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ASYMPTOTIC BEHAVIOR OF A WEAKLY FORCED DRY FRICTION OSCILLATOR

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. This note is devoted to stick-slip aspects of the motion of a dry friction damped oscillator under weak irregular forcing. Our main result complements [10, Theorem 3.(a)] and is also related to [1], where a non-Lipschitz model for Coulomb friction was consider in the unforced case. We provide sufficient conditions guaranteeing that solutions stabilizing in finite time, but observe also an infinite succession of "stick-slip" behavior. The last section discusses an extension to certain systems of such oscillators.

1. Introduction

Its is well known that the abstract Cauchy problem associated with multivalued monotone (resp. accretive) operators on Hilbert spaces (resp. Banach spaces) may lead to very peculiar strong convergence asymptotic behaviour for its solutions. More precisely, if for instance $X = H$ is a Hilbert space, and $A : D(A) \to \mathcal{P}(H)$ is a maximal monotone operator multivalued at 0 (with $0 \in intD(A)$) then the solution of the

$$\begin{gathered} \frac{du}{dt}(t) + Au(t) \ni f(t) \quad \text{in } X, \\ u(0) = u_0, \end{gathered} \tag{1}$$

possesses the property of extinction in finite time once we assume that f satisfies

$$B(f(t), \epsilon) \subset A0, \quad \text{for a.e. } t \geq t_f, \text{ for some } \epsilon > 0 \text{ and } t_f \geq 0. \tag{2}$$

This result, due to H. Brezis ([4]), has been generalized in many different directions in the last twenty years (see, for instance, the survey [11]). The main goal of this paper is to investigate some simple cases in which the image of 0 under the multivalued operator is not so large as to contain a ball.

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In order to get some insight into this type of difficulties we shall first study the long-term behaviour of solutions of

$$\ddot{x} + x - p(t) \in \operatorname{sgn}(-\dot{x}), \tag{3}$$

where

$$\operatorname{sgn}(y) := \begin{cases} y/|y| & \text{if } y \neq 0 \\ [-1,1] & \text{if } y = 0 \end{cases}$$

represents the damping force due to dry friction and $p \in C([0,\infty),\mathbb{R})$ is an external forcing, which is weak in the sense that $\sup\{|p(t)| : t \in [0,\infty)\} < 1$. (3) is one of a variety of damped oscillator equations modelling dry friction, and our interest in this particular setting arises from the fact that it describes pure dry friction damping, mathematically, the more challenging case. We refer to [1], [2] and [13] for other settings and references, to [10] and [13] for "resonance" under almost periodic forcing, and to [5] and the references therein for dry friction damped wave equations. As for our purposes, it was shown in [10] (formally in an almost periodic setting), that every solution converges to a constant solution as $t \to \infty$ (cf. also Lemma 2.5 (4) below). We are interested in solutions which either are eventually constant (the mass comes to rest in finite time) or show an infinite succession of stick-slip events. We allow temporally irregular forcing and can require without loss of generality that

$$\overline{p} := \limsup_{t\to\infty} p(t) = -\underline{p} := \liminf_{t\to\infty} p(t).$$

Definition 1.1. Let $a,\ b \in \mathbb{R}_+,\ a < b$. An interval $[a,b]$ $\big([a,\infty)\big)$ is called a dead zone of a solution x of (3), if $\dot{x}(t) = 0$ for $t \in [a,b]$ $\big(t \in [a,\infty)\big)$.

Our main result regarding (3) read as follows.

Theorem 1.2. *If x is a solution of (3) with $x_\infty := \lim_{t\to\infty} x(t) < 1 + \overline{p}$. Then one of the following alternatives occurs.*

1. *$\overline{p} - 1 < x_\infty < 1 - \overline{p}$, $t \mapsto x_\infty$ is a constant solution of (3), and x has a dead zone $[\underline{t},\infty)$.*
2. *$|x_\infty| > 1 - \overline{p}$, x is monotone and has a compact dead zone in each neighborhood of infinity.*
3. *If $|x_\infty| = 1 - \overline{p}$, there may or may not be a dead zone.*

We conclude this paper with some partial result concerning the system

$$\begin{gathered} m\ddot{x}_i(t) + k(-x_{i-1}(t) + 2x_i(t) - x_{i+1}(t)) + \mu_\beta \operatorname{sgn}(\dot{x}_i(t)) \ni p_i(t) \\ x_i(0) = u_{0,i},\ \dot{x}_i(0) = v_{0,i} \end{gathered} \tag{4}$$

$i = 1,\dots,N$, where we are assuming that

$$x_0(t) = 0,\ x_{N+1}(t) = 1 \quad \text{for } t \in (0,+\infty),$$

m, μ_β are positive constants and the term $\mu_\beta \operatorname{sgn}(\dot{x}_i(t))$ represents the Coulomb friction. This system arises in the modeling of the vibration of N-particles of equal mass m in a non-inertial coordinate system. Indeed, we denote the locations of the particles, along the interval $(0,1)$ of the x axis, by $x_i(t)$, and we assume that each particle is connected to its neighbors by two harmonic springs of strength k. We also assume that the particles are subject to a resultant friction force (the Coulomb (or solid) friction). Functions $p_i(t)$ correspond to fictitious forces due to the change of variable with respect to an inertial system. We show that, at least, in some special cases the first conclusion of Theorem 1.2 remains true.

2. Preliminaries

It is worth noting that the formally more general equation

$$\ddot{x} + kx - p(t) \in \mu \operatorname{sgn}(-\dot{x}) \tag{5}$$

with k, $\mu \in (0, \infty)$, and $p \in L^\infty([0, \infty), \mathbb{R})$ can be re-scaled to the simpler form (3). Replacing x by $\frac{x}{\mu}$ and p by $\frac{p}{\mu}$ yields $\mu = 1$ without loss of generality. Next, let $\bar{p} := \limsup_{t\to\infty} p(t)$ and $\underline{p} := \liminf_{t\to\infty} p(t)$. The transformation $x \to x - \frac{\bar{p}+\underline{p}}{2k}$ and $p \to p - \frac{\bar{p}+\underline{p}}{2}$ allows us to assume $\underline{p} = -\bar{p}$. Finally, setting $\tau = \frac{t}{\sqrt{k}}$ and $y(\tau) = kx(\frac{t}{\sqrt{k}})$, one obtains $\dot{y}(\tau) = \sqrt{k}\dot{x}(\frac{t}{\sqrt{k}})$ and $\ddot{y}(\tau) = \ddot{x}(\frac{t}{\sqrt{k}})$, hence $\ddot{y} + y - p(\tau) \in -\operatorname{sgn}(-\dot{y})$, thus we arrive at (3) under the additional "symmetry hypotheses" $\bar{p} = -\underline{p}$.

For the rest of this article, we consider (3) under the Hypotheses:

$$p \in L^\infty(\mathbb{R}_+, \mathbb{R}) \cap C(\mathbb{R}_+, \mathbb{R}), \quad \overline{p} = -\underline{p}.$$

Definition 2.1. One calls $x \in W^{2,1}_{\rm loc}([0, \infty), \mathbb{R})$ a solution of (3), if there exists a $u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $u(t) \in \operatorname{sgn}(\dot{x}(t))$ for a.e. $t \in \mathbb{R}_+$ such that $\ddot{x}(t) + x(t) = p(t) - u(t)$ for a.e. $t \in (0, \infty)$.

Note that $|u(t)| \le 1$ for a.e. $t \in (0, \infty)$. The general theory of "multi-valued" ordinary differential equations ([7, §5], [13, section 2.2]) yields the following result.

Proposition 2.2. *The initial-value problem* (3), $x(t_0) = x_0$, $\dot{x}(t_0) = \eta_0$ *has for each* $(t_0, x_0, \eta_0) \in \mathbb{R}^3$ *a (forwardly) unique "global" solution* $x \in W^{2,1}_{\rm loc}([t_0, \infty), \mathbb{R})$.

As mentioned, we are interested in whether solutions develop dead zones. The first example shows that "stronger" forcing limits the length of dead zones.

Example 2.3. Let $p(t) = (1 + \frac{1}{1+t}) \sin(t)$. Then $\bar{p} = 1$, but a solution of (3) cannot have dead zones of length greater than π, since $|p(\frac{\pi}{2} + j\pi) - p(\frac{\pi}{2} + (j-1)\pi)| > 2$ for $j \in \mathbb{N}$.

The next example indicates that the second alternative of Theorem 1.2 can in fact occur.

Example 2.4. Let $\overline{p} \in (0, 1)$, $x_0 \in (1 - \overline{p}, 1 + \overline{p})$, $0 < t_1 < t_2 < t_3$, and

$$p(t) = \begin{cases} \overline{p} & t \in [0, t_1), \\ -\overline{p} & t \in [t_1, t_2), \\ \overline{p} & t \in [t_2, t_3]. \end{cases}$$

If $t_2 - t_1 < \frac{\pi}{4}$ is sufficiently small, then $2\overline{p} - (x_0 - 1 + \overline{p}) \cos(t_2 - t_1) > 0$ in view of $x_0 < 1 + \overline{p}$. Moreover, let

$$t_2^* = t_2 + \arctan\Big(\frac{(x_0 - 1 + \overline{p}) \sin(t_2 - t_1)}{2\overline{p} - (x_0 - 1 + \overline{p}) \cos(t_2 - t_1)}\Big)$$

satisfy $t_2 < t_2^* < t_3$. One verifies that

$$x(t) := \begin{cases} x_0 & t \in [0, t_1], \\ (x_0 - 1 + \overline{p}) \cos(t - t_1) + 1 - \overline{p} & t \in (t_1, t_2), \\ \Big((x_0 - 1 + \overline{p}) \cos(t_2 - t_1) - 2\overline{p} \Big) \cos(t - t_2) & \\ +1 - P - (x_0 - 1 + \overline{p}) \sin(t_2 - t_1) \sin(t - t_2) & t \in (t_2, t_2^*). \end{cases}$$

solves (3) on $[0, t_2^*)$ and satisfies $x(0) = x_0$, $\dot{x}(0) = 0$. In fact, one has $\dot{x}(t) = -(x_0 - 1 + \overline{p}) \sin(t - t_1) < 0$ for $t \in [t_1, t_2]$, $\dot{x}(t) = -\Big((x_0 - 1 + \overline{p}) \cos(t_2 - t_1) - 2\overline{p} \Big) \sin(t - t_2) - (x_0 - 1 + \overline{p}) \sin(t_2 - t_1) \cos(t - t_2) < 0$ for $t \in (t_2, t_2^*)$, and $\dot{x}(t_2^*) = 0$.

Finally, $\big((x_0 - 1 + \overline{p}) \cos(t_2 - t_1) - 2\overline{p} \big) \cos(t_2^* - t_2) + 1 - P - (x_0 - 1 + \overline{p}) \sin(t_2 - t_1) \sin(t_2^* - t_2) \to x_0$ as $t_2 \to t_1$. Therefore, given $0 < \epsilon < x_0 - 1 + \overline{p}$, we can choose t_2 such that $x_0 - x(t_2^*) < \epsilon/2$ and have $x(t) = x(t_2^*)$ for $t \in (t_2^*, t_3]$. We can repeat this process with $\frac{\epsilon}{2^j}$ in the $j-th$ step and obtain a solution of (3) on $[0, \infty)$ with a p that switches between ± 1. The solution converges to an $x_\infty > 1 - \overline{p}$, is not eventually constant, and has infinitely many dead zones near infinity.

It is easy to see how to modify the example in order to find stick-slip behavior for a smooth p. One has to guarantee that p takes on the value -1 in each of the intervals (t_{2j-1}, t_{2j}), which prevents infinitely long dead zones, but that these intervals are so short that $x(t_{2j-1}) - x(t_{2j+1}) < \frac{\epsilon}{2^j}$ for $j \in \mathbb{N}$.

Next we collect some folklore results (cf. also [10] or [13]), which we prove for the reader's convenience.

Lemma 2.5. *Let $\|p\|_\infty < 1$ and x and y be solutions of* (3)*. Then*

1. $t \mapsto \dot{x}(t)^2 + x(t)^2$ *is nonincreasing on $\mathbb{R}_+$ and strictly decreasing on intervals which do not intersect the interior of any dead zone of x.*
2. $\int_0^\infty |\dot{x}(t)|\, dt \le \frac{1}{2(1 - \|p\|_\infty)} \Big[\dot{x}(0)^2 + x(0)^2 \Big]$.
3. $\|\ddot{x}\|_\infty \le 1 + \|p\|_\infty + \sqrt{\dot{x}(0)^2 + x(0)^2}$.
4. $x_\infty := \lim_{t\to\infty} x(t)$ *exists and belongs to* $-1 - \|p\|_\infty \le x_\infty \le 1 + \|p\|_\infty$. *Moreover, $\dot{x}(t) \to 0$ as $t \to \infty$.*
5. $(\dot{x} - \dot{y})^2 + (x - y)^2$ *is nonincreasing on $\mathbb{R}_+$.*
6. *The interval* $[\sup p - 1, \inf p + 1]$ *forms the set of constant solutions of* (3).

Proof. Let $u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $u \in \operatorname{sgn} \circ \dot{x}$ a.e. One has

$$\frac{1}{2} \frac{d}{dt} \Big[\dot{x}(t)^2 + x(t)^2 \Big] = \dot{x}(t) \Big[p(t) - u(t) \Big] \le -|\dot{x}(t)| + |p(t)||\dot{x}(t)| \le -|\dot{x}(t)| \big(1 - \|p\|_\infty \big). \tag{6}$$

This yields the first assertion of 1. Also, if $0 < t_1 < t_2 < \infty$ with $\dot{x}(t_1)^2 + x(t_1)^2 = \dot{x}(t_2)^2 + x(t_2)^2$, then $-|\dot{x}(t)| + p(t)\dot{x}(t) = 0$ for $t \in [t_1, t_2]$, hence $\dot{x}(t) = 0$ for $t \in [t_1, t_2]$ in view of $\|p\|_\infty < 1$.

2. One obtains from (6) that $\int_0^t |\dot{x}(s)|\, ds \le \frac{1}{2(1 - \|p\|_\infty)} \big[\dot{x}(0)^2 + x(0)^2 \big]$.

3. Inequality (6) implies $\|x\|_\infty^2 \le \dot{x}(0)^2 + x(0)^2$, hence (3) yields the L^∞-bound for $\ddot{x}$.

4. Statement 2 and $|x(\overline{t}) - x(\underline{t})| \le \int_{\underline{t}}^{\overline{t}} |\dot{x}|\, dt$ for $0 \le \underline{t} < \overline{t} < \infty$ imply the convergence of $x(t)$ as $t \to \infty$. Since $\dot{x}(t)^2 + x(t)^2$ also converges as $t \to \infty$, $|\dot{x}(t)|$ converges, and its limit is equal to 0 in view of $\dot{x} \in L^1(\mathbb{R}_+, \mathbb{R})$. Finally, let $u \in \operatorname{sgn}(\dot{x})$ such that $\ddot{x}(t) + x(t) = p(t) - u(t)$ for a.e. $t \in \mathbb{R}_+$. Assume that

$x_\infty > 1 + \|p\|_\infty$. Select $\epsilon \in (0, 1 - \|p\|_\infty)$ with $x_\infty > 1 + \|p\|_\infty + 2\epsilon$ and choose $\underline{t} \geq 0$ such that $x(t) \geq x_\infty - \epsilon$ and $|\dot{x}| < \epsilon$ for $t \in [\underline{t}, \infty)$. Since $p(t) - x(t) - u(t) \leq p(t) - x_\infty + \epsilon + 1 \leq p(t) - (1 + \|p\|_\infty + 2\epsilon) + \epsilon + 1 \leq -\epsilon$ for a.e. $t \in [\underline{t}, \infty)$, one has $\ddot{x} < -\epsilon$ a.e. on $[\underline{t}, \infty)$, hence $\dot{x} \leq -\epsilon(t - \underline{t})$ a.e. on $[\underline{t}, \infty)$ which is a contradiction. Likewise, one obtains $x_\infty \geq -1 - \|p\|_\infty$.

5. Let $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $v \in \operatorname{sgn} \circ \dot{y}$, then

$$\frac{1}{2}\frac{d}{dt}\Big[(\dot{x}(t) - \dot{y}(t))^2 + (x(t) - y(t))^2\Big] = -\big(u(t) - v(t)\big)\big(\dot{x}(t) - \dot{y}(t)\big) \leq 0 \tag{7}$$

for $t \in (0, \infty)$ a.e.

6. If $z \in [\sup p - 1, \inf p + 1]$, then $-1 \leq z - p(t) \leq 1$, hence $z - p(t) \in \operatorname{sgn}(0)$ for all $t \in (0, \infty)$, which shows that z is a constant solution of (3). On the other hand, if z is a constant solution of (3), then $z - p(t) \in \operatorname{sgn}(0)$ for all $t \in (0, \infty)$, hence $-1 \leq z - p(t) \leq 1$ for all $t \in (0, \infty)$, thus, $-1 \leq z - \sup p$ and $z - \inf p \leq 1$. $\square$

As for statement 6, the following example shows that one can have solutions with dead zones of the form $[a, \infty)$ which stay away from the set of constant solutions. Obviously, the reason is that $\bar{p} < \sup p$.

Example 2.6. Let

$$p(t) := \begin{cases} \frac{1}{2} - t & 0 \leq t \leq 1; \\ t - \frac{3}{2} & 1 < t \leq \frac{3}{2}; \\ 0 & t > \frac{3}{2}, \end{cases}$$

$$\begin{aligned} X(t) &:= \sin(t)\cos(1/2) - \cos(t)\sin(1/2) + 3/2 - t, \\ Y(t) &:= 2\sin(t) - 4\sin(t)\cos(1/2)^2 + \sin(t)\cos(1/2) - \cos(t)\sin(1/2) \\ &\quad + 4\cos(t)\sin(1/2)\cos(1/2) - 1/2 + t, \\ Z(t) &:= -2\sin(t)\cos(1/2) + 2\sin(t) - 4\sin(t)\cos(1/2)^2 + 4\sin(t)\cos(1/2)^3 \\ &\quad - 4\cos(t)\sin(1/2)\cos(1/2)^2 + 4\cos(t)\sin(1/2)\cos(1/2) + 1, \end{aligned}$$

and $\bar{t}$ be the zero of $\dot{Z}$ in $[2.5, 2.6]$. Then

$$x(t) := \begin{cases} 1 & 0 \leq t \leq \frac{1}{2}, \\ X(t) & \frac{1}{2} < t \leq 1, \\ Y(t) & 1 < t < \frac{3}{2}, \\ Z(t) & \frac{3}{2} < t \leq \bar{t}, \\ Z(\bar{t}) & t > \bar{t}, \end{cases}$$

solves (3) for the above p, and $Z(\bar{t}) > \frac{3}{4}$. Note that $\|p\|_\infty = \frac{1}{2}$, whereas $\bar{p} = 0$. In fact, every constant $t \mapsto \rho$ for $\rho \in [-1, 1]$ solves (3) on $[\frac{3}{2}, \infty)$.

3. Proof of Theorem 1.2

We proceed in several steps.

Step 1. *Let x be a solution of* (3)*, then x cannot have a negative local maximum or a positive local minimum on an interval which does not intersect dead zones.*

In fact, if $a \in (0, \infty)$ and $x(a)$ is a local minimum of x, then $x(a)^2 = x(a)^2 + \dot{x}(a)^2$. If $x(a)$ is positive, then $x(t)^2 \geq x(a)^2$ for $t \in [a, a + \delta)$ and some $\delta > 0$. But, $t \mapsto x(t)^2 + \dot{x}(t)^2$ is nonincreasing by Lemma 2.5(1); hence $x(t) = x(a)$ for $t \in [a, a + \delta)$, i.e. x has a dead zone.

Step 2. *Let x be a solution of* (3) *and $x_\infty := \lim_{t\to\infty} x(t)$, which exists thanks to Lemma 2.5(4). If $|x_\infty| < 1-\overline{p}$, then there exists an $\underline{t} \in [0,\infty)$ with $x(t) = x_\infty$ for $t \in [\underline{t},\infty)$.*

Proof. Select $\epsilon \in (0, 1-\overline{p})$ with $|x_\infty| < 1-\overline{p}-4\epsilon$ and $\tilde{t} \in (0,\infty)$ with $|x(t)-x_\infty| < \epsilon$ and $|\dot{x}(t)| < \epsilon$ for $t \in [\tilde{t},\infty)$.

1-st case: $\dot{x}(\tilde{t}) = 0$. Noting that $|p(t)-x(t)| \le \overline{p}+|x_\infty|+\epsilon \le \overline{p}+[1-\overline{p}-4\epsilon]+\epsilon = 1-3\epsilon$ for $t \in [\tilde{t},\infty)$, we obtain $x(t) = x_\infty$ for $t \in [\tilde{t},\infty)$.

2-nd case: $\dot{x}(\tilde{t}) > 0$. Let $\underline{t} := \sup\{t \in [\tilde{t},\infty) : \dot{x}(\tau) > 0 \text{ for } \tau \in [\tilde{t},t]\}$. Then $\ddot{x}(t) = p(t)-x(t)-1 \le \overline{p}+|x_\infty|+\epsilon-1 = \overline{p}+1-\overline{p}-4\epsilon+\epsilon-1 = -3\epsilon$ for $t \in [\tilde{t},\underline{t})$, hence $|\dot{x}(t)| < \epsilon$ for $t \in [\tilde{t},\infty)$ implies $\underline{t} < \infty$. Since $\dot{x}(\underline{t}) = 0$, we arrive at the first case with $\tilde{t} = \underline{t}$.

Likewise, the last case $\dot{x}(\tilde{t}) < 0$ can be derived.

Step 3. *Let x be a solution of* (3) *with $x_\infty := \lim_{t\to\infty} x(t) \in (1-\overline{p}, 1+\overline{p})$. Then x has a dead zone in every neighborhood of ∞.*

Proof. Otherwise, $\dot{x}$ possesses only isolated zeroes in a neighborhood of ∞. By step 1, x cannot have a positive minimum, thus x has to be monotone. Select $\epsilon > 0$ and $\underline{t} > 0$ such that the following holds:

- $1-\overline{p}+\epsilon < x(t) < 1+\overline{p}-\epsilon$ for $t \ge \underline{t}$;
- $\dot{x}$ has only isolated zeroes in $[\underline{t},\infty)$;
- $|\dot{x}| < \epsilon$ on $[\underline{t},\infty)$.

Assume that x is monotone increasing on $[\underline{t},\infty)$. Then $u(t)$ in (2) is equal to 1 almost everywhere on $[\underline{t},\infty)$. Thus, $\ddot{x}(t) = p(t)-1-x(t) \le \overline{p}-1-(1-\overline{p}+\epsilon) \le -\epsilon$ for a.e. $t \in [\underline{t},\infty)$ which is a contradiction.

Likewise, $\ddot{x}(t) = p(t)-1-x(t) \ge p(t)-1-(1+\overline{p}-\epsilon) \ge \epsilon$ for a.e. $t \in [\underline{t},\infty)$ shows that x cannot be decreasing near ∞.

Clearly, one obtains a corresponding assertion if $\lim_{t\to\infty} x(t) \in (-1-\overline{p}, \overline{p}-1)$.

We remark that by Lemma 2.5(1), $t \mapsto x(t)^2 + \dot{x}(t)^2$ is nonincreasing, hence x cannot increase under the assumptions of step 3., when leaving a dead zone. Thus, x is nonincreasing near ∞.

4. Some partial result for the case of multivalued systems

In what follows, $\mathbf{a}\cdot\mathbf{b}$ denotes the Euclidian scalar product of $\mathbf{a},\mathbf{b} \in \mathbf{R}^N$ and $\|.\|$ the Euclidean norm.

A complete extension of Theorem 1.2 to systems of the form (4) appears to be quite difficult. Here we will merely show that the solution

$$\mathbf{x}(t) := (x_1(t), x_2(t), \dots, x_N(t))^T$$

may develop a final dead zone $[\underline{t},\infty)$.

We require stronger assumptions than those of the one-dimensional case:

$$\mathbf{p}(t)^T \in [-\frac{\mu_\beta}{2k}+\epsilon, \frac{\mu_\beta}{2k}-\epsilon]^N \text{ for a.e. } t \ge T_f, \text{ for some } T_f \text{ and } \epsilon > 0, \tag{8}$$

$$\|\mathbf{p}(t)-\mathbf{p}_\infty\| \to 0 \quad \text{as} t \to +\infty. \tag{9}$$

Theorem 4.1. *We have:*

(i) *Let $(u_0, v_0) \in \mathbb{R}^{2N}$, $p \in L^2(0,\infty : \mathbb{R}^N)$, and (8) be satisfied. Then problem (4) admits a unique weak solution $x \in C^1([0,+\infty), \mathbb{R}^N)$. If, moreover, (9) holds, then there exists a unique equilibrium state $x_\infty \in \mathbb{R}^N$, i.e., x_∞*

satisfies $Ax_\infty - p_\infty \in ([-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]^N)^T$, *such that* $\|\dot{\mathbf{x}}(t)\| + \|x(t) - x_\infty\| \to 0$ as $t \to +\infty$.

(ii) *Assume (8) and (9) hold. Let* x_∞ *be the associate equilibrium state and assume that*

$$|\Delta_i^*| < 1 \text{ where } \Delta_i^* := (\mathbf{A}\mathbf{x}_\infty)_i - p_{i.\infty}, \quad \text{for some } i \in 1, \dots, N.$$

Then there exists $T_i \geq 0$ *such that* $\dot{x}_i(t) = 0$ *for all* $t \geq T_i$.

Proof. We shall adapt in our presentation some arguments of [12]. To reformulate (4) in the framework of nonlinear semi-group operators theory, we introduce the *phase space* $\mathbf{H} = (\mathbb{R}^N, \langle,\rangle_{\mathbf{A}}) \times (\mathbb{R}^N, \cdot)$, with $\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{A}} = \mathbf{A}\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{A}$ is the symmetric positive definite matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... \\ 0 & -1 & 2 & -1 & 0 \\ ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

We also introduce $\mathbf{B} : \mathbf{R}^N \to \mathcal{P}(\mathbb{R}^N)$ as the (multivalued) maximal monotone operator given by $\mathbf{B}(y_1, \dots, y_N) = (\beta(y_1), \dots, \beta(y_N))^T$ where $\beta(s) = Sgn(s)$. Finally, we define the operator $\mathbf{L}$ in $\mathbf{H}$ by

$$\mathbf{L}(\mathbf{x}, \mathbf{y}) = \{-\mathbf{y}\} \times \{\frac{k}{m}\mathbf{A}\mathbf{x} + \frac{\mu_\beta}{m}\mathbf{B}(\mathbf{y})\} \text{ for } (\mathbf{x}, \mathbf{y}) \in \mathbf{H}. \tag{10}$$

It is easy to prove that $\mathbf{L}$ is maximal monotone in H and so, by using results from the theory of maximal monotone operators (see [3]) we get the existence and uniqueness of a solution of (4). Multiplying the equation by $\dot{\mathbf{x}}(t)$ and integrating in time we get the *energy relation*

$$E(t) + \int_0^t [\sum_{i=1}^N \frac{\mu_\beta}{m}|\dot{x}_i(s)| - p_i(s)\dot{x}_i(s)]ds = E(0) \tag{11}$$

where

$$E(t) = \frac{1}{2}\|\dot{\mathbf{x}}(t)\|^2 + \int_0^t \frac{k}{2m}\mathbf{A}\mathbf{x}(s) \cdot \mathbf{x}(s)ds. \tag{12}$$

By (11) and the assumptions on $\mathbf{p}(t)$, the trajectory $(\mathbf{x}(t), \dot{\mathbf{x}}(t))_{t\geq 0}$ is compact in $\mathbf{H}$, so, we can find $\alpha > 0$ such that $\mu_\beta|\dot{x}_i(t)| - p_i(s)\dot{x}_i(t) \geq \alpha|\dot{x}_i(t)|$ for $i = 1, \dots, N$ and all $t \geq 0$. By (11), we conclude that $\dot{\mathbf{x}} \in L^1(\mathbb{R}_+)$ which leads to the existence of the limit $\mathbf{x}_\infty := \lim_{t\to+\infty} \mathbf{x}(t)$ and to $\lim_{t\to+\infty} \dot{\mathbf{x}}(t) = 0$. The uniqueness of $\mathbf{x}_\infty$ is deduced from the strict monotonicity of the operator $\widetilde{\mathbf{L}}(\mathbf{x}, \mathbf{y}) = \{-\mathbf{y}\} \times \{\frac{k}{m}\mathbf{A}\mathbf{x}\}$ for $(\mathbf{x}, \mathbf{y}) \in \mathbf{H}$.

To prove (ii) we recall that, since $\mathbf{x}_\infty$ is an stationary point, we have $(\Delta_i^*)_{i=1}^N \in [-1, 1]^N$. Now, let $0 < \delta << 1$ be fixed. By (i) we can find $t_0 \geq 0$ such that

$$|\Delta_i(t)| \leq (1 - 2\delta) \quad \forall t \geq t_0, \tag{13}$$

If $\dot{x}_i(t_0) = 0$, we conclude that $x_i(t) \equiv x_i(t_0) = x_{\infty i}$ for all $t \geq t_0$ since $\Delta_i(t) \in [-1, 1]$ for all $t \geq t_0$. If not, let $T = \sup\{s \geq t_0,\ |\dot{x}_i(t)| > 0\ \forall t \in [t_0, s)\}$. Multiplying the i-component of (4) by $\dot{x}_i(t)$ and using (9) we obtain

$$\frac{1}{2}\frac{d}{dt}(|\dot{x}_i(t)|^2) + \delta|\dot{x}_i(t)| \leq 0, \quad \text{for a.e. } t \in [t_0, T). \tag{14}$$

Dividing (14) by $|\dot{x}_i(t)|$ we get

$$\frac{d}{dt}(|\dot{x}_i(t)|) + \delta \leq 0 \quad \text{for a.e. } t \in [t_0, T). \tag{15}$$

Integrating, we see that $\dot{x}_i(t_0 + \frac{|\dot{x}_i(t_0)|}{\delta}) = 0$. Thus $T < +\infty$ and we conclude, as before, that $x_i(t) \equiv x_i(T) = x_{\infty i}$ for any $t \geq T$. $\square$

We remark that the behavior exhibited in the above result is different from the case in which the amplitude of $\mathbf{p}(t)$ becomes large. In that case the dynamics may generate a wide range of events leading to chaos (see [6]).

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