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LARGE RADIAL SOLUTIONS OF A POLYHARMONIC EQUATION WITH SUPERLINEAR GROWTH

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. This paper concerns the equation $\Delta^m u = |u|^p$, where $m \in \mathbb{N}$, $p \in (1,\infty)$, and Δ denotes the Laplace operator in $\mathbb{R}^N$, for some $N \in \mathbb{N}$. Specifically, we are interested in the structure of the set $\mathcal{L}$ of all large radial solutions on the open unit ball B in $\mathbb{R}^N$. In the well-understood second-order case, the set $\mathcal{L}$ consists of exactly two solutions if the equation is subcritical, of exactly one solution if it is critical or supercritical. In the fourth-order case, we show that $\mathcal{L}$ is homeomorphic to the unit circle S^1 if the equation is subcritical, to S^1 minus a single point if it is critical or supercritical. For arbitrary $m \in \mathbb{N}$, the set $\mathcal{L}$ is a full $(m-1)$-sphere whenever the equation is subcritical. We conjecture, but have not been able to prove in general, that $\mathcal{L}$ is a punctured $(m-1)$-sphere whenever the equation is critical or supercritical. These results and the conjecture are closely related to the existence and uniqueness (up to scaling) of entire radial solutions. Understanding the geometric and topological structure of the set $\mathcal{L}$ allows precise statements about the existence and multiplicity of large radial solutions with prescribed center values $u(0), \Delta u(0), \dots, \Delta^{m-1}u(0)$.

1. Introduction

This paper is a contribution to the study of polyharmonic equations with superlinear reaction terms. A prototype problem is the equation

$$\Delta^m u = |u|^p, \tag{1.1}$$

where $m \in \mathbb{N}$, $p \in (1,\infty)$, and Δ denotes the Laplace operator in $\mathbb{R}^N$, for some $N \in \mathbb{N}$. Equations of this type arise in many contexts, from differential geometry to quantum mechanics. While the second-order case is by now well understood, numerous open problems remain in the fourth and higher-order cases. We refer to [2, 9, 11, 12, 15, 17, 20, 25, 29] and the references therein for background and recent contributions.

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Here we consider *radial solutions* of Equation (1.1), by which we mean classical noncontinuable solutions that depend only on the distance from the origin and are defined either on open balls centered at the origin or on all of $\mathbb{R}^N$. Given such a solution u, the numbers $u(0), \Delta u(0), \dots, \Delta^{m-1}u(0)$ are called the *center values* of u; when there is no danger of confusion, we also refer to the single number $u(0)$ as the solution's center value. Every radial solution of (1.1) except for the trivial one is either an *entire solution*, that is, a nontrivial solution on $\mathbb{R}^N$, or a *large solution*, that is, an unbounded solution on an open ball. Our main interest is in the structure of the set of all large radial solutions and their blow-up behavior; nonetheless, we need to study entire radial solutions as well.

Due to the homogeneity of its right-hand side, the equation (1.1) enjoys a scaling-property that allows us to confine attention to entire solutions with center values ± 1 and large solutions on the unit ball. In fact, define $q := 2m/(p-1)$ and suppose that u is a solution of (1.1). Then, for every $\lambda \in (0, \infty)$, the function u_λ, defined by $u_\lambda(x) := \lambda^q u(\lambda x)$ for all $x \in \mathbb{R}^N$ such that u is defined at λx, is again a solution of (1.1). We call u_λ a *rescaling* (or more precisely, the λ-*rescaling*) of u and say that two solutions of (1.1) are *scaling-equivalent* if one is a rescaling of the other. Clearly, every entire radial solution of (1.1) is scaling-equivalent to an entire solution with center value 1 or -1, and every large radial solution of (1.1) is a rescaling of a large solution on B, the open ball of radius 1 centered at the origin in $\mathbb{R}^N$.

Our first objective is to understand the structure of the set $\mathcal{L}$ of all large radial solutions of (1.1) on B. Any such solution u is uniquely determined by its center values $u(0), \Delta u(0), \dots, \Delta^{m-1}u(0)$. Hence there is a one-to-one correspondence between $\mathcal{L}$ and the set $\mathcal{H}$ of all points $\xi \in \mathbb{R}^m$ such that the radial solution of (1.1) with center values $\xi_1, \xi_2, \dots, \xi_m$ blows up at $|x| = 1$; this correspondence is a homeomorphism with respect to the natural topology of $\mathcal{L}$ as a subspace of $C^{2m}(B)$ (see Remark 2.1 for details). Understanding the structure of $\mathcal{L}$ thus amounts to understanding the structure of the set $\mathcal{H}$.

We show that $\mathcal{H}$ is homeomorphic to a relatively open subset $\mathcal{S}$ of S^{m-1}, the unit sphere in $\mathbb{R}^m$; the homeomorphism is the restriction to $\mathcal{H}$ of a smooth projection of $\mathbb{R}^m \setminus \{0\}$ onto S^{m-1}, called the *scaling-projection* (see Section 4 for details). The set $S^{m-1} \setminus \mathcal{S}$ is contained in $\mathcal{O} := \{\xi \in \mathbb{R}^m \mid (-1)^{m-i}\xi_i < 0 \ \forall\, i \in \{1, \dots, m\}\}$ (the negative half-axis if $m = 1$, the open fourth quadrant if $m = 2$, an open orthant in $\mathbb{R}^m$ otherwise). If $m \geq 2$, then $\mathcal{H} \cap \mathcal{O}$ is unbounded unless $\mathcal{S} = S^{m-1}$. Since $S^{m-1} \setminus \mathcal{S} \subset \mathcal{O}$, the set $\mathcal{S}$ contains in particular S^{m-1}_+, the intersection of S^{m-1} with the nonnegative cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$. The set $\mathcal{H}_+ := \mathcal{H} \cap \mathbb{R}^m_+$ is a compact unordered manifold that defines an order decomposition of $\mathbb{R}^m_+$ and is homeomorphic to S^{m-1}_+ under the radial projection.

These and other properties of $\mathcal{H}$ have immediate implications for the existence or nonexistence of large radial solutions of (1.1) on B with prescribed center values. For example, the fact that $\mathcal{H}_+$ is homeomorphic to S^{m-1}_+ under the radial projection implies that for every unit vector $\xi \in \mathbb{R}^m_+$ there exists exactly one large radial solution u of (1.1) on B such that the vector $(u(0), \Delta u(0), \dots, \Delta^{m-1}u(0))$ is a positive multiple of ξ.

Regarding the global structure of the hypersurface $\mathcal{H}$, let us discuss two special cases. First consider the familiar second-order case, $m = 1$. Since $S^0 = \{\pm 1\}$, our general result says that $\mathcal{H}$ consists either of exactly one positive number or of exactly two real numbers of opposite sign. In other words, (1.1) has either exactly one large radial solution on B, with a positive center value, or it has exactly two such solutions, one with a positive and one with a negative center value. Well-known facts about second-order elliptic equations imply that the latter possibility occurs if and only if Equation (1.1) is subcritical.

Now consider the fourth-order case, $m = 2$. Our general result implies that $\mathcal{H}$ is homeomorphic, under the scaling-projection, to a relatively open subset $\mathcal{S}$ of the unit circle S^1 that contains at least the segments of S^1 in the first, second, and third quadrants. Invoking recent results on biharmonic equations with power nonlinearities [9, 15, 30], we are able to prove more. If (1.1) is subcritical, then $\mathcal{S} = S^1$, and $\mathcal{H}$ is a closed simple curve. If (1.1) is critical or supercritical, then there is a point $\xi = (\xi_1, \xi_2) \in S^1$ with $\xi_1 > 0$ and $\xi_2 < 0$ such that $\mathcal{S} = S^1 \setminus \{\xi\}$, and $\mathcal{H}$ is an unbounded simple curve that "closes at infinity" in the fourth quadrant. In either case, the origin belongs to the "interior" of $\mathcal{H}$. For a graphical visualization, see Figures 2 and 4 in Section 4, where the set $\mathcal{H}$ is shown in a subcritical and a supercritical case.

For arbitrary $m \in \mathbb{N}$, we prove that the hypersurface $\mathcal{H}$ is compact and a full $(m-1)$-sphere (homeomorphic to S^{m-1} under the scaling-projection) whenever the equation (1.1) is subcritical. In the critical case, by contrast, we know that $\mathcal{H}$ is *not* a full $(m-1)$-sphere and is in fact unbounded (unless $m = 1$, of course). We conjecture that, in generalization of the results for $m = 1$ and $m = 2$, the set $\mathcal{H}$ is a punctured $(m-1)$-sphere (homeomorphic to S^{m-1} minus a single point in the orthant $\mathcal{O}$) whenever (1.1) is critical or supercritical, but have not been able to prove this for $m \geq 3$.

These results and the conjecture are closely related to the question of existence and uniqueness (up to scaling) of entire radial solutions of (1.1). Indeed, our result for the subcritical case is a consequence of the nonexistence of entire radial solutions; the one for the critical case follows from the fact that at least one scaling-equivalence class of entire radial solutions is known explicitly. In the critical/supercritical case, existence and uniqueness (up to scaling) of an entire radial solution would prove our conjecture; but, to the best of our knowledge, this is an open problem for $m \geq 3$ (see Section 3 for details and comments on the relevant literature). This problem and other issues relating to the existence and properties of entire radial solutions of polyharmonic equations are of independent interest and the subject of an ongoing investigation.

Beyond the structure of the set of all large radial solutions of (1.1) on B, we are interested in their asymptotic behavior near the boundary of B. In view of the existing literature on large solutions of elliptic equations and systems (see, for example, [1, 7, 8, 10, 18, 26] and the references therein), one would expect any large solution u of (1.1) on B to satisfy

$$u(x) \sim \frac{C}{(1-|x|)^{\gamma}} \quad \text{as } |x| \to 1,$$

for some real numbers $C > 0$ and $\gamma > 0$. (Given two real-valued functions u_1 and u_2 on B, we write $u_1(x) \sim u_2(x)$ if the ratio $u_1(x)/u_2(x)$ converges to 1 as $|x| \to 1$.) A simple calculation shows that if u is a radial solution with this kind of asymptotic behavior, then, necessarily, $\gamma = q$ and $C = Q$, where

$$q := \frac{2m}{p-1} \quad \text{and} \quad Q := \big(q(q+1)\dots(q+2m-1)\big)^{q/(2m)}.$$

These numbers are independent of the space dimension N; the number q appeared already in the definition of a λ-rescaling.

A detailed analysis of the blow-up behavior of large radial solutions of (1.1) on B will be carried out in a companion paper [6]. In many cases, we find precisely the "expected behavior" described above. However, if $m \geq 6$ and p exceeds a certain critical value depending only on m, then there exist large radial solutions that oscillate, more and more rapidly near the boundary of B, about the "expected asymptotic profile." Still, as $|x| \to 1$, the ratio of $u(x)$ and $Q/(1-|x|)^q$ remains bounded from above and from below by positive constants depending only on m and p. For a special case, the biharmonic equation with critical exponent, a result in the same direction appears in [12].

Some of the basic ideas underlying the present paper and [6] were already developed in our earlier work [5], where we studied a very special elliptic system, equivalent to a biharmonic equation, which nonetheless behaves in many ways like Equation (1.1) in the second-order case. At least in principle, our approach works just as well if the nonlinearity $|u|^p$ in Equation (1.1) is replaced by $\pm u|u|^{p-1}$ or, more generally, by $f(u)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a p-homogeneous function. However, in this more general situation, one encounters large radial solutions that are neither eventually positive nor eventually negative, but wildly oscillatory and neither bounded from below nor from above. In fact, this is the only possible kind of blow-up behavior in the case of a *decreasing* nonlinearity $f(u)$; it does not occur if $m = 1$, in which case no large radial solutions exist if f is decreasing, but does occur if $m = 2$. Oscillatory blow-up of this kind is also possible (albeit non-generic) in the case of an *increasing* nonlinearity, but only if $m \geq 3$. In all these cases, the analysis is more delicate with regard to both, the structure of the set of large radial solutions and the asymptotic behavior of its members, and several questions remain open. We hope to address these issues in future work.

The present paper is organized as follows. In Section 2 we reduce the problem to the study of a system of first-order ODEs and give a complete classification of its solutions. In Section 3 we collect the relevant information regarding entire radial solutions of (1.1). Section 4 contains our main results about the set of all large radial solutions of (1.1) on B, along with a detailed discussion of the fourth-order case.

2. The ODE System for Radial Solutions

With $v_i := \Delta^{i-1} u$ for $i \in \{1, \dots, m\}$ and $v_{m+1} := |v_1|^p$, Equation (1.1) is equivalent to the system

$$\Delta v_i = v_{i+1}, \quad i \in \{1, \dots, m\}.$$

Radial solutions, as defined in the introduction, correspond to maximal forward solutions of the ODE system

$$v_i'' + \frac{N-1}{r} v_i' = v_{i+1}, \quad i \in \{1, \dots, m\},$$

starting at $r = 0$ with $v_i'(0) = 0$ for all $i \in \{1, \dots, m\}$. Letting $\mu := N - 1$, the above system is equivalent to the first-order system

$$v_i' = w_i, \quad w_i' + \frac{\mu}{r} w_i = v_{i+1}, \quad i \in \{1, \dots, m\}, \tag{2.1}$$

which we will study for arbitrary $\mu \in \mathbb{R}_+$.

We begin by collecting a few basic facts regarding the existence, uniqueness, maximal continuation, and regularity of solutions of (2.1). These facts are analogous to standard results for regular ODE systems and can be established with the same means.

By a *forward solution* of the system (2.1), we mean a continuous $\mathbb{R}^{2m}$-valued function $(v, w) = (v_1, w_1, \dots, v_m, w_m)$ on an interval of the form $[r_0, r_\infty)$ with $0 \le r_0 < r_\infty \le \infty$, differentiable on (r_0, r_∞), such that the differential equations (2.1) are satisfied for all $r \in (r_0, r_\infty)$. The interval $[r_0, r_\infty)$ is called the solution's *interval of existence* and the point r_0 its *starting-point*.

A forward solution (v, w) on $[r_0, r_\infty)$ is called *maximal* if it cannot be continued beyond r_∞, that is, if there is no forward solution starting at r_0 that is a proper extension of (v, w). In this case, we refer to the point r_∞ as the solution's *exit point*.

Given a starting-point $r_0 \in \mathbb{R}_+$ and initial values $v_i(r_0)$, $w_i(r_0) \in \mathbb{R}$, with $w_i(r_0) = 0$ if $r_0 = 0$ and $\mu > 0$, there exists a unique maximal forward solution starting at r_0, which depends continuously on the initial values and on the parameters p and μ. In particular, the solution's exit point depends lower-semicontinuously on initial values and parameters: if (v, w) is a maximal forward solution on $[r_0, r_\infty)$ and if $r_1 \in (r_0, r_\infty)$, then every maximal forward solution starting at r_0 with initial values and parameters sufficiently close to those of (v, w) exists on an interval containing $[r_0, r_1]$.

A maximal forward solution with interval of existence $[r_0, r_\infty)$ is called *global* if $r_\infty = \infty$. If $r_\infty < \infty$, the solution is necessarily unbounded; in this case, we call the solution *explosive*, refer to r_∞ as its *blow-up point*, and say that the solution *blows up* at r_∞.

Even though the system (2.1) is singular at $r = 0$ if $\mu > 0$, forward solutions have the maximal regularity determined by the right-hand side of the last equation, $v_{m+1} = |v_1|^p$, and depend not only continuously, but differentiably on initial values and parameters. In particular, since the mapping $s \mapsto |s|^p$ is at least C^1 (recall that $p > 1$), any forward solution is necessarily C^2 on its interval of existence, including the starting-point. More precisely, if (v, w) is a forward solution of (2.1) on $[r_0, r_\infty)$, then $v_i \in C^{2(m+1-i)+1}([r_0, r_\infty))$ and $w_i \in C^{2(m+1-i)}([r_0, r_\infty))$ for all $i \in \{1, \dots, m\}$. Note that if $\mu > 0$, then any forward solution (v, w) starting at 0 necessarily satisfies $w_i(0) = 0$, $w_i'(0) = v_{i+1}(0)/(\mu + 1)$, and $w_i''(0) = 0$ for all $i \in \{1, \dots, m\}$.

In the sequel we will use the term "solution" as a synonym for "maximal forward solution." Further, a solution (v, w) of (2.1) will be called *even* if it starts at $r = 0$ with $w(0) = 0$. Note that if r_∞ is the exit point of such a solution, then the even extension $\tilde{v}$ of v and odd extension $\tilde{w}$ of w determine a C^2-function $(\tilde{v}, \tilde{w})$ on $(-r_\infty, r_\infty)$ that satisfies the differential equations (2.1) on $(-r_\infty, r_\infty) \setminus \{0\}$. The even solutions of (2.1) are uniquely determined by the initial values of their v-components and depend continuously (in fact, differentiably) on those values. Given a point $\xi \in \mathbb{R}^m$, we will denote the even solution with $v(0) = \xi$ by (v^ξ, w^ξ) and refer to it as the *even solution starting at* ξ.

Finally, we will call a solution of (2.1) *nonnegative*, *positive*, *nondecreasing*, *increasing*, et cetera, if each of its components has the respective property throughout the solution's interval of existence, and we will say that the solution *approaches infinity* if each of its components does so.

Remark 2.1. The even solutions of (2.1) with $\mu = N-1$ correspond to radial solutions of Equation (1.1) as defined in the introduction. More precisely, nontrivial global even solutions of (2.1) correspond to *entire* radial solutions, explosive even solutions of (2.1) to *large* radial solutions of (1.1). In particular, large radial solutions of (1.1) on the unit ball B are in one-to-one correspondence with even solutions of (2.1) that blow up at $r = 1$.

Now let $\mathcal{H}$ denote the subset of $\mathbb{R}^m$ consisting of all points ξ such that the even solution of (2.1) starting at ξ blows up at $r = 1$; further, let $\mathcal{L}$ denote the set of all large radial solutions of (1.1) on B. As a consequence of the preceding discussion, $\mathcal{L}$ is contained in $C^{2m}(B)$ (a complete metrizable space under the C^{2m}-topology on compact subsets of B), and the mapping $u \mapsto (u(0), \Delta u(0, \dots, \Delta^{m-1}u(0))$ is a homeomorphism from $\mathcal{L} \subset C^{2m}(B)$ onto $\mathcal{H} \subset \mathbb{R}^m$ (assuming that $\mu = N-1$, of course). In this sense, the structure of $\mathcal{L}$ is determined by that of $\mathcal{H}$.

Remark 2.2. As long as $v_1 \geq 0$, the system (2.1) satisfies the well-known Kamke condition and hence a comparison principle. This can be proved in the same way as for regular ODE systems; we refer to [27] for the methodology and to Lemma 3.2 in [16] for an analogous result involving a singular system. A precise statement requires some additional terminology.

Given a, $b \in \mathbb{R}^m$, we write $a \leq b$ or $b \geq a$ ($a < b$ or $b > a$) if the respective inequalities hold componentwise. If $a \leq b$ or $a \geq b$ ($a < b$ or $a > b$), we call a and b *ordered* (*strictly ordered*); if $a \geq 0$ ($a > 0$), we call *a nonnegative* (*positive*).

By a *subsolution* (*supersolution*) of (2.1), we mean a continuous $\mathbb{R}^{2m}$-valued function $(v, w) = (v_1, w_1, \dots, v_m, w_m)$, defined on a nontrivial interval $I \subset [0, \infty)$, differentiable at least in the interior of I, and satisfying the differential inequalities obtained from the differential equations in (2.1) by replacing "=" with "$\leq$" ("$\geq$"). If $\mu > 0$, any subsolution (supersolution) on an interval starting at 0 necessarily satisfies $w(0) \leq 0$ ($w(0) \geq 0$).

Now, suppose that $(\underline{v}, \underline{w})$ and $(\overline{v}, \overline{w})$ are a subsolution and a supersolution of (2.1), respectively, on a common interval $[r_0, r_\infty)$ with $0 \leq r_0 < r_\infty \leq \infty$, such that $\underline{v}_1 \geq 0$ throughout and $\underline{v}(r_0) \leq \overline{v}(r_0)$, $\underline{w}(r_0) \leq \overline{w}(r_0)$. Then $\underline{v}(r) \leq \overline{v}(r)$ and $\underline{w}(r) \leq \overline{w}(r)$ for all $r \in [r_0, r_\infty)$. Moreover, $\underline{v}(r) < \overline{v}(r)$ and $\underline{w}(r) < \overline{w}(r)$ for all $r \in (r_0, r_\infty)$, unless $\underline{v}(r_0) = \overline{v}(r_0)$ and $\underline{w}(r_0) = \overline{w}(r_0)$.

Remark 2.3. Since the constant 0 is a trivial solution of (2.1), the comparison principle implies that the nonnegative cone $\mathbb{R}^{2m}_+$ of $\mathbb{R}^{2m}$ is forward-invariant in a strong sense: any solution of (2.1) starting in $\mathbb{R}^{2m}_+$ will remain in $\mathbb{R}^{2m}_+$ and will, in fact, immediately enter the interior of $\mathbb{R}^{2m}_+$, unless it is the trivial solution. Further, it follows from the differential equations that for any nontrivial nonnegative solution (v, w), the components v_i and the functions $r \mapsto r^\mu w_i(r)$, for $i \in \{1, \dots, m\}$, are strictly increasing.

Remark 2.4. Sub- and supersolutions of (2.1) for $\mu > 0$ can be constructed from solutions of the autonomous system

$$v_i' = w_i, \quad w_i' = v_{i+1}, \quad i \in \{1, \dots, m\}. \tag{2.2}$$

Clearly, if (v, w) is a solution (or supersolution) of (2.2) with $w \geq 0$, then (v, w) is a supersolution of (2.1) for every $\mu \in \mathbb{R}_+$. Now suppose that (v, w) is a solution (or subsolution) of (2.2) on an interval $I \subset [0, \infty)$ such that $w(r) \leq rw'(r)$ for all $r \in I$. (This condition is satisfied, for example, if the interval I starts at 0 and if $v(0) \geq 0$ and $w(0) = 0$, which implies that w' is nondecreasing.) Given $\mu \in \mathbb{R}_+$, let $\nu := 1/\sqrt{\mu+1}$ and define $(\tilde{v}, \tilde{w})$ by $\tilde{v}(r) := v(\nu r)$ and $\tilde{w}(r) := \nu w(\nu r)$ for $r \in \tilde{I} := \{r \in [0,\infty) \,|\, \nu r \in I\}$. A short computation shows that $(\tilde{v}, \tilde{w})$ is a subsolution of (2.1) on $\tilde{I}$. Implicitly, this was already observed in [28].

Estimates in [28] show that all eventually positive solutions of the autonomous system (2.2) are explosive and approach infinity at the blow-up point. Using the subsolutions from Remark 2.4, the same can then be proved for eventually positive solutions of (2.1) with arbitrary $\mu \in \mathbb{R}_+$. We will give a different proof, based on uniform a-priori bounds, which yields the continuity of the blow-up point as a function of initial data and parameters. The following lemma gathers some preliminary information about the solutions of (2.1). (Recall that "solution" means "maximal forward solution.")

Lemma 2.5. *Let (v, w) be an arbitrary solution of* (2.1)*.*
(a) *Every component of (v, w) is bounded from below by a polynomial.*
(b) *If any component of (v, w) is bounded from above by a polynomial, then the same is true for all components, and (v, w) is a global solution.*
(c) *If (v, w) is not a global solution, then (v, w) approaches infinity.*
(d) *If (v, w) is a nontrivial global solution that does not approach infinity and if w is initially zero, then every component of v is strictly monotonic and vanishes at infinity. More precisely, the functions v_i and $r \mapsto r^\mu w_i(r)$, for $i \in \{1, \dots, m\}$, are increasing if $m-i$ is even, decreasing if $m-i$ is odd, and in either case, $v_i(r) \to 0$ as $r \to \infty$.*
(e) *If (v, w) is a nontrivial solution with nonnegative initial values, then (v, w) approaches infinity.*

Proof. Throughout, suppose that (v, w) is a solution of (2.1) with interval of existence $[r_0, r_\infty)$. Integrating the differential equations, we have

$$v_i(r) = v_i(r_0) + \int_{r_0}^{r} w_i(s)\,ds, \quad r^\mu w_i(r) = r_0^\mu w_i(r_0) + \int_{r_0}^{r} s^\mu v_{i+1}(s)\,ds$$

for all $r \in [r_0, r_\infty)$ and $i \in \{1, \dots, m\}$. It follows that if w_i or v_{i+1} is bounded from above (from below) by a polynomial, then so is v_i or w_i, respectively. Using the fact that $v_{m+1} = |v_1|^p$ is bounded from below by 0 and repeatedly applying the preceding argument, we see that every component of (v, w) is bounded from below by a polynomial; this proves (a).

Now suppose that some component of (v, w) is bounded from above by a polynomial. Our earlier argument then shows that the same is true for all the "preceding" components and in particular for v_1. Due to (a), v_1 is also bounded from below by a polynomial. Hence we have a polynomial bound for $|v_1|$ and also for $v_{m+1} = |v_1|^p$. Applying the earlier argument again, we obtain polynomial upper bounds for all the components of (v, w). Together with (a), this implies that the norm of (v, w) is polynomially bounded. In particular, (v, w) is not explosive, and this finishes the proof of (b).

Next, suppose that (v, w) is explosive, that is, $r_\infty < \infty$. Further assume that v_{i+1} is eventually nonnegative for some $i \in \{1, \dots, m\}$. Since $\frac{d}{dr}\big(r^\mu w_i(r)\big) = r^\mu v_{i+1}(r)$ for all $r \in (r_0, r_\infty)$, the function $r \mapsto r^\mu w_i(r)$ is eventually nondecreasing, necessarily without bound (else, since $r_\infty < \infty$, w_i itself would be bounded, contradicting (b)). Thus, $r^\mu w_i(r) \to \infty$, whence $w_i(r) \to \infty$, as $r \to r_\infty$. In particular, w_i

is eventually positive. But then, v_i is eventually increasing, necessarily without bound; that is, $v_i(r) \to \infty$ as $r \to r_\infty$. In particular, v_i is eventually positive. Since $v_{m+1} = |v_1|^p$ is nonnegative throughout, repeated application of the preceding argument shows that $v_i(r)$, $w_i(r) \to \infty$ as $r \to r_\infty$ for all $i \in \{1, \dots, m\}$. This proves (c).

In preparation for the proof of (d), let us verify the following statement, which holds for *any* solution (v, w): $(*)$ If $i \in \{1, \dots, m\}$ and v_{i+1} is eventually positive (negative) and bounded away from zero, then w_i and v_i approach infinity (negative infinity). In view of (c), it suffices to consider the case of a global solution. So, suppose that $i \in \{1, \dots, m\}$, $\varepsilon \in (0, \infty)$, $r_i \in [r_0, \infty)$, and $v_{i+1} \ge \varepsilon$ on $[r_i, \infty)$. For $r \in [r_i, \infty)$, we then have

$$r^\mu w_i(r) = r_i^\mu w_i(r_i) + \int_{r_i}^r s^\mu v_{i+1}(s)\, ds \ge r_i^\mu w_i(r_i) + \frac{\varepsilon}{\mu + 1}\bigl(r^{\mu+1} - r_i^{\mu+1}\bigr),$$

whence

$$w_i(r) = \Bigl(\frac{r_i}{r}\Bigr)^\mu w_i(r_i) + \frac{\varepsilon}{\mu + 1}\Bigl(r - r_i\Bigl(\frac{r_i}{r}\Bigr)^\mu\Bigr),$$

and thus, $w_i(r) \to \infty$ as $r \to \infty$. Since $v_i' = w_i$, this implies that v_i approaches infinity as well. The same reasoning shows that w_i and v_i will both approach negative infinity if $v_{i+1} \le -\varepsilon$ on $[r_i, \infty)$, and this finishes the proof of $(*)$.

Now suppose that (v, w) is a nontrivial global solution with $w(r_0) = 0$ and that (v, w) does not approach infinity. Further, suppose that v_{i+1} is nonnegative (nonpositive) for some $i \in \{1, \dots, m\}$. Then the function $r \mapsto r^\mu w_i(r)$ is nondecreasing (nonincreasing); in fact, it must be strictly increasing (strictly decreasing). (If it were constant on a nontrivial interval $I \subset (r_0, \infty)$, then all components of (v, w) would vanish on I, which is impossible as (v, w) is not the trivial solution.) Since $w_i(r_0) = 0$, it follows that $w_i > 0$ ($w_i < 0$) on (r_0, ∞), and thus, v_i is strictly increasing (strictly decreasing) as well. Let L_i denote the (proper or improper) limit of $v_i(r)$ as $r \to \infty$. We claim that $L_i = 0$. Suppose that $L_i > 0$. Then v_i is eventually positive and bounded away from zero. Due to $(*)$, all the "preceding" components of (v, w), if any, approach infinity. In any case, v_1 is eventually positive and bounded away from zero, and so is $v_{m+1} = |v_1|^p$. Applying $(*)$ again, we see that all components of (v, w) must approach infinity, contradicting our assumption. Now suppose that $L_i < 0$. Then v_i is eventually negative and bounded away from zero. Using $(*)$ in the same way as before, we infer that also v_1 is eventually negative and bounded away from zero. But then, $v_{m+1} = |v_1|^p$ is eventually positive and bounded away from zero, and we arrive at a contradiction just as before. In conclusion, we have $L_i = 0$.

Since $v_{m+1} = |v_1|^p$ is nonnegative, the preceding argument shows $r \mapsto r^\mu w_m(r)$ and v_m are strictly increasing with $v_m(r) \to 0$ as $r \to \infty$. In particular, v_m is nonpositive, and the remaining assertions in (d) follow by iteration.

Finally, to prove (e), suppose that (v, w) is a nontrivial solution with nonnegative initial values and that (v, w) is global (else, the claim would follow from (c)). By Remark 2.3, all components of (v, w) are positive on (r_0, r_∞), and there is no loss of generality in assuming that already the initial values are positive. Now suppose (v, w) did not approach infinity. Then, by virtue of the comparison principle in Remark 2.2, neither would the nontrivial global solution $(\tilde{v}, \tilde{w})$ starting at r_0 with $\tilde{v}(r_0) = v(r_0)$ and $\tilde{w}(r_0) = 0$. But this is impossible, since (d) would imply that

$\tilde{v}_m$ is strictly increasing with limit zero. The contradiction shows that (v,w) does approach infinity, and this concludes the proof of the lemma. $\square$

Lemma 2.6. *For every $\delta \in (0,\infty)$ there exists a constant $C_\delta \in (0,\infty)$ such that for every positive solution (v,w) of (2.1) on $[r_0, r_\infty)$ with $r_0 \ge \delta$ and $r_\infty > r_0+\delta$, we have*

$$v_1 w_1 \dots v_m w_m < C_\delta \quad \text{on} \quad [r_0, r_\infty - \delta), \tag{2.3}$$

where $r_\infty - \delta = \infty$ if $r_\infty = \infty$.

Proof. Let $\delta \in (0,\infty)$, define $\varepsilon := \big(2mq + (2m-1)m\big)^{-1}$ with $q := 2m/(p-1)$, and consider the autonomous scalar ODE

$$\zeta' = \Big(\zeta^\varepsilon - \frac{m\mu}{\delta}\Big)\zeta. \tag{2.4}$$

The constant $(m\mu/\delta)^{1/\varepsilon}$ is an equilibrium; solutions above this equilibrium are explosive and approach infinity. Choose $C_\delta > (m\mu/\delta)^{1/\varepsilon}$ such that the maximal forward solution of (2.4) with $\zeta(0) = C_\delta$ exists precisely on the interval $[0,\delta)$.

Now let (v,w) be a positive solution of the equation (2.1) on $[r_0, r_\infty)$ and define $\eta := v_1 w_1 \dots v_m w_m$. Differentiating and using the differential equations for the components of (v,w), we get

$$\eta' = \Big(\frac{w_1}{v_1} + \frac{v_2}{w_1} + \dots + \frac{w_m}{v_m} + \frac{v_{m+1}}{w_m}\Big)\eta - \frac{m\mu}{r}\,\eta. \tag{2.5}$$

Moreover, since $p\,q = q + 2m$, we have

$$\Big(\frac{w_1}{v_1}\Big)^{q+2m-1}\Big(\frac{v_2}{w_1}\Big)^{q+2m-2}\dots\Big(\frac{w_m}{v_m}\Big)^{q+1}\Big(\frac{v_{m+1}}{w_m}\Big)^{q} = \frac{v_{m+1}^q}{v_1^{q+2m}}\,\eta = \eta,$$

whence

$$\Big(\frac{w_1}{v_1}\Big)^{\varepsilon(q+2m-1)}\Big(\frac{v_2}{w_1}\Big)^{\varepsilon(q+2m-2)}\dots\Big(\frac{w_m}{v_m}\Big)^{\varepsilon(q+1)}\Big(\frac{v_{m+1}}{w_m}\Big)^{\varepsilon q} = \eta^\varepsilon. \tag{2.6}$$

Since, by the definition of ε, the exponents on the left-hand side of (2.6) add up to 1, the convexity of the exponential function yields

$$\begin{aligned}
&\Big(\frac{w_1}{v_1}\Big)^{\varepsilon(q+2m-1)}\Big(\frac{v_2}{w_1}\Big)^{\varepsilon(q+2m-2)}\dots\Big(\frac{w_m}{v_m}\Big)^{\varepsilon(q+1)}\Big(\frac{v_{m+1}}{w_m}\Big)^{\varepsilon q}\\
&\le \varepsilon(q+2m-1)\frac{w_1}{v_1} + \varepsilon(q+2m-2)\frac{v_2}{w_1} + \dots + \varepsilon(q+1)\frac{w_m}{v_m} + \varepsilon q\frac{v_{m+1}}{w_m}\\
&\le \frac{w_1}{v_1} + \frac{v_2}{w_1} + \dots + \frac{w_m}{v_m} + \frac{v_{m+1}}{w_m}\,.
\end{aligned}$$

Combining this with (2.5) and (2.6), we obtain

$$\eta' \ge \Big(\eta^\varepsilon - \frac{m\mu}{r}\Big)\eta. \tag{2.7}$$

Now suppose that $r_0 \ge \delta$ and $r_\infty > r_0 + \delta$. Assuming (2.3) to be false, choose $r_1 \in [r_0, r_\infty - \delta)$ such that $\eta(r_1) \ge C_\delta$. Due to (2.7) we then have

$$\eta' \ge \Big(\eta^\varepsilon - \frac{m\mu}{\delta}\Big)\eta \quad \text{on } [r_1, r_\infty) \quad \text{and} \quad \eta(r_1) \ge C_\delta.$$

Let ζ be the maximal forward solution of (2.4) starting at r_1 with $\zeta(r_1) = C_\delta$. Then ζ exists on $[r_1, r_1+\delta)$ and blows up at $r_1 + \delta < r_\infty$. On the other hand, the comparison principle implies that $\zeta \le \eta$ on $[r_1, r_1+\delta)$, and η is continuous on $[r_1, r_1+\delta]$. This contradiction proves (2.3). $\square$

Proposition 2.7. *Let (v, w) be an eventually positive solution of* (2.1) *with interval of existence $[r_0, r_\infty)$. Then $r_\infty < \infty$ and (v, w) approaches infinity. Moreover, given $r_1 \in (r_\infty, \infty)$, any solution of* (2.1) *starting at r_0 with initial values sufficiently close to those of (v, w) blows up before r_1.*

Proof. Without loss of generality, assume that $r_0 > 0$ and that (v, w) is positive throughout. From Lemma 2.5(e), we know that (v, w) approaches infinity. Assuming $r_\infty = \infty$, Lemma 2.6 would yield the boundedness of $v_1 w_1 \dots v_m w_m$ and hence a contradiction. Thus, $r_\infty < \infty$.

Now, let $r_1 \in (r_\infty, \infty)$, fix $\delta \in (0, r_0]$ such that $r_\infty + \delta \le r_1$, and choose C_δ as in Lemma 2.6. Assuming the final assertion of the proposition to be false, choose a sequence of solutions $(v^{(n)}, w^{(n)})$ of (2.1), starting at r_0 with exit points $r_\infty^{(n)} \ge r_1$, such that $(v^{(n)}(r_0), w^{(n)}(r_0)) \to (v(r_0), w(r_0))$ as $n \to \infty$. Since the initial values of (v, w) are strictly positive, the same can be assumed for $(v^{(n)}, w^{(n)})$, and this implies that $(v^{(n)}, w^{(n)})$ is positive throughout. But then, since we have $r_0 + \delta < r_\infty + \delta \le r_1 \le r_\infty^{(n)}$, Lemma 2.6 shows that $v_1^{(n)} w_1^{(n)} \dots v_m^{(n)} w_m^{(n)} < C_\delta$ on $[r_0, r_\infty^{(n)} - \delta)$ and, in particular, on $[r_0, r_\infty)$. Continuous dependence on initial data now implies that $v_1 w_1 \dots v_m w_m \le C_\delta$ on $[r_0, r_\infty)$, contradicting the fact that (v, w) approaches infinity. The proposition is proved. ☐

Remark 2.8. Combining Lemma 2.5(c) and Proposition 2.7, we see that a solution of (2.1) is explosive if and only if it is eventually positive. Moreover, the last part of Proposition 2.7 implies the upper semicontinuity, and hence continuity, of the blow-up point as a function of initial values (recall that lower semicontinuity is a consequence of standard continuous-dependence results). Observing that the constant C_δ in Lemma 2.6 varies continuously with the parameters μ and p, we obtain, in fact, the continuity of the blow-up point as a function of both initial data and parameters.

The close connection between continuity of the blow-up point and uniform a-priori bounds (as in Lemma 2.6) has been observed and exploited in other types of blow-up problems; see, for example, [21].

In the sequel we will need a few additional properties of the blow-up point of explosive solutions of (2.1) as a function of initial values. These properties have to do with the scaling-law, already mentioned in connection with Equation (1.1), which obviously extends to the system (2.1). For a precise statement, we need some notation.

For $\lambda \in (0, \infty)$ let Λ denote the linear isomorphism of $\mathbb{R}^m$ defined by

$$\Lambda(x_1, x_2, \dots, x_m) := (x_1, \lambda^2 x_2, \dots, \lambda^{2(m-1)} x_m)$$

for $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Given a solution (v, w) of (2.1) on $[r_0, r_\infty)$ and a number $\lambda \in (0, \infty)$, let $r_{\lambda 0} := r_0/\lambda$, $r_{\lambda\infty} := r_\infty/\lambda$ and define

$$v_\lambda(r) := \lambda^q \Lambda v(\lambda r), \quad w_\lambda(r) := \lambda^{q+1} \Lambda w(\lambda r)$$

for $r \in [r_{\lambda 0}, r_{\lambda\infty})$. Then (v_λ, w_λ) is a solution of (2.1) on $[r_{\lambda 0}, r_{\lambda\infty})$. Consistent with the terminology used in the introduction, we call (v_λ, w_λ) a *rescaling* (more precisely, the λ-*rescaling*) of (v, w) and say that two solutions of (2.1) are *scaling-equivalent* if one is a rescaling of the other. Clearly, this defines an equivalence relation.

Recall that we are mostly interested in *even* solutions of (2.1), that is, solutions (v, w) starting at $r = 0$ with $w(0) = 0$. Given $x \in \mathbb{R}^m$, we denote by (v^x, w^x) the even solution of (2.1) starting at x, that is, the one with $v(0) = x$. For $\lambda \in (0, \infty)$, the λ-rescaling $(v^x_\lambda, w^x_\lambda)$ of (v^x, w^x) is nothing but the even solution of (2.1) starting at $\lambda^q \Lambda x$. This suggests to call the point

$$\sigma_x(\lambda) := \lambda^q \Lambda x$$

a *rescaling* (more precisely the λ-*rescaling*) of x and to say that two points in $\mathbb{R}^m$ are *scaling-equivalent* if one is a rescaling of the other. While the scaling-equivalence class of 0 is trivial, that of any point $x \in \mathbb{R}^m \setminus \{0\}$ is a smooth simple curve, given by

$$\Sigma_x := \{\sigma_x(\lambda) \mid \lambda \in (0, \infty)\}.$$

We call Σ_x the *scaling-parabola through* x. Note that each scaling-parabola intersects the boundary of every ball centered at the origin exactly once. In particular, each point $x \in \mathbb{R}^m \setminus \{0\}$ is scaling-equivalent to a unique point $\pi(x)$ on S^{m-1}, the unit sphere in $\mathbb{R}^m$. We refer to π, a smooth mapping of $\mathbb{R}^m \setminus \{0\}$ onto S^{m-1}, as the *scaling-projection*.

For $x \in \mathbb{R}^m$, let $\rho(x)$ denote the exit point of the solution (v^x, w^x). We then have $\rho(\sigma_x(\lambda)) = \rho(x)/\lambda$ for all $\lambda \in (0, \infty)$. Thus, along every scaling-parabola Σ_x with $x \in \mathbb{R}^m \setminus \{0\}$, the function ρ is either strictly decreasing from ∞ to 0 or identically equal to ∞. As a consequence of Lemma 2.5(d), the latter is possible only if x belongs to the set

$$\mathcal{O} := \{\xi \in \mathbb{R}^m \mid (-1)^{m-i}\xi_i < 0 \ \ \forall\, i \in \{1, \dots, m\}\}$$

(the negative half-axis if $m = 1$, the open fourth quadrant if $m = 2$, an open orthant in $\mathbb{R}^m$ otherwise). Further properties of ρ are gathered in the following proposition.

Proposition 2.9.
(a) *The set* $\{x \in \mathbb{R}^m \mid \rho(x) < \infty\} = \bigcup \{\Sigma_x \mid x \in S^{m-1},\ \rho(x) < \infty\}$ *is open; the set* $\{x \in \mathbb{R}^m \mid \rho(x) = \infty\} = \{0\} \cup \bigcup \{\Sigma_x \mid x \in S^{m-1},\ \rho(x) = \infty\}$ *is closed and contained in* $\{0\} \cup \mathcal{O}$*; and* ρ *is a continuous mapping of* $\mathbb{R}^m$ *onto* $(0, \infty]$.
(b) *Let* $x, y \in \mathbb{R}^m$ *with* $x \le y$ *and* $x \ne y$ *and suppose that* $v_1^x \ge 0$ (*that is, the first component of the even solution of* (2.1) *starting at* x *is nonnegative*). *Then* $\rho(x) > \rho(y)$.
(c) *Suppose that* $m \ge 2$ *and let* $\mathrm{e}^{(i)}$ *denote the* i*-th standard unit vector in* $\mathbb{R}^m$*, for some* $i \in \{1, \dots, m\}$*. Then, for every* $a \in \mathbb{R}^m$*, the function* $t \mapsto \rho(a + t\mathrm{e}^{(i)})$ *vanishes as* $t \to \pm\infty$.

Proof. All the claims in (a) follow directly from Lemma 2.5(c)/(d) and Proposition 2.7 (see Remark 2.8).

By the comparison principle in Remark 2.2, the assumptions in (b) guarantee that the solutions (v^x, w^x) and (v^y, w^y) are strictly ordered except possibly at 0, that is, $v^x < v^y$ and $w^x < w^y$ on the open interval $(0, \rho(y))$; clearly, $\rho(x) \ge \rho(y)$. We claim that $\rho(y)$ is finite. To see this, suppose that $\rho(x) = \rho(y) = \infty$. Then, by Lemma 2.5(d), both v_1^x and v_1^y vanish at infinity, and it follows that

$$\int_0^\infty w_1^x(r)\, dr = -v_1^x(0) = -x_1 \ge -y_1 = -v_1^y(0) = \int_0^\infty w_1^y(r)\, dr,$$

contradicting the fact that $w_1^x < w_1^y$ on $(0, \infty)$. Thus $\rho(y)$ is finite, and nothing is left to prove unless $\rho(x)$ is finite as well. So, assume that both (v^x, w^x) and

(v^y, w^y) are explosive and hence eventually positive. Choose $r_0 \in (0, \rho(x))$ such that $v^x(r_0)$, $w^x(r_0) > 0$ and suppose that $r_0 < \rho(y)$ (else, we are done). We then have $0 < v^x(r_0) < v^y(r_0)$ and $0 < w^x(r_0) < w^y(r_0)$, which allows us to choose a number $\lambda \in (1, \infty)$ such that

$$0 < v^x(r_0) < \lambda^q \Lambda v^x(r_0) < v^y(r_0), \quad 0 < w^x(r_0) < \lambda^{q+1} \Lambda w^x(r_0) < w^y(r_0).$$

The λ-rescaling $(v^x_\lambda, w^x_\lambda)$ of (v^x, w^x) satisfies $v^x_\lambda(r_0/\lambda) = \lambda^q \Lambda v^x(r_0)$ and $w^x_\lambda(r_0/\lambda) = \lambda^{q+1} \Lambda w^x(r_0)$ and blows up at $\rho(x)/\lambda$. For r in the interval $[r_0, r_0 + (\rho(x) - r_0)/\lambda)$, define

$$\phi(r) := v^x_\lambda(r + (1/\lambda - 1)r_0) \quad \text{and} \quad \psi(r) := w^x_\lambda(r + (1/\lambda - 1)r_0).$$

Since $r + (1/\lambda - 1)r_0 < r$, the pair (ϕ, ψ) is a subsolution of (2.1); it satisfies $\phi(r_0) = \lambda^q \Lambda v^x(r_0)$ and $\psi(r_0) = \lambda^{q+1} \Lambda w^x(r_0)$ and blows up at the point $r_0 + (\rho(x) - r_0)/\lambda$. By the comparison principle, the subsolution (ϕ, ψ) and the solution (v^y, w^y) are ordered where both are defined, which implies that $\rho(y) \le r_0 + (\rho(x) - r_0)/\lambda < \rho(x)$. This finishes the proof of (b).

Adopting the assumptions in (c), note that for all t, $\lambda \in (0, \infty)$ we have

$$\sigma_{a \pm te^{(i)}}(\lambda) = \sigma_a(\lambda) \pm t\, \sigma_{e^{(i)}}(\lambda) = \sigma_a(\lambda) \pm t\, \lambda^{q+2i-2} e^{(i)}.$$

Letting $\lambda := t^{-1/(q+2i-2)}$ and $t \to \infty$, we get

$$\sigma_{a \pm te^{(i)}}(\lambda) = \sigma_a(\lambda) \pm e^{(i)} \to \pm e^{(i)},$$

which, due to the continuity of ρ, implies that

$$\rho(\sigma_{a \pm te^{(i)}}(\lambda)) \to \rho(\pm e^{(i)}).$$

Note that $\rho(\pm e^{(i)}) < \infty$ (any nontrivial global even solution of (2.1) starts in the open orthant $\mathcal{O}$ if $m \ge 2$). Since $\rho(\sigma_{a \pm te^{(i)}}(\lambda)) = \rho(a \pm te^{(i)})/\lambda$, it follows that

$$\rho(a \pm te^{(i)}) = \lambda \rho(\sigma_{a \pm te^{(i)}}(\lambda)) \to 0 \quad \text{as } t \to \infty,$$

and this completes the proof of the proposition. $\square$

Remark 2.10. The last part of Proposition 2.9 implies in particular that if $m \ge 2$, then the function ρ attains a global maximum value on every line parallel to one of the coordinate axes in $\mathbb{R}^m$; this value is finite unless the line passes through the origin or intersects a scaling-parabola Σ_x with $\rho(x) = \infty$, necessarily in the open orthant $\mathcal{O}$. We conjecture, but have not been able to prove in general, that ρ attains its global maximum value at exactly one point on each such line and is strictly monotonic on either side of that point. This would have interesting implications for the geometric structure of the set $\mathcal{H}$, as defined in Remark 2.1, and hence for the set of all large radial solutions of Equation (1.1) on the unit ball (see Remark 4.6).

Note that the conjecture is trivially true for the coordinate axes (since the positive and negative half-axes are scaling-parabolae). Moreover, it follows from Proposition 2.9(b) that if $a \in \mathbb{R}^m$ is such that the solution (v^a, w^a) has a nonnegative first component, then ρ is strictly decreasing on the half-line $\{a + te^{(i)} \mid t \in [0, \infty)\}$ for every $i \in \{1, \dots, m\}$. This holds in particular if $a \in \mathbb{R}^m_+$.

3. Entire Radial Solutions

Combining Lemma 2.5 and Proposition 2.7, we obtain a complete classification of all nontrivial even solutions of (2.1): any such solution is either *explosive* and approaches infinity; or it is *global* and of the form described in Lemma 2.5(d). If $\mu = N-1$, solutions of the former type correspond to *large radial solutions* of the equation (1.1), solutions of the latter type to *entire radial solutions* of (1.1). Thus every nontrivial radial solution of (1.1) is either *large* and scaling-equivalent to a large solution on the unit ball or *entire* and scaling-equivalent to an entire solution with center-value $(-1)^m$. Note that, as a consequence of Lemma 2.5(d), if u is an entire radial solution of (1.1), then $\tilde{u} := (-1)^m u$ is a positive solution of the equation $(-\Delta)^m\,\tilde{u} = \tilde{u}^p$ on $\mathbb{R}^N$ that vanishes at infinity. With slight abuse of language, such solutions are frequently referred to as *ground states* (although they may not have "finite energy," which would impose a condition on the solutions' rate of decay at infinity).

As discussed in the introduction, entire radial solutions of Equation (1.1) are a subject of independent interest (see, for example, [9, 17, 25, 29]). Their existence and uniqueness (up to scaling) or nonexistence is of critical importance in our study of the structure of the set of all large radial solutions of (1.1). In this section, we gather the relevant information, which we will state for the system (2.1) with arbitrary $\mu \in \mathbb{R}_+$. For convenience, nontrivial global even solutions of (2.1) will be called entire solutions.

Most of the results in this section could be extracted from the literature, but brief proofs are included for the sake of completeness. The following proposition provides rather detailed a-priori information about the entire solutions of (2.1). Much more could be said about their asymptotic behavior, at least in the second and fourth-order cases (see [9]); but the decay estimates in (e) are sufficient for our purposes.

Proposition 3.1. *Every entire solution (v,w) of (2.1) satisfies the following conditions for every $i \in \{1,\dots,m\}$:*
(a) *the mapping $r \mapsto r^{\mu-2(m-i)} w_i(r)$ is strictly monotonic, increasing (decreasing) if $m-i$ is even (odd);*
(b) *the mapping $r \mapsto r^{\mu-2(m-i)-1} v_i(r)$ is strictly monotonic, decreasing (increasing) if $m-i$ is even (odd);*
(c) $2(m-i)|w_i(r)| < r|v_{i+1}(r)| < (\mu+1)|w_i(r)|$ *for all* $r \in (0,\infty)$*;*
(d) $r|w_i(r)| < \big(\mu - 2(m-i) - 1\big)|v_i(r)|$ *for all* $r \in (0,\infty)$*;*
(e) $|v_i(r)| < \big(\sqrt{\mu^2-1}/r\big)^{q+2i-2}$ *and* $|w_i(r)| < \sqrt{\frac{\mu-1}{\mu+1}}\big(\sqrt{\mu^2-1}/r\big)^{q+2i-1}$ *for all* $r \in (0,\infty)$.

Proof. Suppose that (v,w) is an entire solution of (2.1). Since, by Proposition 2.7, (v,w) does not approach infinity, Lemma 2.5(d) applies and will be invoked repeatedly without further reference. Also, in all the subsequent estimates, we assume without saying that $r \in (0,\infty)$.

First we verify the conditions (a)–(d) for $i = m$. Note that in this case (a) is already proved. We also know that v_1 is strictly monotonic with limit 0, which implies that $v_{m+1} = |v_1|^p$ is strictly decreasing and positive. Consequently,

$$r^\mu w_m(r) = \int_0^r s^\mu v_{m+1}(s)\,ds > v_{m+1}(r)\int_0^r s^\mu ds = v_{m+1}(r)\,r^{\mu+1}/(\mu+1)$$

and hence

$$0 < rv_{m+1}(r) < (\mu+1)w_m(r),$$

which proves (c). Further, since v_m vanishes at infinity and $r \mapsto r^\mu w_m(r)$ is strictly increasing, we have

$$-v_m(r) = \int_r^\infty w_m(s)\,ds = \int_r^\infty s^{-\mu}s^\mu w_m(s)\,ds > r^\mu w_m(r)\int_r^\infty s^{-\mu}ds,$$

which implies that $\mu > 1$ and $-v_m(r) > rw_m(r)/(\mu-1)$. Since $w_m(r) > 0$, this yields

$$0 < rw_m(r) < -(\mu-1)v_m(r)$$

and hence (d). Finally, substituting $w_m(r) = v_m'(r)$ in the last inequality and multiplying by $r^{\mu-2}$, we see that $r^{\mu-1}v_m'(r) + (\mu-1)r^{\mu-2}v_m(r) < 0$, that is, $\frac{d}{dr}\big(r^{\mu-1}v_m(r)\big) < 0$. This proves (b).

Now suppose that the conditions (a)–(d) hold for some $i \in \{2,\dots,m\}$. We will verify that the same conditions then hold with i replaced by $i-1$. For definiteness, assume that $m-i$ is even. In this case, we know that v_i is strictly increasing and negative. It follows that

$$r^\mu w_{i-1}(r) = \int_0^r s^\mu v_i(s)\,ds < v_i(r)\int_0^r s^\mu ds = v_i(r)\,r^{\mu+1}/(\mu+1)$$

and hence

$$0 > rv_i(r) > (\mu+1)w_{i-1}(r),$$

which proves the second inequality in (c). Also, by our inductive assumption, $r \mapsto r^{\mu-2(m-i)-1}v_i(r)$ is strictly decreasing, which implies that

$$\begin{aligned} r^\mu w_{i-1}(r) &= \int_0^r s^{2(m-i)+1}s^{\mu-2(m-i)-1}v_i(s)\,ds \\ &> r^{\mu-2(m-i)-1}v_i(r)\,r^{2(m-i)+2}/(2(m-i)+2). \end{aligned}$$

Since $w_{i-1}(r) < 0$, this yields

$$rv_i(r) < 2(m-i+1)w_{i-1}(r) < 0,$$

which proves the first inequality in (c). Since $v_i(r) = w_{i-1}'(r) + \frac{\mu}{r}\,w_{i-1}(r)$, multiplication of the above inequality by $r^{\mu-2(m-i+1)}$ leads to

$$r^{\mu-2(m-i+1)+1}w_{i-1}'(r) + \big(\mu-2(m-i+1)\big)r^{\mu-2(m-i+1)}w_{i-1}(r) < 0,$$

that is, $\frac{d}{dr}\big(r^{\mu-2(m-i+1)}w_{i-1}(r)\big) < 0$, proving (a). Using this and the fact that v_{i-1} vanishes at infinity, we get

$$-v_{i-1}(r) = \int_r^\infty w_{i-1}(s)\,ds < r^{\mu-2(m-i+1)}w_{i-1}(r)\int_r^\infty s^{-\mu+2(m-i+1)}ds,$$

whence $\mu > 2(m-i+1)+1$ and $-v_{i-1}(r) < rw_{i-1}(r)/(\mu-2(m-i+1)-1)$. Since $w_{i-1}(r) < 0$, this yields

$$0 > rw_{i-1}(r) > -\big(\mu-2(m-i+1)-1\big)v_{i-1}(r),$$

proving (d). Noting that $w_{i-1} = v_{i-1}'$ and multiplying by $r^{\mu-2(m-i+1)-2}$, we get

$$r^{\mu-2(m-i+1)-1}v_{i-1}'(r) + \big(\mu-2(m-i+1)-1\big)r^{\mu-2(m-i+1)-2}v_{i-1}(r) > 0,$$

that is, $\frac{d}{dr}\big(r^{\mu-2(m-i+1)-1}v_{i-1}(r)\big) > 0$, which proves (b). This finishes the inductive step in the case where $m-i$ is even; the other case is dealt with analogously, essentially by reversing all the inequalities.

Now we can prove (e). Iteratively applying the inequalities

$$r|v_{i+1}(r)| < (\mu+1)|w_i(r)| \quad \text{and} \quad r|w_i(r)| < (\mu-1)|v_i(r)|, \tag{3.1}$$

which follow directly from (c) and (d), we see that

$$r^{2m}|v_1(r)|^p = r^{2m}|v_{m+1}(r)| < (\mu+1)^m(\mu-1)^m|v_1(r)|,$$

which implies $r^{2m}|v_1(r)|^{p-1} < (\mu^2-1)^m$, whence $r^q|v_1(r)| < (\mu^2-1)^{q/2}$, and thus

$$|v_1(r)| < \big(\sqrt{\mu^2-1}/r\big)^q.$$

Again applying the inequalities (3.1), we obtain the remaining estimates in (e). $\square$

Remark 3.2. The inequality (d) in Proposition 3.1 shows that the existence of an entire solution (v,w) of (2.1) requires that $\mu > 2m-1$. Moreover, by the decay estimates in (e), we have $w_m(r) = O(r^{-(q+2m-1)})$ as $r\to\infty$, which implies that $r^\gamma w_m(r) \to 0$ for every $\gamma \in [0, q+2m-1)$, so that $r \mapsto r^\gamma w_m(r)$ cannot be monotonic. Since we know that $r\mapsto r^\mu w_m(r)$ is monotonic, it follows that $\mu \ge q+2m-1$. Well-known integral identities, the prototypes of which are due to Rellich and Pohožaev, allow a further sharpening of this result. The following is an explicit "radial" version of the Rellich-type identity (2.16) in [17], valid for any $\mathbb{R}^{2m}$-valued function $(v,w) = (v_1, w_1, \dots, v_m, w_m)$ on an interval $[0,R]$ such that $v_1 \in C^{2m}([0,R])$, $v_{i+1} = v_i'' + \frac{\mu}{r}v_i'$ for $i\in\{2,\dots,m\}$, $w_i = v_i'$ for $i \in \{1,\dots,m\}$, and $w(0)=0$, where $m\in\mathbb{N}$, $R\in(0,\infty)$, and $\mu\in\mathbb{R}_+$:

$$\begin{aligned}
&2\int_0^R r^{\mu+1}v_{m+1}(r)w_1(r)\,dr\\
&= (2m-\mu-1)\int_0^R r^\mu v_{m+1}(r)v_1(r)\,dr\\
&\quad + R^{\mu+1}\sum_{i=1}^m w_i(R)w_{m-i+1}(R) + (\mu-1)R^\mu\sum_{i=1}^m v_i(R)w_{m-i+1}(R)\\
&\quad + 2R^\mu\sum_{i=2}^m (i-1)\big(v_i(R)w_{m-i+1}(R) - w_i(R)v_{m-i+1}(R)\big)\\
&\quad - R^{\mu+1}\sum_{i=2}^m v_i(R)v_{m-i+2}(R).
\end{aligned}$$

Specifically, if (v,w) is a solution of (2.1), that is, if $v_{m+1} = |v_1|^p$, an integration by parts shows that

$$\begin{aligned}
&(p+1)\int_0^R r^{\mu+1}v_{m+1}(r)w_1(r)\,dr\\
&= R^{\mu+1}v_{m+1}(R)\,v_1(R) - (\mu+1)\int_0^R r^\mu v_{m+1}(r)v_1(r)\,dr,
\end{aligned}$$

and substituting this into the preceding identity, we obtain

$$\begin{aligned}
&\frac{m}{q+m}\,(\mu+1-2q-2m)\int_0^R r^\mu v_1(r)|v_1(r)|^p\,dr \\
&= R^{\mu+1}\Big\{\sum_{i=1}^m w_i(R)w_{m-i+1}(R) + (\mu-1)\sum_{i=1}^m R^{-1}v_i(R)w_{m-i+1}(R) \\
&\quad + 2\sum_{i=2}^m (i-1)\big(R^{-1}v_i(R)w_{m-i+1}(R) - R^{-1}w_i(R)v_{m-i+1}(R)\big) \\
&\quad - \sum_{i=2}^m v_i(R)v_{m-i+2}(R) - \frac{q}{q+m}v_1(R)|v_1(R)|^p\Big\}.
\end{aligned}$$

Now, if (v,w) is an *entire* solution of (2.1), then this identity holds for every $R \in (0,\infty)$, and the decay estimates in Proposition 3.1(e) show that each term within the braces on the right-hand side is $O(R^{-(2q+2m)})$ as $R \to \infty$. Assuming $\mu+1 < 2q+2m$, it would follow that $\int_0^\infty r^\mu v_1(r)|v_1(r)|^p\,dr = 0$, which is impossible since v_1 is either positive or negative. In conclusion, the system (2.1) does not have any entire solutions unless $\mu \geq 2q+2m-1$.

Consistent with standard terminology in the theory of elliptic PDEs, we define $\mu^* := 2q+2m-1$ and call the system (2.1) *subcritical*, *critical*, or *supercritical*, depending on whether $\mu < \mu^*$, $\mu = \mu^*$, or $\mu > \mu^*$. A simple computation shows that these conditions are equivalent to $p < p^*$, $p = p^*$, and $p > p^*$, respectively, where

$$p^* := \begin{cases} \frac{\mu+1+2m}{\mu+1-2m} & \text{if } \mu+1 > 2m, \\ \infty & \text{if } \mu+1 \leq 2m. \end{cases}$$

If $\mu = N-1$, then p^* is the Sobolev critical exponent associated with Equation (1.1), that is, the critical exponent for the embedding of $H^m(B)$ into $L^{p+1}(B)$. In this terminology, the preceding observation regarding entire solutions of (2.1) simply says that no such solutions exist if the system is subcritical. This is, of course, a well-established fact, at least in the case of integer μ (see [17, 23, 29]).

The result is optimal in the sense that a family of entire solutions of (2.1) is known explicitly in the critical case. In fact, if $\mu = 2q+2m-1$, then the function $r \mapsto (-1)^m\,2^q Q/(1+r^2)^q$ is the first component of such a solution, and additional ones are obtained by rescaling. If $\mu = N-1$, these solutions correspond to "minimum-energy solutions" of Equation (1.1) on $\mathbb{R}^N$ and determine the norm of the embedding of the associated "finite-energy space" into $L^{p^*+1}(\mathbb{R}^N)$ (see [24]).

Remark 3.3. We call a nontrivial even solution (v,w) of (2.1) a *solution of the Dirichlet problem* if there exists a point in the solution's interval of existence where the first m components of $(v,w) = (v_1,w_1,\dots,v_m,w_m)$ vanish simultaneously. More precisely, if R is a common zero of the first m components, we say that (v,w) solves the Dirichlet problem for (2.1) on the interval $[0,R]$. Clearly, if $\mu = N-1$, any such solution corresponds to a nontrivial radial solution of Equation (1.1) satisfying Dirichlet conditions on the boundary of the ball $B_R(0)$.

Let $R \in (0,\infty)$ and suppose that (v,w) solves the Dirichlet problem for (2.1) on $[0,R]$. If $m = 2k-1$ for some $k \in \mathbb{N}$, then v_1 is negative and increasing on $[0,R]$, and while $v_1, w_1, \dots, v_{k-1}, w_{k-1}, v_k$ vanish at R, the next component, w_k, does not; similarly, if $m = 2k$ for some $k \in \mathbb{N}$, then v_1 is positive and decreasing on $[0,R]$, and while $v_1, w_1, \dots, v_k, w_k$ vanish at R, v_{k+1} does not (see [14, Theorem 3.3]). Applying the Rellich-type identity from Remark 3.2 to the solution (v, w), we see that all but one of the terms on the right-hand side vanish; in fact, we have

$$\begin{aligned} &\frac{m}{q+m}\,(\mu+1-2q-2m)\int_0^R r^\mu v_1(r)|v_1(r)|^p\,dr \\ &= \begin{cases} R^{\mu+1}w_k^2(R) > 0 & \text{if } m = 2k-1,\ k \in \mathbb{N}, \\ -R^{\mu+1}v_{k+1}^2(R) < 0 & \text{if } m = 2k,\ k \in \mathbb{N}. \end{cases} \end{aligned}$$

In either case, it follows that $\mu+1-2q-2m < 0$; that is, solutions of the Dirichlet problem do not exist unless (2.1) is subcritical. Like the complementary result for entire solutions in Remark 3.2, this is well known, at least for integer μ (see [19] for the supercritical case and [23] for the critical case).

Remark 3.4. According to Remark 3.2, the system (2.1) has no entire solutions if it is subcritical and at least one scaling-equivalence class of entire solutions if it is critical. Moreover, the explicitly known solutions in the critical case can be shown to be the only entire solutions with *finite energy*. For integer μ, this was done by Swanson [24, 25] and again by Wei and Xu [29], who claim implicitly that *every* entire solution has finite energy. This is correct for $m \le 2$ (see [15, 30] or the discussion below) and probably for any m; but the argument in [29] (specifically the proof of Lemma 4.3 ibidem) appears to be inconclusive, and we have not been able to close the gap (see, however, the "note added in proof" at the end of the paper).

We conjecture, in fact, that (2.1) has exactly one scaling-equivalence class of entire solutions not only in the critical, but also in the supercritical case. This result is extractable from the literature (see below) if $m \le 2$, but appears to be a wide open problem if $m \ge 3$. We include short proofs in the cases $m = 1$ and $m = 2$ for the sake of completeness.

First suppose that $m = 1$. Then uniqueness is trivial, since any entire solution of (2.1) must be scaling-equivalent to the unique even solution starting at -1. Furthermore, the first component of this solution is either negative throughout, in which case the solution is global, or it has a zero, in which case the solution solves the Dirichlet problem. By Remark 3.3, the latter is impossible if (2.1) is critical or supercritical, so that in this case, the even solution starting at -1 is an entire solution.

Now suppose that $m = 2$. Then any entire solution of (2.1) is scaling-equivalent to an even solution (v, w) with $v_1(0) = 1$ and v_1 positive. Since $v_2(0)$ is the only free parameter, any two solutions satisfying these conditions are ordered, and Proposition 2.9(b) implies that not both can be global unless they are equal. This proves uniqueness. Now, given $\alpha \in \mathbb{R}$, let (v^α, w^α) denote the even solution of (2.1) starting at $(1, \alpha)$; further, define $I := \{\alpha \in \mathbb{R} \,|\, v_1^\alpha \ge 0\}$. By the comparison principle, I is an interval containing $\mathbb{R}_+$. Since the solution (v^α, w^α) depends continuously on α, I is closed. For the same reason, 0 is not a lower bound for I (note that v_1^0 is increasing and thus satisfies $v_1^0 \ge 1$). However, I is bounded from below, else Proposition 2.9(b) would imply that the exit point of (v^α, w^α) increases as $\alpha \to -\infty$, contradicting Proposition 2.9(c). In conclusion, I has a negative minimum α^*. Denote by (v^*, w^*) the even solution starting at $(1, \alpha^*)$. Its first component, v_1^*, is either monotonically decreasing throughout, or it decreases to a global minimum, necessarily with value 0, and increases to infinity thereafter.

In the first case, (v^*, w^*) is an entire solution; in the second case, it solves the Dirichlet problem. If (2.1) is critical or supercritical, the latter is impossible, by Remark 3.3, and (v^*, w^*) is an entire solution. (For the supercritical case, a similar proof was given in [9]; more general existence and nonexistence results can be found in [4, 22].)

In conclusion, (2.1) has exactly one scaling-equivalence class of entire solutions if the system is critical or supercritical and if $m \le 2$. If $m \ge 3$, the above arguments fail, with regard to both existence and uniqueness. As for uniqueness, note that Proposition 2.9(b) implies that (2.1) cannot have two distinct global even solutions that are initially ordered and have nonnegative first components. Since entire solutions have *positive* first components if m is *even*, but *negative* first components if m is *odd*, this observation is relevant if m is even, but does not imply uniqueness unless $m = 2$.

The following theorem and corollary gather the main results of this section.

Theorem 3.5. *Let $\mu^* := 2q + 2m - 1$. The system* (2.1)*, for arbitrary $m \in \mathbb{N}$, has no entire solutions if $\mu < \mu^*$ and at least one scaling-equivalence class of entire solutions if $\mu = \mu^*$. For $m \in \{1, 2\}$, the system has exactly one scaling-equivalence class of entire solutions if $\mu \ge \mu^*$.*

Corollary 3.6. *Equation* (1.1)*, for arbitrary $m \in \mathbb{N}$, has no entire radial solutions in the subcritical and at least one scaling-equivalence class of entire radial solutions in the critical case. For $m = 1$ or $m = 2$, there is exactly one scaling-equivalence class of entire radial solutions if the equation is critical or supercritical.*

4. Large Radial Solutions

Recall from Section 2 that we denote the even solution of (2.1) starting at $x \in \mathbb{R}^m$ by (v^x, w^x), its exit point by $\rho(x)$. The rescalings of a point $x \in \mathbb{R}^m$ are defined by $\sigma_x(\lambda) := \lambda^q \Lambda x := \lambda^q(x_1, \lambda^2 x_2, \dots, \lambda^{2(m-1)} x_m)$ for $\lambda \in (0, \infty)$. If $x \ne 0$, the curve Σ_x parametrized by σ_x is called the scaling-parabola through x; it intersects the unit sphere S^{m-1} in exactly one point, denoted by $\pi(x)$. We call the mapping $\pi : \mathbb{R}^m \setminus \{0\} \to S^{m-1}$ the scaling-projection.

Along each scaling-parabola, the exit point ρ is either identically equal to ∞ or strictly decreasing from ∞ to 0; in fact, $\rho(\sigma_x(\lambda)) = \rho(x)/\lambda$ for all $x \in \mathbb{R}^m$ and $\lambda \in (0, \infty)$. It follows that if $\rho(x) < \infty$, then the curve Σ_x intersects the set $\mathcal{H} := \{\xi \in \mathbb{R}^m \mid \rho(\xi) = 1\}$ exactly once, at the point $\sigma_x(\rho(x))$. On the other hand, if $\rho(x) = \infty$, then Σ_x does not intersect $\mathcal{H}$. We conclude that

$$\mathcal{H} = \{\sigma_\xi(\rho(\xi)) \mid \xi \in S^{m-1},\ \rho(\xi) < \infty\}.$$

Define $\mathcal{S} := \{\xi \in S^{m-1} \mid \rho(\xi) < \infty\}$. By Proposition 2.9(a), the set $\mathcal{S}$ is relatively open in S^{m-1}, and the mapping $\xi \mapsto \sigma_\xi(\rho(\xi))$ is a homeomorphism from $\mathcal{S}$ onto $\mathcal{H}$; its inverse is the scaling-projection π, restricted to $\mathcal{H}$.

Not only is $\mathcal{H}$ homeomorphic to an open subset of S^{m-1} under the scaling-projection; it is in fact a "separating hypersurface" in $\mathbb{R}^m$, namely, the common boundary of the connected open sets $\mathcal{A}$ and $\mathcal{B}$, defined by

$$\begin{aligned} \mathcal{A} &:= \{\xi \in \mathbb{R}^m \mid \rho(\xi) > 1\} = \{0\} \cup \{\sigma_\xi(\lambda) \mid \xi \in S^{m-1},\ \lambda \in (0, \rho(\xi))\}, \\ \mathcal{B} &:= \{\xi \in \mathbb{R}^m \mid \rho(\xi) < 1\} = \{\sigma_\xi(\lambda) \mid \xi \in S^{m-1},\ \lambda \in (\rho(\xi), \infty)\}, \end{aligned}$$

the "inside" and "outside" of $\mathcal{H}$, relative to the scaling-projection. Note, however, that $\mathcal{H}$ is a closed subset of $\mathbb{R}^m$ and thus, cannot be bounded unless $\mathcal{S}$ is compact; if $m \geq 2$, this requires that $\mathcal{S} = S^{m-1}$. In any case (including $m = 1$), the set $\mathcal{A}$ is bounded if and only if $\mathcal{S} = S^{m-1}$.

Recall from Proposition 2.9(a) that $S^{m-1} \setminus \mathcal{S}$ is contained in the open orthant

$$\mathcal{O} := \{\xi \in \mathbb{R}^m \mid (-1)^{m-i}\xi_i < 0 \;\forall\, i \in \{1, \dots, m\}\}$$

(the negative half-axis if $m = 1$). This implies that $\mathcal{H} \cap (\mathbb{R}^m \setminus \mathcal{O})$ is compact and, in particular, bounded; but if $m \geq 2$, then $\mathcal{H} \cap \mathcal{O}$ is bounded if and only if $\mathcal{S} = S^{m-1}$. (If $m = 1$, then $\mathcal{H} \cap \mathcal{O}$ is empty unless $\mathcal{S} = S^0 = \{\pm 1\}$.)

In particular, $\mathcal{S}$ contains $S^{m-1}_+ := S^{m-1} \cap \mathbb{R}^m_+$, and $\mathcal{H}_+ := \mathcal{H} \cap \mathbb{R}^m_+$ is compact and homeomorphic to S^{m-1}_+ under the scaling-projection. Moreover, it follows from Proposition 2.9(b) that every half-line in $\mathbb{R}^m_+$, emanating from the origin, intersects $\mathcal{H}_+$ exactly once. Thus, $\mathcal{H}_+$ is homeomorphic to S^{m-1}_+ also under the *radial* projection.

As another consequence of Proposition 2.9(b), the set $\mathcal{H}_+$ is the boundary of an order decomposition of $\mathbb{R}^m_+$. By this we mean a pair (A, B) of nonempty closed sets A, $B \subset \mathbb{R}^m_+$ with $A \cup B = \mathbb{R}^m_+$ and $\operatorname{int}(A \cap B) = \emptyset$ such that A is lower-closed (that is, $\{\xi \in \mathbb{R}^m_+ \mid \xi \leq x\} \subset A$ for every $x \in A$) and B is upper-closed (that is, $\{\xi \in \mathbb{R}^m_+ \mid \xi \geq x\} \subset B$ for every $x \in B$). The set $A \cap B$, the common boundary of A and B relative to $\mathbb{R}^m_+$, is called the boundary of the order decomposition (A, B). (These notions, due to Hirsch [13], are useful in the theory of monotone dynamical systems.) The set $\mathcal{H}_+$ is the boundary of the order decomposition $(\bar{\mathcal{A}}_+, \bar{\mathcal{B}}_+)$ of $\mathbb{R}^m_+$, given by $\bar{\mathcal{A}}_+ := \bar{\mathcal{A}} \cap \mathbb{R}^m_+ = \{\xi \in \mathbb{R}^m_+ \mid \rho(\xi) \geq 1\}$ (lower-closed) and $\bar{\mathcal{B}}_+ := \bar{\mathcal{B}} \cap \mathbb{R}^m_+ = \{\xi \in \mathbb{R}^m_+ \mid \rho(\xi) \leq 1\}$ (upper-closed). Moreover, $\mathcal{H}_+ = \bar{\mathcal{A}}_+ \cap \bar{\mathcal{B}}_+$ is *unordered*, that is, it does not contain any two distinct points that are ordered.

We collect the basic properties of $\mathcal{H}$ in the following proposition.

Proposition 4.1.
(a) *The set $\mathcal{H} := \{\xi \in \mathbb{R}^m \mid \rho(\xi) = 1\}$ is a closed subset of $\mathbb{R}^m$; it is homeomorphic, under the scaling-projection, to the relatively open subset $\mathcal{S}$ of S^{m-1}, defined by $\mathcal{S} := \{\xi \in S^{m-1} \mid \rho(\xi) < \infty\}$; and it is the common boundary of the connected open sets $\mathcal{A} := \{\xi \in \mathbb{R}^m \mid \rho(\xi) > 1\}$ and $\mathcal{B} := \{\xi \in \mathbb{R}^m \mid \rho(\xi) < 1\}$ (the "inside" and "outside" of $\mathcal{H}$, relative to the scaling-projection).*
(b) *The set $S^{m-1} \setminus \mathcal{S}$ is contained in $\mathcal{O} := \{\xi \in \mathbb{R}^m \mid (-1)^{m-i}\xi_i < 0 \;\forall\, i \in \{1, \dots, m\}\}$; $\mathcal{H} \cap (\mathbb{R}^m \setminus \mathcal{O})$ is compact; and if $m \geq 2$, then $\mathcal{H} \cap \mathcal{O}$ is bounded if and only if $\mathcal{S} = S^{m-1}$.*
(c) *The set $\mathcal{H}_+ := \mathcal{H} \cap \mathbb{R}^m_+$ is compact, homeomorphic to $S^{m-1}_+ := S^{m-1} \cap \mathbb{R}^m_+$ under the radial projection, unordered, and the boundary of the order decomposition $(\bar{\mathcal{A}}_+, \bar{\mathcal{B}}_+)$ of $\mathbb{R}^m_+$ with $\bar{\mathcal{A}}_+ := \bar{\mathcal{A}} \cap \mathbb{R}^m_+$ and $\bar{\mathcal{B}}_+ := \bar{\mathcal{B}} \cap \mathbb{R}^m_+$.*

A complete characterization of the set $\mathcal{H}$, beyond the general properties gathered above, requires the determination of the set $\mathcal{S}$. Note that the points of $S^{m-1} \setminus \mathcal{S}$ are in one-to-one correspondence with the scaling-equivalence classes of entire radial solutions of (2.1). Recalling Theorem 3.5, we conclude that $\mathcal{S} = S^{m-1}$ whenever (2.1) is subcritical. For $m \leq 2$, we infer that $\mathcal{S} = S^{m-1} \setminus \{\xi_0\}$, for some point $\xi_0 \in S^{m-1} \cap \mathcal{O}$, whenever (2.1) is critical or supercritical. In view of our conjecture regarding entire solutions of (2.1) (see Remark 3.4), we expect the latter to hold for $m \geq 3$ as well. This leads to the following theorem and conjecture.

Theorem 4.2.
(a) *Suppose that $m = 1$. Then the set $\mathcal{H}$ consists of exactly two real numbers of opposite sign if* (2.1) *is subcritical, of exactly one positive number if* (2.1) *is critical or supercritical.*
(b) *Suppose that $m = 2$. Then the set $\mathcal{H}$ is a closed simple curve in $\mathbb{R}^2$ if* (2.1) *is subcritical, an unbounded simple curve, asymptotic to a unique scaling-parabola in the fourth quadrant of $\mathbb{R}^2$, if* (2.1) *is critical or supercritical; in either case, the origin belongs to the interior of the curve.*
(c) *Suppose that $m \geq 3$ and that* (2.1) *is subcritical. Then the set $\mathcal{H}$ is a closed hypersurface in $\mathbb{R}^m$, homeomorphic to S^{m-1} under the scaling-projection.*

Conjecture 4.3. *Suppose that $m \geq 3$ and that* (2.1) *is critical or supercritical. Then the set $\mathcal{H}$ is an unbounded hypersurface in $\mathbb{R}^m$, and there exists a point $\xi_0 \in S^{m-1} \cap \mathcal{O}$ such that $\mathcal{H}$ is homeomorphic to $S^{m-1} \setminus \{\xi_0\}$ under the scaling-projection.*

Remark 4.4. While the conjecture is completely open in the supercritical case, Theorem 3.5 and Proposition 4.1 yield at least a partial result if (2.1) is critical: the set $\mathcal{H}$ is then an unbounded hypersurface in $\mathbb{R}^m$, and there exists a nonempty closed set $X_0 \subset S^{m-1} \cap \mathcal{O}$ such that $\mathcal{H}$ is homeomorphic to $S^{m-1} \setminus X_0$ under the scaling-projection.

Remark 4.5. As discussed in Remark 2.1, the set $\mathcal{H}$ (for $\mu = N-1$) is homeomorphic to the set $\mathcal{L}$ of all large radial solutions of Equation (1.1) on the unit ball B, endowed with the natural topology of the space $C^{2m}(B)$. Thus, by Theorem 4.2, $\mathcal{L}$ is a full $(m-1)$-sphere whenever (1.1) is subcritical; if (1.1) is critical or supercritical, and if $m \leq 2$, then $\mathcal{L}$ is a punctured $(m-1)$-sphere. If proved, Conjecture 4.3 would imply the latter to hold without the restriction $m \leq 2$.

Remark 4.6. In Remark 2.10 we conjectured that on every line parallel to one of the coordinate axes in $\mathbb{R}^m$, the function ρ attains its global maximum value at exactly one point and is strictly monotonic on either side of that point. This is equivalent to saying that any such line intersects the set $\mathcal{H}$ at most twice, which would shed further light on the geometric structure of $\mathcal{H}$, with interesting implications regarding the exact multiplicity of large radial solutions of Equation (1.1). In fact, given any $m-1$ of the values $u(0), \Delta u(0), \dots, \Delta^{m-1}u(0)$, there would be at most two large radial solutions of (1.1) on B with these prescribed center values.

In the remainder of this section, we illustrate our results and derive further information in the second and fourth-order cases. We note that the accompanying graphs are not schematic drawings; they are based on high-accuracy numerical computations (the method will be described in [6]). The depictions of typical large radial solutions illustrate qualitative features that are consequences of the present analysis, as discussed below. These graphs do not and cannot resolve fine points of the solutions' blow-up behavior; those will be addressed in [6].

First consider the familiar second-order case, $m = 1$. If the equation (1.1) is subcritical, there are exactly two large radial solutions on the unit ball, one with a positive, the other with a negative center value. If (1.1) is critical or supercritical, there is exactly one such solution, with a positive center value. In either case, the solutions are strictly increasing with respect to the radial variable. Figure 1 shows the large radial solutions of (1.1) on B in space dimension $N = 3$ for $p = 3$, $p = 4$, and $p = 4.5$ (subcritical, two solutions) as well as for $p = 5$ (critical, one solution).

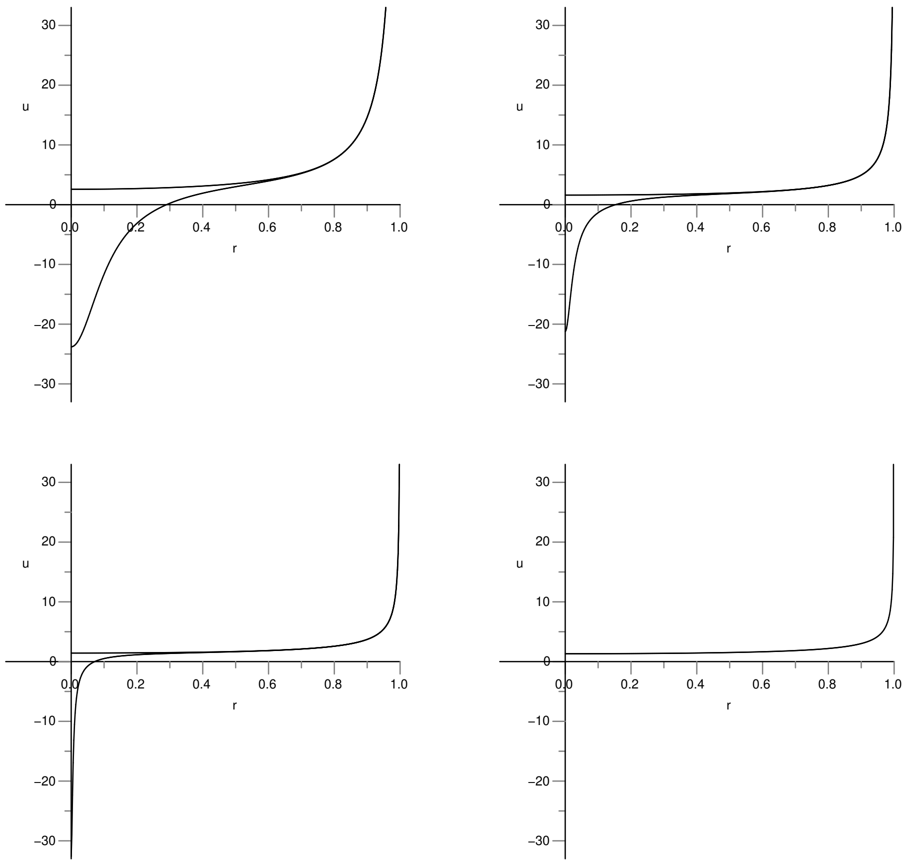

FIGURE 1. Large radial solutions of (1.1) on B for $m = 1$ and $N = 3$, in the subcritical cases $p = 3$ (top-left), $p = 4$ (top-right), $p = 4.5$ (bottom-left), and in the critical case $p = 5$ (bottom-right).

Now consider the fourth-order case, $m = 2$. If Equation (1.1) is subcritical, the set $\mathcal{H}$ is a closed simple curve in $\mathbb{R}^2$, containing the origin in its interior. Hence there exist numbers $\underline{\alpha}$, $\overline{\alpha} \in \mathbb{R}$ with $\underline{\alpha} < 0 < \overline{\alpha}$ such that, given $\alpha \in \mathbb{R}$, Equation (1.1) has no large radial solution on B with center value α if $\alpha < \underline{\alpha}$ or $\alpha > \overline{\alpha}$; at least one such solution if $\alpha = \underline{\alpha}$ or $\alpha = \overline{\alpha}$; and at least two such solutions if $\underline{\alpha} < \alpha < \overline{\alpha}$. (As noted in Remark 4.6, we could say "exactly" instead of "at least" if our conjecture regarding the monotonicity of ρ on lines parallel to the coordinate axes were proved.)

Figure 2 depicts the set $\mathcal{H}$ for $p = 3$ and $N = 3$, along with the scaling-parabolae passing through the four extremal points of $\mathcal{H}$. The close-up in the graph on the right reveals that there are indeed two extremal points in the fourth quadrant, albeit very close to each other. Figure 3 shows typical large radial solutions of (1.1) on B, including the ones with the smallest and largest center values (top-left graph).

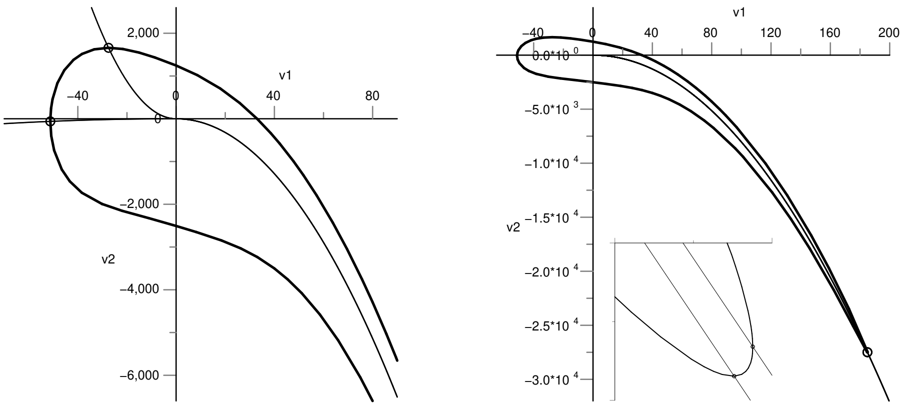

FIGURE 2. The set $\mathcal{H}$ for $m = 2$, $p = 3$, $N = 3$ (subcritical, $\mathcal{H}$ compact).

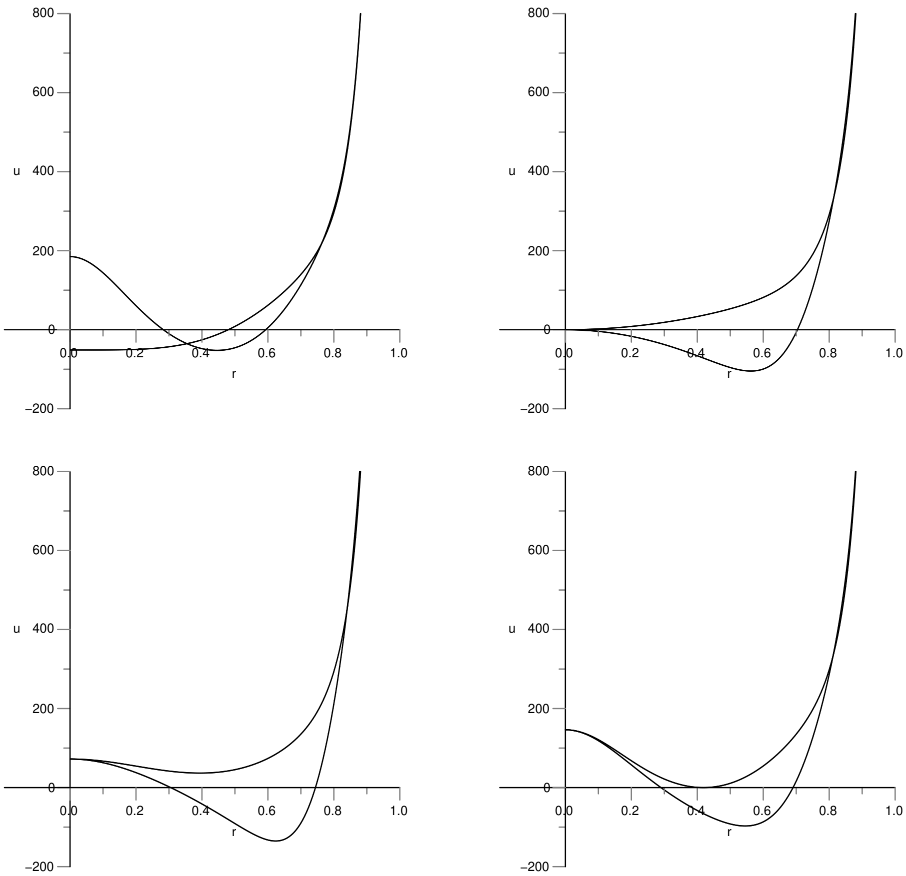

FIGURE 3. Large radial solutions of (1.1) on B for $m = 2$, $p = 3$, $N = 3$ (subcritical).

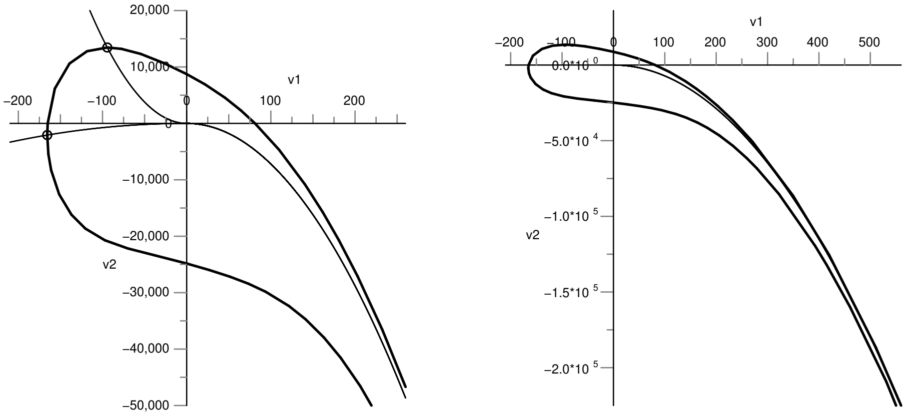

FIGURE 4. The set $\mathcal{H}$ for $m = 2$, $p = 3$, $N = 13$ (supercritical, $\mathcal{H}$ unbounded).

If Equation (1.1) is critical or supercritical, the set $\mathcal{H}$ is an unbounded simple curve in $\mathbb{R}^2$, asymptotic to a unique scaling-parabola in the fourth quadrant and containing the origin in its "interior." Hence there exists a number $\underline{\alpha} \in \mathbb{R}$ with $\underline{\alpha} < 0$ such that, given $\alpha \in \mathbb{R}$, Equation (1.1) has no large radial solution on B with center value α if $\alpha < \underline{\alpha}$; at least one such solution if $\alpha = \underline{\alpha}$; and at least two such solutions if $\alpha > \underline{\alpha}$. (Again, we could say "exactly" instead of "at least" if the conjecture in Remark 2.10 were proved.)

Figure 4 depicts the set $\mathcal{H}$ for $p = 3$ and $N = 13$, along with the scaling-parabolae passing through the two extremal points of $\mathcal{H}$ and the unique scaling-parabola in the fourth quadrant that does not intersect $\mathcal{H}$. Figure 5 shows typical large radial solutions of (1.1) on B, including the one with the smallest center value (on the left). Despite its appearance, the graph on the right contains six solutions — two for each of the center values, one positive, the other sign-changing.

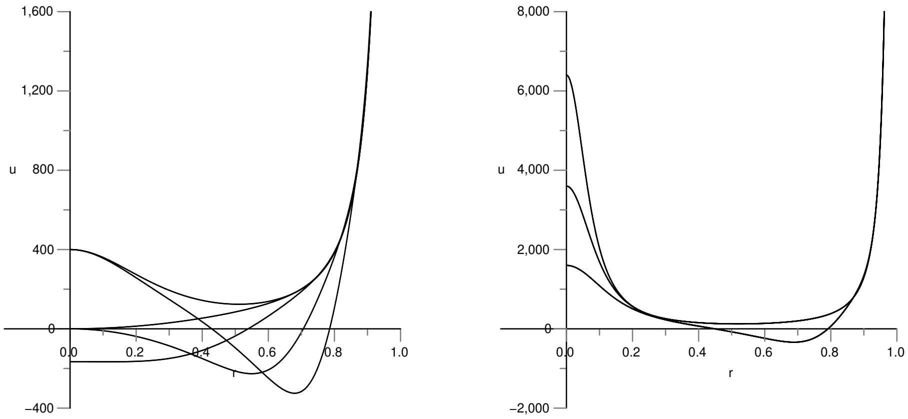

FIGURE 5. Large radial solutions of (1.1) on B for $m = 2$, $p = 3$, $N = 13$ (supercritical).

All large radial solutions of (1.1) on B that start in the upper half-plane or on the horizontal axis (that is, solutions u with $\Delta u(0) \geq 0$) are strictly increasing with respect to the radial variable; those starting in the lower half-plane (that is, solutions u with $\Delta u(0) < 0$) are strictly decreasing to a global minimum and strictly increasing thereafter. In the subcritical case, there exists a solution of the second kind whose minimum value is 0 and which, therefore, solves the Dirichlet problem for Equation (1.1) on a ball centered at the origin. This solution necessarily starts at a point (α^*, β^*) on the "upper" part of $\mathcal{H}$ in the fourth quadrant.

The segment of $\mathcal{H}$ in the first quadrant and its continuation into the fourth quadrant, down to the point (α^*, β^*), is the locus of the starting-points of the nonnegative large radial solutions of (1.1) on B. As a consequence of Proposition 2.9(b), this segment is unordered and thus the graph of a strictly decreasing continuous function $\phi\colon [0, \alpha^*] \to \mathbb{R}$ with $\phi(0) =: \beta_* > 0$ and $\phi(\alpha^*) = \beta^* < 0$. It follows that, given $\alpha \in \mathbb{R}$, Equation (1.1) has a unique nonnegative large radial solution u on B with center value $u(0) = \alpha$ if and only if $0 \leq \alpha \leq \alpha^*$. As α increases from 0 to α^*, the second center value $\Delta u(0)$ decreases from $\beta_* > 0$ to $\beta^* < 0$. Further, the solution is strictly positive except if $\alpha = 0$ or $\alpha = \alpha^*$.

In the critical or supercritical case, radial solutions of (1.1) starting on the unique scaling-parabola in the fourth quadrant that misses the set $\mathcal{H}$ are entire solutions and positive. Consequently, the segment of $\mathcal{H}$ "above" this parabola is comprised of starting-points of positive large radial solutions of (1.1) on B. Proposition 2.9(b) implies that this segment, including its end-point above the origin, is unordered and thus the graph of a strictly decreasing continuous function $\phi\colon [0, \infty) \to \mathbb{R}$ with $\phi(0) =: \beta_* > 0$ and $\phi(\alpha) \to -\infty$ as $\alpha \to \infty$. We conclude that Equation (1.1) has a unique nonnegative large radial solution u on B with $u(0) = \alpha$ for every center value $\alpha \in [0, \infty)$. As α increases from 0 to ∞, the second center value $\Delta u(0)$ decreases from $\beta_* > 0$ to $-\infty$, and the solution is strictly positive except if $\alpha = 0$. In the critical case, similar results were obtained in [12].

Figure 6 shows typical nonnegative large radial solutions of (1.1) on B in a subcritical and a supercritical case. Included are the extremal solutions with center value 0 (in both cases) and the one with center value α^* in the subcritical case.

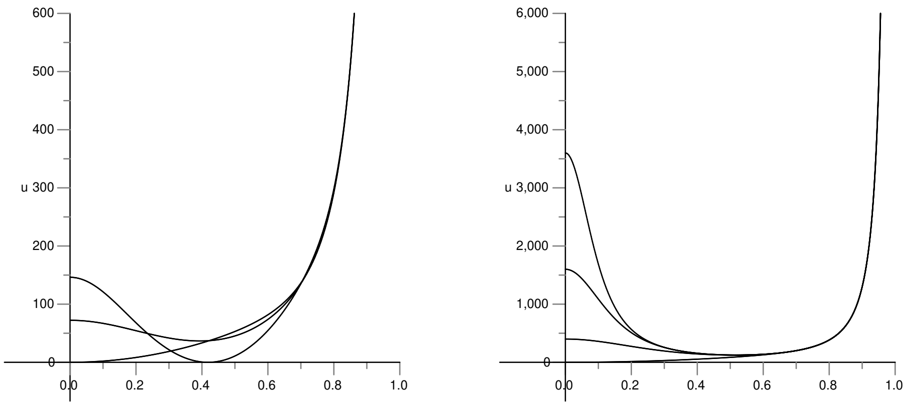

FIGURE 6. Nonnegative large radial solutions of (1.1) on B for $m = 2$ and $p = 3$, in the cases $N = 3$ (subcritical, left) and $N = 13$ (supercritical, right).

Note Added in Proof. After this paper was accepted and edited for publication, we became aware of Reference [3], which appears to close the gap in [29] discussed at the beginning of Remark 3.4. It would follow that, in the critical case with arbitrary $m \in \mathbb{N}$, Equation (1.1) has exactly one scaling-equivalence class of entire radial solutions, and then the set of all large radial solutions on the unit ball is homeomorphic to a punctured $(m-1)$-sphere. This lends further credence to Conjecture 4.3. We thank Tobias Weth of the University of Giessen for drawing our attention to [3].

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