lG$. □

lem4.7 **Lemma 4.7.** *Let $u\in D(\Omega)\cap C^1(\bar\Omega)$. Then the function G provided by Lemma 4.6 belongs to $L^\infty(\Omega)$.*

Proof. It is inspired from Moser's iteration technique as given for instance in [11, Theorem 8.15]. For $\beta\geq 1$ and $M>0$, let $H\in C^1[0,+\infty[$ be defined by setting $H(r)=r^\beta$ for $r\in[0,M]$ and taking H to be linear for $r\geq M$. Put $v(x):=\int_0^{G(x)}[H'(s)]^2ds$. One has that $v\in H^1(\Omega,d^2)$ since

$$v(x)=\begin{cases}\frac{\beta^2}{2\beta-1}G(x)^{2\beta-1} & \text{if } G(x)\leq M,\\ \frac{\beta^2}{2\beta-1}M^{2\beta-1}+\beta^2M^{2\beta-2}(G(x)-M) & \text{if } G(x)>M,\end{cases}$$

$\nabla v=(H'(G))^2\nabla G$ and $G\in H^1(\Omega,d^2)$. So v is an admissible test function in (3.3) and consequently

$$\int_\Omega u^2\langle A\nabla G+aG,\nabla v\rangle=0.$$

Using the inequality $2rs\leq(\varepsilon r)^2+(s/\varepsilon)^2$, one obtains from the above that

$$\int_\Omega u^2|\nabla(H(G))|^2\leq c_1\int_\Omega u^2(H'(G))^2G^2 \tag{4.14}$$ eq4.14

where $c_1=c_1(A,a)$. On the other hand $H(G)\in H^1(\Omega,d^2)$ and so, fixing p with $2<p\leq 2+4/N$, one has by Lemma 4.3

$$\|H(G)\|_{L^p(\Omega,d^2)}\leq c_2\|H(G)\|_{H^1(\Omega,d^2)} \tag{4.15}$$ eq4.15

where $c_2=c_2(\Omega,p)$. Combining (4.14) and (4.15) and using the fact that $u\in D(\Omega)$, it follows

$$\|H(G)\|_{L^p(\Omega,d^2)}\leq c\left(\|H(G)\|_{L^2(\Omega,d^2)}+\|H'(G)G\|_{L^2(\Omega,d^2)}\right) \tag{4.16}$$ eq4.16

where c depends on L,Ω,u,p but does not depend on G,β,M. The function H above depends on M, i.e. $H=H_M$, and when $M\to+\infty$, one has that for each $r\geq 0$, $H_M(r)\to\tilde H(r)$ and $H_M'(r)\to\tilde H'(r)$ in a nondecreasing way, where $\tilde H(r)=r^\beta$. The monotone convergence theorem can thus be applied to (4.16), which shows that (4.16) still holds with H replaced by $\tilde H$. This means that

$$\|G^\beta\|_{L^p(\Omega,d^2)}\leq c(1+\beta)\|G^\beta\|_{L^2(\Omega,d^2)};$$

i.e.,

$$\|G\|_{L^{p\beta}(\Omega,d^2)}\leq[c(1+\beta)]^{1/\beta}\|G\|_{L^{2\beta}(\Omega,d^2)}, \tag{4.17}$$ eq4.17

where c is the same constant as in (4.16). A priori the above quantities might be $+\infty$, but a simple iteration of (4.17), where one takes successively $\beta=1$ (for which the right-hand side of (4.17) is finite), $\beta=p/2$, $\beta=p^2/4,\dots$, $\beta=(p/2)^j,\ldots\to+\infty$ shows that $G\in L^q(\Omega,d^2)$ for all $1\leq q<\infty$.

We now consider another iteration of (4.17) for which the constants will be controlled. Take $\beta = (p/2)^j/2$ for $j = j_0, j_0+1, \dots$ with $j_0 \in \mathbb{N}$ chosen so that $(p/2)^{j_0}/2 \geq 1$. One gets

$$\|G\|_{L^{(p/2)^{j+1}}(\Omega,d^2)} \leq c^{2\sum_{i=j_0}^{j}(2/p)^i} \prod_{i=j_0}^{j} [1+(p/2)^i/2]^{\frac{1}{(p/2)^i/2}} \|G\|_{L^{(p/2)^{j_0}}(\Omega,d^2)}. \tag{4.18}$$
eq4.18

Note that $G \in L^{(p/2)^{j_0}}(\Omega, d^2)$ as previously observed. The exponent of c in (4.18) converges as $j \to +\infty$ since it is part of a convergent geometric series. Calling q_j the product $\prod_{i=j_0}^{j} \dots$ in (4.18), one has

$$\log q_j = \sum_{i=j_0}^{j} 2(2/p)^i \log[1+(p/2)^i/2];$$

since

$$\log[1+(p/2)^i/2] = i\log(p/2) + \log[(2/p)^i + (1/2)] \leq i\log(p/2)$$

for i sufficiently large, and since the series $\sum_{i=1}^{\infty}(2/p)^i i$ converges, one sees that there exists $\bar{q}$ such that $q_j \leq \bar{q}$ for all $j \geq j_0$. It thus follows from (4.18) that

$$\|G\|_{L^{(p/2)^{j+1}}(\Omega,d^2)} \leq \bar{c}\|G\|_{L^{p/2}(\Omega,d^2)} < +\infty$$

for all $j \geq j_0$, with a constant $\bar{c}$ independent of j. Letting $j \to +\infty$, one deduces that G belongs to $L^\infty(\Omega)$. $\square$

lem4.8 **Lemma 4.8.** *Let $u \in D(\Omega) \cap C^1(\bar{\Omega})$. Then the function G provided by Lemma 4.6 satisfies*

$$G \geq \delta \quad \text{a.e. in } \Omega \tag{4.19}$$
eq4.19

for some constant $\delta > 0$.

Proof. We will first consider the particular case where the vector field a in (1.2) satisfies

$$\langle a, \nu \rangle < 0 \quad \text{on } \partial\Omega. \tag{4.20}$$
eq4.20

Claim. For any $l \geq 0$ there exists $\varepsilon = \varepsilon(a,u,l) > 0$ such that

$$\int_\Omega u^2 \langle A\nabla(G-c) + a(G-c),\ \nabla\varphi\rangle + \int_\Omega l u^2 (G-c)\varphi \geq 0$$

for all constants $c \geq 0$ and all $\varphi \in H^1(\Omega, d^2) \cap L^\infty(\Omega)$ with $\varphi \geq 0$ and $\operatorname{supp}\varphi \subset \Gamma_\varepsilon$ (Γ_ε was defined in Lemma 3.4).

Proof of the Claim. Using equation (3.3) for G and the divergence theorem from Lemma 3.4, one obtains

$$\int_\Omega u^2\langle A\nabla(G-c) + a(G-c), \nabla\varphi\rangle + \int_\Omega lu^2(G-c)\varphi \geq \int_\Omega cu[2\langle\nabla u, a\rangle + u \operatorname{div} a - lu]\varphi.$$

Since (4.20) implies $\langle a, \nabla u\rangle > 0$ on $\partial\Omega$ and since u vanishes on $\partial\Omega$, the bracket in the last integral is > 0 on $\partial\Omega$, and consequently is ≥ 0 on Γ_ϵ for some $\varepsilon = \varepsilon(a,u,l) > 0$. The inequality of the claim thus follows. $\square$

We now turn to the proof of (4.19) in the particular case where (4.20) holds. Let us fix l sufficiently large so that (4.4) holds on $H^1(\Omega, d^2)$ and let $\varepsilon = \varepsilon(a,u,l)$ be given by the above claim. Call

$$\delta = \operatorname*{ess\,inf}_{x \in \Omega_{\varepsilon/2}} G(x),$$

which is > 0 by Lemma 4.6, and let $\varphi = (G-\delta)^-$. Clearly $\varphi \geq 0$, with supp $\varphi \subset \Gamma_\varepsilon$ since $G \geq \delta$ on $\Omega_{\varepsilon/2}$; moreover $\varphi \in H^1(\Omega, d^2) \cap L^\infty(\Omega)$ since G belongs to that space (by Lemma 4.7). Applying the inequality of the claim with $c = \delta$ and φ as above gives

$$0 \leq B_l(G-\delta, \varphi) = B_l(-\varphi, \varphi)$$

Inequality (4.4) then implies $\varphi = 0$ a.e. in Ω, i.e. $G \geq \delta$ a.e. in Ω, and the lemma is proved (in the case where (4.20) holds).

Let us now consider the general case where (4.20) possibly does not hold. Call φ_1 a positive eigenfunction associated to the principal eigenvalue λ_1 of $-\Delta$ on $H_0^1(\Omega)$. Put $w = r\varphi_1 + 1$ where $r \geq 0$ is chosen so that $\langle a, \nu\rangle + r\langle A\nabla\varphi_1, \nu\rangle$ is < 0 on $\partial\Omega$, which is clearly possible since $\langle A\nabla\varphi_1, \nu\rangle$ is < 0 on $\partial\Omega$. Write G as Hw. It follows that $H = G/w \in H^1(\Omega, d^2) \cap L^\infty(\Omega)$ and satisfies

$$\int_\Omega (u^2 w)\langle A\nabla H + (A\frac{\nabla w}{w} + a)H, \nabla\varphi\rangle = 0 \tag{4.21}$$ eq4.21

for all $\varphi \in H^1(\Omega, d^2)$, with moreover $H > 0$ a.e. in Ω. Equation (4.21) is of the same type as (3.3) : u is replaced by $u\sqrt{w}$ (which still belongs to $D(\Omega) \cap C^1(\bar\Omega)$) and a is replaced by $A(\nabla w/w) + a$ (which still belongs to $C^{0,1}(\bar\Omega)$ but now satisfies (4.20) by the choice of r and the fact that $w \equiv 1$ on $\partial\Omega$). It follows that H is the solution provided by applying Lemma 4.6 to this new equation (4.21). By that part of Lemma 4.8 which has already been proved, one deduces that H satisfies (4.19) for some $\delta > 0$. Since $w \geq 1$, one gets that $G = Hw$ also satisfies (4.19). This completes the proof of Lemma 4.8. □

Lemma 3.3 clearly follows from the previous Lemmas 4.6, 4.7 and 4.8.

5. Antimaximum principle

It is our purpose in this section to present the AMP in the previous framework, i.e. for some nonselfadjoint problems with a weight in $L^\infty(\Omega)$. The assumptions on L, m and Ω are the same as in section 2. We directly deal with the general case where a_0 may not be ≥ 0.

theo5.1 **Theorem 5.1.** *Suppose $m^+ \not\equiv 0$. Assume also the existence of a principal eigenvalue for (1.1) and let λ^* be the largest of these principal eigenvalues (cf. Proposition 2.6). Take $h \in L^p(\Omega)$ with $p > N$ and $h \geq 0$, $h \not\equiv 0$. Then there exists $\delta = \delta(h) > 0$ such that for $\lambda \in]\lambda^*, \lambda^* + \delta[$, any solution u of (1.3) satisfies $u < 0$ in Ω and $\partial u/\partial\nu > 0$ on $\partial\Omega$.*

The proof of Theorem 5.1 is based on a preliminary nonexistence result, which reads as follows.

lem5.2 **Lemma 5.2.** *Let λ^* be as above and take $h \in L^p(\Omega)$ with $1 < p < \infty$ and $h \geq 0$, $h \not\equiv 0$. Then problem (1.3) has no solution $u \geq 0$ if $\lambda > \lambda^*$, and no solution at all if $\lambda = \lambda^*$.*

The proof of the above two results can be carried out through a rather standard adaptation to the present Dirichlet situation of the arguments developed in [13] in the case of the Neumann-Robin boundary conditions, and we will omit it. The general philosophy of this adaptation consists in replacing the space $C(\bar\Omega)$ by the space $C_0^1(\Omega)$, the condition $u > 0$ on $\bar\Omega$ by the condition $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$, and the restriction $h \in L^p(\Omega)$ with $p > N/2$ by the restriction $h \in L^p(\Omega)$

with $p > N$. One should also remark that the assumption $a_0 \geq 0$ in [13] is used there only to guarantee the existence of a principal eigenvalue.

As before the case of the smallest principal eigenvalue can be reduced to the case covered by Theorem 5.1 by changing m into $-m$.

6. Nonuniformity of the antimaximum principle

The assumptions on L, m and Ω in this section are those of section 3. Our purpose is to show that the AMP of Theorem 5.1 is not uniform, i.e. that a $\delta > 0$ independent of h cannot be found.

prop6.1 **Proposition 6.1.** *Assume $m^+ \not\equiv 0$ and let $\lambda^* > 0$ be as in Theorem 5.1. Suppose that $\lambda \in \mathbb{R}$ enjoys the following property : (*) for any $h \in C_c^\infty(\Omega)$, $h \geq 0$, $h \not\equiv 0$, problem* (1.3) *has a solution u which satisfies $u < 0$ in Ω. Then $\lambda \leq \lambda^*$.*

The proof of Proposition 6.1 uses the minimax formula of section 3. It is again an adaptation of arguments developed in [13] in the case of the Neumann-Robin boundary conditions. However the adaptation here is not so standard as in section 5 since it involves the introduction of spaces with weights and of the set $D(\Omega)$. It seems consequently useful to sketch part of the arguments, and the rest of this section will be devoted to that.

Recall that by Theorem 3.5,

$$\lambda^* = \min_{u \in U} \frac{\Lambda(u) - \inf Q_u}{\int_\Omega m u^2} \tag{6.1}$$ eq6.1

where as before $\inf Q_u$ stands for $\inf\{Q_u(w) : w \in H^1(\Omega, d^2)\}$.

We start with the following lemma whose proof is similar to that of inequality (3.6) in section 3. In fact, with respect to (3.6), (6.2) below involves λ instead of λ^* and has an extra term $-\int_\Omega \frac{h}{v} u^2$.

lem6.2 **Lemma 6.2.** *Let $\lambda \in \mathbb{R}$ be such that for some $h \in C_c^\infty(\Omega)$, the problem $Lv = \lambda m v + h$ in Ω, $v = 0$ on $\partial\Omega$ has a solution v with $v > 0$ in Ω. Then*

$$\lambda \int_\Omega m u^2 \leq \Lambda(u) - \inf Q_u - \int_\Omega \frac{h}{v} u^2 \tag{6.2}$$ eq6.2

for any $u \in U$.

The objective is to prove that if λ enjoys property (*), then (6.2) holds without the extra term $-\int_\Omega \frac{h}{v} u^2$. Once this is done, the conclusion of Proposition 6.1 follows by using (6.1). As an intermediate step towards this objective one has the following

lem6.3 **Lemma 6.3.** *Suppose λ enjoys property (*). Then*

$$\lambda \int_\Omega m u^2 \leq \Lambda(u) - \inf Q_u \tag{6.3}$$ eq6.3

for any $u \in H^1(\Omega)$ such that $0 \leq u(x) \leq c_u d(x)$ in Ω for some constant c_u, $\int_\Omega m u^2 > 0$, and u vanishes on some ball $B_u \subset \Omega$.

The proof of Lemma 6.3 can be adapted from [13, Lemma 5.5]. The main modifications consist in using now as approximates for u the functions

$$u_j = \max\{u(x), \frac{d(x)}{j}\}$$

for $j = 1, 2, \dots$ and in replacing [13, Lemma 5.4] by the following lem6.4 **Lemma 6.4.** *For any u such that*

$$u(x) \le c_u d(x) \tag{6.4}$$ eq6.4

for some constant c_u and a.e. $x \in \Omega$, one has $\inf Q_u > -\infty$. Moreover if u with (6.4) and $w \in H^1(\Omega, d^2)$ vary in such a way that $\|u\|_\infty$ remains bounded and $Q_u(w)$ remains bounded from above, then $\|u\nabla w\|_2$ remains bounded.

The idea now to prove Proposition 6.1 is to approximate any $u \in U$ by functions as those in Lemma 6.3 and go to the limit in (6.3). Here are some details. Given $u \in U$, there exists $u_k \in H^1(\Omega)$ such that $0 \le u_k \le u$, $u_k \to u$ in $H^1(\Omega)$, $u_k = 0$ on some hall B_k, with in addition, for any $\Omega' \subset\subset \Omega$, $u_k \equiv u$ on Ω' for k sufficiently large. One can for instance take $u_k = u\psi_k$ where the functions ψ_k are given by Lemma 6.6 below. Note that the proof that $u_k \to u$ in $H^1(\Omega)$ uses the fact that u satisfies an estimate near $\partial\Omega$ of the type (6.4). Note also that it is at the moment of this approximation that in the case of the Neumann-Robin boundary conditions, one had to impose in [13] the restriction $N \ge 2$, restriction which is not necessary here. With these approximations u_k at our disposal, the proof of Proposition 6.1 can be completed by following the same lines as on pages 106-107 from [13]. The main modifications in this last part consist in introducing the weight d^2 in the spaces to which the functions $w_k, w, \nabla w_k, \nabla w$ from [13] belong, and in replacing ultimately [13, Lemma 5.6] by the following

lem6.5 **Lemma 6.5.** *Let $u \in L^2_{\mathrm{loc}}(\Omega)$ with $\nabla u \in L^2(\Omega, d^2)$. Then $u \in L^2(\Omega, d^2)$.*

The proof of lemma 6.5 uses the Poincaré inequality of Lemma 4.2.

lem6.6 **Lemma 6.6.** *There exists a sequence $\psi_k \in C^1_c(\Omega)$ such that (i) $0 \le \psi_k \le 1$ in Ω, (ii) $\psi_k \equiv 1$ on $\Omega_{2/k}$, (iii) supp $\psi_k \subset \Omega_{1/k}$, (iv) $d|\nabla\psi_k| \le K$ on Ω for some constant K and all k, where $\Omega_\eta := \{x \in \Omega : d(x, \partial\Omega) > \eta\}$.*

Proof. Take $\psi \in C^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi(x) = 0$ for $x \le 1$, $\psi(x) = 1$ for $x \ge 2$. For $k = 1, 2, \dots$, define $\psi_k(x) = \psi(kd(x))$ for $x \in \bar\Omega \setminus \Omega_{2/k}$ and $\psi_k(x) = 1$ for $x \in \Omega_{2/k}$. Since Ω is of class C^2, it follows from [11, Lemma 14.16] that $\psi_k \in C^2(\bar\Omega)$ for k sufficiently large. Properties (i), (ii), (iii) clearly hold, and (iv) is easily certified using the fact that $2/k \le d(x) \le 1/k$ where $\nabla\psi_k(x) \ne 0$. $\square$

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