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SOME REMARKS ON INFINITE-DIMENSIONAL NONLINEAR ELLIPTIC PROBLEMS

PHILIPPE CLÉMENT, MARTA GARCÍA-HUIDOBRO, RAÚL F. MANÁSEVICH

Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. We discuss some nonlinear problems associated with an infinite dimensional operator L defined on a real separable Hilbert space H. As the operator L we choose the Ornstein-Uhlenbeck operator induced by a centered Gaussian measure μ with covariance operator Q.

1. Introduction

The goal of this note is to present some results for nonlinear problems associated with an infinite dimensional operator L defined on a real separable Hilbert space H. As the operator L we choose the Ornstein-Uhlenbeck operator induced by a centered Gaussian measure μ with covariance operator Q (see [8]).

In the first part we consider existence and uniqueness of solutions for a problem of the form

$$-Lu + \beta(u) = f, \tag{1.1}$$

where β satisfies

(H1) β is a strictly increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, $\beta(0) = 0$,

and $f \in L^2(H, \mu)$ is given. As a consequence of the existence part we can show that the operator $L(\beta^{-1})$, with an appropriate domain, has an m-dissipative closure in $L^1(H, \mu)$. Thus, in view of the Crandall-Liggett Theorem, see [7] (and also [6]), it generates a nonlinear contraction semigroup on the closure of its domain in $L^1(H, \mu)$.

In the second part we make the additional assumption that β is odd and we consider the nonlinear eigenvalue problem

$$-Lu + \beta(u) = \lambda u, \quad \lambda \geq 0, \tag{1.2}$$

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where $u \in L^2(H,\mu)$, $\|u\|_{L^2(H,\mu)} = R$, with $R > 0$ given.

By using results in [4] and [9], we obtain the existence of an infinite sequence $\{(\lambda_n, u_n)\}_{n\in\mathbb{N}}$ of solutions to (1.2) with $\lambda_n \to \infty$ as $n \to \infty$. This implies the existence of infinitely many solution pairs (λ, u) with non constant u. Moreover, we discuss the existence of solutions with nonnegative and non constant u.

2. Preliminaries

In this section we establish the notation that we will use throughout this work. Most of it is taken from [8] and we refer the reader to this book. H will denote a finite or infinite dimensional real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Throughout the paper $\mu = N_Q$ will denote the centered Gaussian measure on H with covariance Q, (see [8, page 12]), where Q denotes a positive symmetric operator of trace class in H with $Ker(Q) = \{0\}$. Also, $\{e_k\}_{k\in\mathbb{N}}$ will denote a complete orthonormal system of eigenvectors of Q with corresponding eigenvalues $\{\gamma_k\}_{k\in\mathbb{N}}$ satisfying

$$0 < \gamma_{k+1} \le \gamma_k. \tag{2.1}$$

We recall here that the Ornstein-Uhlenbeck semigroup "associated with μ" is given by

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad x \in H,\ t > 0,$$

and $\varphi \in B_b(H)$ (Borel bounded functions on H). Here $A = -\frac{1}{2}Q^{-1}$ and

$$Q_t x = \int_0^t e^{2sA}x\,ds = Q(I - e^{2tA})x, \quad x \in H,\ t > 0.$$

As a consequence of [8, Proposition 10.22], R_t can be uniquely extended to a strongly continuous contraction semigroup in $L^2(H,\mu)$, which we still denote by R_t, and μ is the unique invariant measure of R_t and for $x \in H$,

$$\lim_{t\to\infty} R_t\varphi(x) = \int_H \varphi(y)d\mu(y) = \overline{\varphi}.$$

Moreover, from [8, Th5.8], R_t can be uniquely extended to a strongly continuous positive contraction semigroup in $L^p(H,\mu)$ for all $1 \le p < \infty$.

We shall denote by L_p the infinitesimal generator of R_t in $L^p(H,\mu)$. In particular, L_1 is m-dissipative in $L^1(H,\mu)$ hence it satisfies

$$\int_H (L_1u)(x)\,\mathrm{sgn}(u(x))d\mu \le 0, \quad \text{for every } u \in D(L_1), \tag{2.2}$$

see e.g. [3, Lemma 2], where we have used the notation

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0, \\ 0 & t = 0, \\ -1 & t < 0. \end{cases}$$

Moreover, $-L_2$ is a nonnegative self adjoint operator in $L^2(H,\mu)$ with domain

$$D(L_2) = \{u \in W^{2,2}(H,\mu) : \int_H |(-A)^{1/2}Du|^2 d\mu < \infty\}, \tag{2.3}$$

see [8, Propositions 10.22 and 10.34] and

$$L_2\varphi(x) = \frac{1}{2}\mathrm{Tr}[D^2\varphi](x) + \langle x, AD\varphi(x)\rangle, \tag{2.4}$$

where $\varphi \in \mathcal{E}_A(H)$, which is defined to be the linear span in $C_b(H)$ (continuous bounded functions in H) of real and imaginary parts of φ_h, where $\varphi_h(x) = e^{i\langle h,x\rangle}$, $h \in D(A)$, and D, D^2 are the differential operators introduced in [8, Proposition 10.3 and 10.32]. We also introduce $\mathcal{E}(H)$ as the linear span in $C_b(H)$ of real and imaginary parts of φ_h, where $\varphi_h(x) = e^{i\langle h,x\rangle}$, and now $h \in H$. Finally we note also that the null space $N(L_p) = \{\text{const.}\}$, $1 \le p < \infty$.

We also consider the Dirichlet form $a : W^{1,2}(H,\mu)\times W^{1,2}(H,\mu) \to \mathbb{R}$ defined by

$$a(\varphi,\psi) = \frac{1}{2}\int_H \langle D\varphi, D\psi\rangle d\mu.$$

The linear space $W^{1,2}(H,\mu)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}(H,\mu)} = \langle\varphi,\psi\rangle_{L^2(H,\mu)} + 2a(\varphi,\psi)$$

is a real separable Hilbert space with

$$W^{1,2}(H,\mu) \hookrightarrow L^2(H,\mu) \quad \text{compact}, \tag{2.5}$$

see [8, Theorem 10.16]. Finally, we recall that

$$a(\varphi,\psi) = -\int_H \langle L_2\varphi, \psi\rangle d\mu \tag{2.6}$$

for all $\varphi \in D(L_2)$, and all $\psi \in W^{1,2}(H,\mu)$, see [8, Section 10.4].

Remark 2.1. We want to note that in this work the operator L_2 is defined as the generator of the semigroup R_t in $L^2(H,\mu)$, while in [8] the operator L_2 is defined on page 151 via the Lax-Milgram Theorem. In view of [8, Proposition 10.22 (iv)], they are the same.

3. An infinite dimensional porous media type operator

The aim of this section is to construct an infinite dimensional nonlinear second order elliptic operator which is of porous media type $\Delta(\beta^{-1})$, following the approach of [5]

Let β satisfy (H1) and consider problem (1.1).

Proposition 3.1. *(a) For every $f \in L^2(H,\mu)$ there exists a unique $u \in D(L_2)$ such that $\beta(u) \in L^2(H,\mu)$ and u satisfies* (1.1) *with $L = L_2$.*
(b) If

(H2) *$\beta(u) = \varepsilon u + \gamma(u)$ for some $\varepsilon > 0$ and some continuous monotone increasing function $\gamma : \mathbb{R} \to \mathbb{R}$, $\gamma(0) = 0$,*

then for any $f \in L^1(H,\mu)$ there exists a unique $u \in D(L_1)$ with $\beta(u) \in L^1(H,\mu)$ satisfying (1.1) *with $L = L_1$.*

Proof. We start by proving (a). Set $A := -L_2$, $Bu(x) := \beta(u(x))$, where

$$D(B) = \{u \in L^2(H,\mu) : \beta(u) \in L^2(H,\mu)\}$$

and write (1.1) as

$$Au + Bu = f, \quad f \in L^2(H,\mu).$$

We claim that A is maximal monotone and that it is the subdifferential of the convex l.s.c functional $J_a : L^2(H,\mu) \mapsto [0,\infty]$ defined by

$$J_a(\varphi) = \begin{cases} a(\varphi,\varphi), & \varphi \in W^{1,2}(H,\mu), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Indeed, for $u \in D(L_2)$ and $h \in W^{1,2}(H,\mu)$, by (2.6), we have that

$$\begin{aligned} J_a(u+h) &= J_a(u) + \int_H \langle Du, Dh\rangle d\mu + J_a(h) \\ &\geq J_a(u) - \int_H \langle L_2 u, h\rangle d\mu, \end{aligned}$$

which implies that $u \in D(\partial J_a)$ and $-L_2 u \in \partial J_a(u)$. Note that since J_a is convex, it follows that ∂J_a is monotone, moreover, since $-L_2$ is nonnegative and selfadjoint in $L^2(H,\mu)$, it follows that it is maximal monotone. Hence $-L_2 = \partial J_a$ by the maximal monotonicity of $-L_2$. Also, B is the subdifferential of

$$J_b(u) = \begin{cases} \int_H b(u)d\mu, & \text{if } \int_H b(u)d\mu < \infty \\ \infty & \text{otherwise,} \end{cases}$$

where

$$b(t) := \int_0^t \beta(s)ds. \tag{3.2}$$

Therefore A and B satisfy all the assumptions in [2, Example 1] implying that

$$\operatorname{Int}(R(A+B)) = \operatorname{Int}(R(A)+R(B)).$$

Since $R(B) = L^2(H,\mu)$, we conclude that $R(A+B) = L^2(H,\mu)$. Finally, the uniqueness assertion follows from the strict monotonicity of β.

Next we prove (b). In order to achieve this we write (1.1) as

$$(\varepsilon - L_1)u + \gamma(u) = f, \quad f \in L^1(H,\mu). \tag{3.3}$$

Hence in view of Theorem 1 in [3] it is sufficient to see that the operator $A := \varepsilon - L_1$ satisfies (I), (II), and (III) in [3]. Since by definition L_1 generates a linear contraction C_0 semigroup in $L^1(H,\mu)$, so does $L_1 - \varepsilon = -A$, which yields (I). Also, from the dissipativity of L_1 we have that

$$\varepsilon\|u\|_{L^1(H,\mu)} \leq \|\varepsilon u - L_1\|_{L^1(H,\mu)} = \|A\|_{L^1(H,\mu)},$$

implying that (III) is also satisfied. Finally we prove (II). Let $\lambda > 0$ and $f \in L^1(H,\mu)$. Since the semigroup generated by L_1 is positive, we have that

$$(I+\lambda A)^{-1} f \leq (I+\lambda A)^{-1} f^+$$

and hence

$$\operatorname{ess\,sup}(I+\lambda A)^{-1} f \leq \operatorname{ess\,sup}(I+\lambda A)^{-1} f^+. \tag{3.4}$$

Since L_p generates a linear contraction C_0 semigroup in $L^p(H,\mu)$ for all $1 \leq p < \infty$, so does $L_p - \varepsilon$, hence

$$\|(I+\lambda A)^{-1} f^+\|_{L^p(H,\mu)} \leq \|f^+\|_{L^p(H,\mu)}, \tag{3.5}$$

provided that $f^+ \in L^p(H,\mu)$. Assuming $f^+ \in L^\infty(H,\mu)$, by letting $p \to \infty$ in (3.5) we obtain

$$\begin{aligned}\operatorname{ess\,sup}(I+\lambda A)^{-1} f^+ &= \|(I+\lambda A)^{-1} f^+\|_{L^\infty(H,\mu)}\\ &\le \|f^+\|_{L^\infty(H,\mu)} = \operatorname{ess\,sup} f^+\\ &= \max\{0, \operatorname{ess\,sup} f\}.\end{aligned}$$

Therefore, using (3.4) we conclude that

$$\operatorname{ess\,sup}(I+\lambda A)^{-1} f \le \max\{0, \operatorname{ess\,sup} f\},$$

which is exactly assumption (II) in [3]. □

We are now in a position to define a "porous media type" operator, which we denote by L_ϕ, where $\phi = \beta^{-1}$ in $L^1(H,\mu)$:

$$D(L_\Phi) := \{u \in L^1(H,\mu) : \phi(u) \in D(L_1)\},$$

and for $u \in D(L_\Phi)$ we set

$$L_\phi u := L_1(\phi(u)).$$

We have the following result.

Theorem 3.2. (i) *The closure of L_ϕ is a nonlinear (possibly multivalued) m-dissipative operator in $L^1(H,\mu)$.*

(ii) *If β satisfies assumption (H2), then L_ϕ is itself a nonlinear m-dissipative operator in $L^1(H,\mu)$.*

(iii) *If $\phi \in C^2(\mathbb{R})$, then $\overline{D(L_\phi)} = L^1(H,\mu)$.*

Remark 3.3. We do not claim that the last two assertions in Theorem 3.2 are optimal.

Proof of Theorem 3.2. (i) We will first prove the dissipativity of L_ϕ in $L^1(H,\mu)$. Let $u,\ v \in D(L_\phi)$ and let $\bar u = \phi(u)$, $\bar v = \phi(v)$. By assumption, $\bar u$ and $\bar v$ belong to $D(L_1)$. In view of the dissipativity of L_1 in $L^1(H,\mu)$ we have

$$\int_H L_1(\bar u - \bar v)\operatorname{sgn}(\bar u - \bar v)d\mu \le 0, \tag{3.6}$$

and in view of (H1),

$$\operatorname{sgn}(u-v) = \operatorname{sgn}(\bar u - \bar v). \tag{3.7}$$

Hence, replacing (3.7) into (3.6), and using the definition of $\bar u$, $\bar v$ we get

$$\int_H (L_1(\phi(u) - \phi(v))\operatorname{sgn}(u-v)d\mu \le 0,$$

which implies the dissipativity of L_ϕ. We prove now that $R(I - L_\phi)$ is dense in $L^1(H,\mu)$. Let $f \in L^2(H,\mu)$. Then by Proposition 3.1 (a), there exists $v \in D(L_2)$, with $\beta(v) \in L^2(H,\mu)$, such that

$$-L_2 v + \beta(v) = f,$$

hence setting $u = \beta(v)$ we obtain $v = \phi(u)$ and

$$u - L_2\phi(u) = f,$$

hence $f \in R(I - L_\phi)$ (since $L_2 \subset L_1$). We conclude that $L^2(H,\mu) \subseteq R(I - \lambda L_\phi)$ and the claim follows from the density of $L^2(H,\mu)$ in $L^1(H,\mu)$.

It is a well known fact that if $\overline{L_\phi}$ denotes the closure of L_ϕ, then $\overline{L_\phi}$ is dissipative (possibly multivalued) and $R(I-\overline{L_\phi})$ is closed, hence equal to $L^1(H,\mu)$. Therefore $\overline{L_\phi}$ is m-dissipative in $L^1(H,\mu)$.
(ii) It follows from Proposition 3.1 that if β is of the form (H2) then the range

$$R(I-L_\phi)=L^1(H,\mu),$$

hence in this case L_ϕ is m-dissipative.
(iii) It is sufficient to show that $\mathcal{E}_A(H)\subseteq D(L_\phi)$, since $\mathcal{E}_A(H)$ is dense in $L^2(H,\mu)$. If $v\in\mathcal{E}_A(H)$, then there exists $N\geq 1$, $h_1,h_2,\dots,h_N,k_1,k_2,\dots,k_N\in D(A)$ $\alpha_1,\alpha_2,\dots,\alpha_N,\beta_1,\beta_2,\dots,\beta_N\in\mathbb{R}$ such that

$$v(x)=\sum_{i=1}^{N}\Big(\alpha_i\cos\langle h_i,x\rangle+\beta_i\sin\langle k_i,x\rangle\Big),\quad x\in H. \tag{3.8}$$

We will prove that $\phi(v)\in D(L_2)$. In view of (2.3), with first verify that $\phi(v)\in W^{2,2}(H,\mu)$. Since $v\in C_b(H)$, we have that $\phi(v),\phi'(v)$ and $\phi''(v)$ are in $C_b(H)$. In particular, $\phi(v)\in L^2(H,\mu)$.

From the definition of $W^{2,2}(H,\mu)$ in [8, Section 10.6, page 161], we need to compute $D_jD_\ell\phi(v)$, $j,\ell\in\mathbb{N}$. Since D_jv and $D_\ell v$ are bounded and continuous, from

$$D_\ell\phi(v)=\phi'(v)D_\ell v,$$

and

$$D_jD_\ell\phi(v)(x)=\phi'(v)D_jD_\ell v(x)+\phi''(v)D_jv(x)D_\ell v(x), \tag{3.9}$$

we obtain that $D_jD_\ell\phi(v)\in C_b(H)\subseteq L^2(H,\mu)$.

Next we show that

$$\sum_{j,\ell=1}^{\infty}\int_H|D_jD_\ell\phi(v)|^2d\mu<\infty. \tag{3.10}$$

From (3.9) it is sufficient to show that

$$\sum_{j,\ell=1}^{\infty}\int_H|D_jD_\ell v|^2d\mu<\infty\quad\text{and}\quad\sum_{j,\ell=1}^{\infty}\int_H|D_jv(x)D_\ell v(x)|^2d\mu<\infty. \tag{3.11}$$

Indeed, the first assertion in (3.11) follows from the fact that $v\in\mathcal{E}_A(H)\subseteq\mathcal{E}(H)\subseteq W^{2,2}(H,\mu)$. For the second one we note that

$$|D_jv(x)|\leq C\sum_{i=1}^{N}(|\langle h_i,e_j\rangle|+|\langle k_i,e_j\rangle|) \tag{3.12}$$

where C is a positive constant depending only on $\alpha_1,\dots,\alpha_N,\beta_1,\dots,\beta_N$, hence

$$\sum_{j,\ell=1}^{\infty}|D_jv(x)D_\ell v(x)|^2\leq 4N^2C^4\Big(\sum_{i=1}^{N}|h_i|^2+|k_i|^2\Big)^2, \tag{3.13}$$

implying that the second assertion in (3.11) holds and therefore $\phi(v)\in W^{2,2}(H,\mu)$. Finally we will prove that

$$\int_H|(-A)^{1/2}D\phi(v)|^2d\mu<\infty. \tag{3.14}$$

First we prove that $D\phi(v)(x) \in D((-A)^{1/2})$. Since $A = -\frac{1}{2}Q^{-1}$, $w \in H$ belongs to $D((-A)^{1/2})$ if and only if

$$\sum_{j=1}^{\infty} \gamma_j^{-1} \langle w, e_j\rangle^2 < \infty. \tag{3.15}$$

Now, $D\phi(v)(x) = \phi'(v)D_j v(x)$ and $|\phi'(v)| \le C_0$ for some positive constant C_0, hence from (3.12) we find that

$$|D_j\phi(v)(x)|^2 \le C_0^2 |D_j v(x)|^2 \le 2NC_0^2C^2 \sum_{i=1}^{N} (|\langle h_i, e_j\rangle|^2 + |\langle k_i, e_j\rangle|^2)$$

where $h_i,\ k_i \in D(A)$, $i = 1, \dots, N$, that is,

$$\sum_{j=1}^{\infty} \gamma_j^{-2} |\langle h_i, e_j\rangle|^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma_j^{-2} |\langle k_i, e_j\rangle|^2 < \infty. \tag{3.16}$$

Hence from (3.16),

$$\sum_{j=1}^{\infty} \gamma_j^{-1} |D_j\phi(v)(x)|^2 \le 2NC_0^2C^2 \sum_{i=1}^{N}\sum_{j=1}^{\infty} \gamma_j^{-1} (|\langle h_i, e_j\rangle|^2 + |\langle k_i, e_j\rangle|^2) < \infty,$$

since by (2.1) $\gamma_j^{-1} \le \gamma_j^{-2}$ for large j. This implies that $D\phi(v)(x) \in D((-A)^{1/2})$ for any $x \in H$ and

$$\begin{aligned}
|(-A)^{1/2} D\phi(v)|^2(x) &= \sum_{j=1}^{\infty} \langle D\phi(v)(x), (-A)^{1/2} e_j\rangle^2 \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \langle D\phi(v)(x), \gamma_j^{-1/2} e_j\rangle^2 \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \gamma_j^{-1} \langle D\phi(v)(x), e_j\rangle^2,
\end{aligned}$$

implying that the integrand in (3.14) is Borel measurable and bounded and thus (3.14) holds. This completes the proof of part (3). □

We end this section by giving some properties of the nonlinear semigroup generated by $\overline{L_\phi}$.

Proposition 3.4. *Let β satisfy (H1) and $S_t : \overline{D(\overline{L_\phi})} \to \overline{D(\overline{L_\phi})}$ be the nonlinear semigroup generated by $\overline{L_\phi}$. Then the following hold.*

- (i) *For any $c \in \mathbb{R}$, $c \in D(L_\phi)$, and $S_t(c) = c$.*
- (ii) *Let $f,\ g \in \overline{D(\overline{L_\phi})}$ such that $f \le g$. Then $S_t(f) \le S_t(g)$ for all $t > 0$.*
- (iii) *For any $f \in \overline{D(\overline{L_\phi})}$,*

$$\int_H S_t f d\mu = \int_H f d\mu \quad \textit{for all } t > 0.$$

Proof. From Proposition 3.1, for any $h > 0$ there is a unique $u \in L^2(H, \mu)$ such that

$$-L_2 u + \frac{1}{h}\beta(u) = \frac{1}{h} f, \tag{3.17}$$

hence

$$(I - h\overline{L_\phi}))^{-1} f = \beta(u). \tag{3.18}$$

Proof of (i). If $f = c$, we have $\beta(u) = c$ and thus by induction it follows that

$$(I - h\overline{L_\phi})^{-m} c = c \quad \text{for all } m \in \mathbb{N}, \tag{3.19}$$

therefore, for any $t > 0$ we have

$$S_t(c) = \lim_{m\to\infty} (I - \frac{t}{m}\overline{L_\phi})^{-m} c = c$$

Proof of (ii). Let now $f_1,\ f_2 \in L^2(H,\mu)$, with $f_1 \le f_2$, and for $h > 0$ and $\varepsilon > 0$, and $i = 1, 2$, let u_i^ε satisfy

$$\varepsilon u_i^\varepsilon - L_2 u_i^\varepsilon + \frac{1}{h}\beta(u_i^\varepsilon) = \frac{1}{h} f_1.$$

From [1, Proposition 4.7 (iv) implies (i)] with

$$\varphi(u) = \begin{cases} 0 & u \ge 0 \\ +\infty & \text{otherwise,} \end{cases}$$

we obtain $u_1^\varepsilon \le u_2^\varepsilon$. By letting $\varepsilon \to 0$ we obtain $u_1 \le u_2$ where u_i satisfy

$$-L_2 u_i + \frac{1}{h}\beta(u_i) = f_i, \quad i = 1, 2.$$

Hence, $\beta(u_1) \le \beta(u_2)$ and thus

$$(I - h\overline{L_\phi}))^{-1} f_1 \le (I - h\overline{L_\phi}))^{-1} f_2.$$

Therefore, by induction,

$$(I - h\overline{L_\phi}))^{-m} f_1 \le (I - h\overline{L_\phi}))^{-m} f_2. \tag{3.20}$$

Since $L^2(H,\mu)$ is dense in $L^1(H,\mu)$, (3.20) holds also for $f_1,\ f_2 \in L^1(H,\mu)$. By taking $f_1,\ f_2 \in \overline{D(\overline{L_\phi})}$, we obtain as before that $S_t(f_1) \le S_t(f_2)$.

Proof of (iii). Arguing as before, it is sufficient to prove that

$$\int_H (I - h\overline{L_\phi})^{-1} f d\mu = \int_H f d\mu$$

for all $h > 0$ and $f \in L^2(H,\mu)$. This follows by integrating (3.17) over H to obtain

$$\int_H \beta(u) d\mu = \int_H f d\mu,$$

hence our claim follows by integrating now (3.18) over H. □

4. A nonlinear eigenvalue problem associated with the Ornstein-Uhlenbeck operator

In this section we consider the nonlinear eigenvalue problem

$$-L_2 u + \beta(u) = \lambda u, \tag{4.1}$$

where β satisfies (H1) and is odd. By a solution to this equation we mean a pair $(\lambda, u) \in \mathbb{R} \times L^2(H,\mu)$ satisfying $u \in W^{2,2}(H,\mu)$, $\beta(u) \in L^2(H,\mu)$. Clearly, for any $\lambda \in \mathbb{R}$, $(\lambda, 0)$ is a solution to (4.1). Set

$$\lambda^* := \sup\{\lambda \in \mathbb{R} : s \mapsto \beta(s) - \lambda s \text{ is strictly increasing in } \mathbb{R}\}$$

We have that $0 \le \lambda^* < \infty$. If $\lambda < \lambda^*$, then $s \mapsto \beta(s) - \lambda s$ is strictly increasing and hence, from Proposition 3.1 (a) we have that $(\lambda, 0)$ is the only solution to (4.1). For $\lambda \in \mathbb{R}$ let us consider the functional $J_\lambda : L^2(H,\mu) \to [-\infty, \infty]$ defined by

$$J_\lambda(u) = \begin{cases} J_a(u) + J_b(u) - \frac{\lambda}{2}\|u\|^2_{L^2(H,\mu)}, & u \in W^{1,2}(H,\mu),\ \int_H b(u)d\mu < \infty \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

We observe that for $\lambda < \lambda^*$, J_λ is strictly convex, l.s.c. and nonnegative, and 0 is its global minimizer.

Next we investigate the positive constant solutions to (4.1) $u(x) \equiv c$. Then $\beta(c) = \lambda c$. We have the following result.

Proposition 4.1. *Assume that*

$$t \mapsto \beta(t)/t \textit{ is strictly increasing on } (0,\infty). \tag{4.3}$$

Then for all $c > 0$ the pair $(\beta(c)/c, c)$ is a solution to (4.1) *and $u = c$ minimizes the functional J_0 on the set*

$$S_c := \{u \in W^{1,2}(H,\mu) : \|u\|_{L^2(H,\mu)} = c\}.$$

Furthermore, $u = c$ is the unique nonnegative minimizer of J_0 on S_c.

Proof. From (4.3) we obtain that the mapping $t \mapsto b(\sqrt{t})$, $t > 0$, is strictly convex, hence for any $u \in D(J_0)$ we have by Jensen's inequality ([10, Theorem 2.2(a)]) that

$$J_0(u) \ge \int_H b(\sqrt{|u|^2})d\mu \ge b\Big(\sqrt{\int_H |u|^2 d\mu}\Big) = b(c) = J_0(c), \tag{4.4}$$

implying that $u = c$ is a minimizer for J_0. On the other hand, if u is a minimizer, then from (4.4) and the fact that $J_0(c) \ge J_0(u)$, we obtain that

$$\int_H b(\sqrt{|u|^2})d\mu = b\Big(\sqrt{\int_H |u|^2 d\mu}\Big),$$

hence by [10, Theorem 2.2(b)] we deduce that u^2 must be a constant, hence $u = c$ since u is nonnegative. ☐

We will now state and prove our existence results.

Theorem 4.2.
(i) *For any $R > 0$ there exists a solution (λ, u) to* (4.1) *satisfying $u \ge 0$ and u minimizes J_0 on S_R.*
(ii) *For any $R > 0$ there exists a sequence of solutions $\{(\lambda_n, u_n)\}_{n\in\mathbb{N}}$ to* (4.1) *such that $u_n \in S_R$ and*

$$\lambda_n > 0 \quad \textit{for } n \in \mathbb{N}, \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \lambda_n = \infty. \tag{4.5}$$

Proof. (ii) We will apply Theorem 1 in [4], see also [9]. As the real infinite dimensional separable Hilbert space we choose $E = L^2(H,\mu)$. Let $\varphi : E \to [0,\infty]$ be defined by $\varphi(u) := J_{-1}$, $u \in E$. Then clearly φ is convex, even, and $\varphi(0) = 0$. Moreover, in view of the compactness of the imbedding (2.5), the convex set

$$\{u \in E : \varphi(u) \le \rho\}$$

is compact in E for all $\rho \ge 0$. Moreover, since $\mathcal{E}(H) \subseteq C_b(H) \cap W^{1,2}(H,\mu)$ we have that $\mathcal{E}(H) \subseteq D(\varphi)$. The density of $\mathcal{E}(H)$ in E implies the density of $D(\varphi)$ in E.

Hence, all the assumptions of [4, Theorem 1] are satisfied and therefore there exists a sequence $(\nu_k, u_k) \in \mathbb{R} \times E$, $k \in \mathbb{N}$ such that $\|u_k\|_E = R$, $\partial J_{-1}(u_k) \ni \nu_k u_k$ and $\sup_{k\geq 1} \varphi(u_k) = \infty$. We claim that

$$D(\partial J_{-1}) = D(L_2) \cap D(B),$$

and

$$\partial J_{-1}(u) = -L_2 u + Bu + u, \quad u \in D(\partial J_{-1}).$$

Indeed,

$$J_{-1} = J_a + J_{\tilde{b}},$$

where $\tilde{b}(t) = b(t) + \frac{1}{2}t^2$ and we have

$$\partial J_a = -L_2, \quad \text{and} \quad \partial J_{\tilde{b}} = B + I.$$

In view of Proposition 3.1 (a), we have

$$R(-L_2 + B + 2I) = E,$$

which implies that $-L_2 + (B + I)$ is maximal monotone. From [1, page 41] we have

$$\partial J_{-1} = \partial J_a + \partial J_{\tilde{b}},$$

which proves the claim. Therefore

$$-L_2 u_k + \beta(u_k) = (\nu_k - 1)u_k, \quad k \in \mathbb{N}.$$

Set $\lambda_k = \nu_k - 1$, $k \in \mathbb{N}$. By taking inner product with u_k and taking into account that $\|u_k\|_E = R > 0$ we have that $\lambda_k > 0$. Finally, since

$$\varphi(u_k) \leq \langle \partial\varphi(u_k), u_k \rangle = \nu_k R^2,$$

we have $\sup_{k\in\mathbb{N}} \lambda_k = \infty$. and thus (4.5) holds.

(i) In this part we shall use that $u \in W^{1,2}(H,\mu)$ implies that $|u| \in W^{1,2}(H,\mu)$, $J_a(|u|) = J_a(u)$, and moreover, since β is odd, we also have $J_b(|u|) = J_b(u)$. We will apply Theorem 3 in [4]. To this end we set

$$P := \{v \in L^2(H,\mu) : \ v \geq 0\}, \qquad I_P(u) = \begin{cases} 0 & u \in P \\ +\infty & \text{otherwise,} \end{cases}$$

and define $\varphi_+ : E \to [0, \infty]$ by $\varphi_+(u) = \varphi(u) + I_P(u)$, $u \in E$. We have that φ_+ is convex, l.s.c., the set $\{u \in E : \varphi_+(u) \leq \rho\}$ is compact for every $\rho \geq 0$, and $\varphi_+(0) = 0$. We claim that $\overline{D(\varphi_+)} = P$. Indeed, let $u \in P$. By the density of $D(\varphi)$ in E, there exists $\{u_n\} \subseteq D(\varphi)$ such that $u_n \to u$ in E. Hence, $|u_n| \in D(\varphi_+)$ and $|u_n| \to |u| = u$ in E.

Let $R > 0$. From [4, Theorem 3] there exists $(\nu, u) \in \mathbb{R}^+ \times P$, with $\|u\|_E = R$, $\nu u \in D(\partial\varphi_+)$, $\nu u \in \partial\varphi_+(u)$ and

$$\varphi_+(u) = \inf_{v \in S_R} \varphi_+(v).$$

It follows that

$$\varphi_+(v) \geq \varphi_+(u) + \langle \nu u, v - u \rangle \quad \text{for all } v \in D(\varphi).$$

Since $u \in P$, we have $\varphi_+(u) = \varphi(u)$, hence

$$\varphi_+(v) \geq \varphi(u) + \langle \nu u, v - u \rangle \quad \text{for all } v \in D(\varphi).$$

Moreover, for all $v \in P \cap D(\varphi)$ we have

$$\varphi(v) \geq \varphi(u) + \langle \nu u, v - u \rangle,$$

hence for all $v \in D(\varphi)$ we have

$$\varphi(|v|) \geq \varphi(u) + \langle \nu u, |v| - u \rangle.$$

Since $\varphi(|v|) = \varphi(v)$, we obtain

$$\varphi(v) \geq \varphi(u) + \langle \nu u, v - u \rangle + \langle \nu u, |v| - v \rangle \geq \varphi(u) + \langle \nu u, v - u \rangle,$$

hence $\nu u \in D(\partial\varphi(u)$ and $\nu u = -L_2 + Bu + u$. Setting now $\lambda = \nu - 1$ we get

$$-L_2 u + Bu = \lambda u.$$

Finally, we have

$$\begin{aligned} J_0(u) &= \varphi(u) - \frac{1}{2}R^2 = \varphi_+(u) - \frac{1}{2}R^2 \\ &= \inf_{v \in S_R} \varphi_+(v) - \frac{1}{2}R^2 = \inf_{v \in S_R} \varphi(|v|) - \frac{1}{2}R^2 \\ &= \inf_{v \in S_R} \varphi(v) - \frac{1}{2}R^2 \\ &= \inf_{v \in S_R} J_0(v). \end{aligned}$$

□

We complete this note by exhibiting a class of functions β for which the minimum of J_0 on S_R is not attained at the constants for R small.

Proposition 4.3. *Assume that β satisfies the extra conditions*

$$\lim_{s \to 0} \frac{b(s)}{s^2} = \infty, \quad \textit{and} \quad \lim_{s \to \infty} \frac{b(s)}{s^2} = 0; \tag{4.6}$$

$$\textit{there exists } C > 0 \textit{ such that } b(st) \leq C b(s) b(t) \textit{ for all } s, t > 0. \tag{4.7}$$

Then, there exists $R_0 > 0$ such that for any $R \in (0, R_0)$ J_0 does not achieve its minimum on S_R at the constants.

Proof. For $n \in \mathbb{N}$, we set

$$\tilde{u}_n(t) = \begin{cases} -n\alpha_n(|t| - \frac{1}{n}) & |t| \leq \frac{1}{n} \\ 0 & |t| > \frac{1}{n} \end{cases}$$

where α_n will be chosen later. We define $u_n : H \to \mathbb{R}$ by $u_n(x) := \tilde{u}_n(\langle x, e_1 \rangle)$ and we choose α_n so that $\|u_n\|_{L^2(H,\mu)} = R$. We observe also that $u_n \in W^{1,2}(H, \mu)$. One verifies that

$$C_1 R\sqrt{n} \leq \alpha_n \leq C_2 R\sqrt{n}, \tag{4.8}$$

for some positive constants C_1, C_2. We will show now that if n is chosen large enough and $R > 0$ is small enough, then

$$J_0(u_n) < J_0(R) = b(R).$$

Indeed, it follows from the definition of μ that

$$\frac{1}{2}\int_H |Du_n|^2 d\mu \leq K_0 \int_0^{1/n} |\tilde{u}_n'|^2 dt, \quad \int_H b(u_n) d\mu \leq K_0 \int_0^{1/n} b(\tilde{u}_n) dt \tag{4.9}$$

for some positive constant K_0. Now, from (4.8) we have

$$\int_0^{1/n} |\tilde{u}_n'|^2 dt = n\alpha_n^2 \leq C_2^2 n^2 R^2, \tag{4.10}$$

and from (4.8) and (4.7) we get

$$\int_0^{1/n} b(\tilde{u}_n)dt = \frac{1}{n\alpha_n}\int_0^{\alpha_n} b(s)ds \leq \frac{b(\alpha_n)}{n} \leq \frac{b(C_2R\sqrt{n})}{n} \leq Cb(R)\frac{b(C_2\sqrt{n})}{n}.$$

Using now the second condition in (4.6) to find $n_0 \in \mathbb{N}$ so that

$$CK_0\frac{b(C_2\sqrt{n_0})}{n_0} < \frac{1}{4},$$

from the second inequality in (4.9) we obtain

$$\int_H b(u_n)d\mu \leq \int_H b(u_{n_0})d\mu < \frac{1}{4}b(R). \tag{4.11}$$

Finally, in view of the first assumption in (4.6) we can find $R_0 > 0$ such that for any $R \in (0, R_0)$

$$K_0C_2^2n_0^2\frac{R^2}{b(R)} \leq \frac{1}{4},$$

therefore from the first inequality in (4.9) and (4.10), we have

$$\frac{1}{2}\int_H |Du_n|^2 d\mu \leq K_0C_2^2n_0^2\frac{R^2}{b(R)}b(R) \leq \frac{1}{4}b(R). \tag{4.12}$$

Hence, from (4.11) and (4.12) we conclude that for any $R \in (0, R_0)$,

$$\inf_{v\in S_R} J_0(v) \leq J_0(u_{n_0}) \leq \frac{1}{2}b(R) = \frac{1}{2}J_0(R).$$

This completes the proof of the proposition. □

Remark 4.4. We note that $\beta(s) = |s|^{p-1}s$, $0 < p < 1$, satisfies all the assumptions of Proposition 4.3.

As a last comment, we mention that as a consequence of Theorem 4.2 and Proposition 4.3 we have shown the existence of a nonnegative nonconstant solution to (4.1). It is worth observing that a function u of the form

$$u(x) = \tilde{u}(\langle x, e_1\rangle, \langle x, e_2\rangle, \dots, \langle x, e_N\rangle),$$

where $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ is a solution to (4.1) with $H = \mathbb{R}^N$ with the usual inner product and

$$L_2 = \frac{1}{2}\Delta + \langle b(x), \nabla\rangle,$$

where $b_i(x) = -\frac{x_i}{2\gamma_i}$, $1 \leq i \leq N$, is also a solution to the infinite dimensional problem. It is an open problem to know whether (4.1) possesses solutions depending on infinitely many variables.

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Philippe Clément
Mathematical Institute, Leiden University, P.O. Box 9512, NL-2300, RA Leiden, and EEMCS/ DIAM, TU Delft, P.O. Box 5031, NL-2600, GA Delft, The Netherlands
E-mail address: `clement@math.leidenuniv.nl`

Marta García-Huidobro
Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile
E-mail address: `mgarcia@mat.puc.cl`

Raúl F. Manásevich
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile
E-mail address: `manasevi@dim.uchile.cl`