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EXISTENCE OF SOLUTIONS FOR ELLIPTIC SYSTEMS INVOLVING OPERATORS IN DIVERGENCE FORM

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. In this paper, we obtain some results about the existence of solutions to the system

$$-\sum_{k,j=1}^{N}\frac{\partial}{\partial x_k}\big(\rho_{kj,i}\frac{\partial u_i}{\partial x_j}\big)+q_iu_i=\mu_im_iu_i+g_i(x,u_1,\dots,u_n),$$

for $i=1,\dots,n$ defined in $\mathbb{R}^N$.

[section]

1. Introduction

1.1. The problem settings. We study the elliptic system

$$-\sum_{k,j=1}^{N}\frac{\partial}{\partial x_k}\big(\rho_{kj,i}\frac{\partial u_i}{\partial x_j}\big)+q_iu_i=\mu_im_iu_i+g_i(x,u_1,\dots,u_n)\quad\text{in }\mathbb{R}^N,\tag{1.1}$$

for $i=1,\dots,n$. We consider the following hypotheses for each $i=1,\dots,n$:

- (H1) $q_i\in L^2_{\rm loc}(\mathbb{R}^N)\cap L^{\frac{p}{2}}_{\rm loc}(\mathbb{R}^N)$, $p>N$, such that $\lim_{|x|\to+\infty}q_i(x)=+\infty$ and $q_i\geq \text{const}>0$.
- (H2) For all $j,k=1,\dots,N$, $\rho_{kj,i}=\rho_{jk,i}$ and there exists positive constants α_i,β_i such that for all $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$,
$$\alpha_i|\xi|^2\leq\sum_{k,j=1}^{N}\rho_{kj,i}\xi_j\xi_k\leq\beta_i|\xi|^2.$$
- (H3) $m_i\in L^\infty(\mathbb{R}^N)$, $m_i\geq\text{const}>0$.

We will specify later the form and the hypotheses on each g_i and we denote by μ_i real parameters for $i=1,\dots,n$. The variational space is denoted by $V_{q_1}(\mathbb{R}^N)\times$

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$\dots \times V_{q_n}(\mathbb{R}^N)$ where for $i = 1, \dots, n$, $V_{q_i}(\mathbb{R}^N)$ is the completion of $D(\mathbb{R}^N)$, the set of C^∞ functions with compact support, under the norm

$$\|u\|_{\rho_i,q_i} = \Big(\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + q_i u^2] \Big)^{1/2}. \tag{1.2}$$

Due to hypothesis (H2), $V_{q_i}(\mathbb{R}^N)$ is also the completion of $D(\mathbb{R}^N)$ under the norm

$$\|u\|_{q_i} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + q_i u^2 \Big)^{1/2}. \tag{1.3}$$

We recall that the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact. We denote by

$$\|u\|_{m_i} = \Big(\int_{\mathbb{R}^N} m_i u^2 \Big)^{1/2} \tag{1.4}$$

for all $u \in L^2(\mathbb{R}^N)$. According to the hypothesis (H3), $\|\cdot\|_{m_i}$ is a norm in $L^2(\mathbb{R}^N)$, equivalent to the usual norm so the embedding of $V_{q_i}(\mathbb{R}^N)$ into $(L^2(\mathbb{R}^N), \|\cdot\|_{m_i})$ is still compact. We denote by M_i the operator of multiplication by m_i in $L^2(\mathbb{R}^N)$ and by L_{ρ_i} the operator defined by

$$L_{\rho_i} u := - \sum_{k,j=1}^N \frac{\partial}{\partial x_k} (\rho_{kj,i} \frac{\partial u}{\partial x_j}). \tag{1.5}$$

The operator $(L_{\rho_i} + q_i)^{-1} M_i : (L^2(\mathbb{R}^N), \|\cdot\|_{m_i}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{m_i})$ is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence tending to 0. We denote by λ_i the inverse of the first eigenvalue and by ϕ_i the corresponding eigenfunction which satisfy

$$(L_{\rho_i} + q_i)\phi_i = \lambda_i m_i \phi_i \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

$\lambda_i > 0$ and $\|\phi_i\|_{m_i} = 1$. (We recall that λ_i is simple and $\phi_i > 0$ (see for example [1, 2, 4, 9, 21, 24] and Proposition 1.1 below.) By the Courant-Fischer formulas,

$$\lambda_i = \inf \Big\{ \frac{\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} + q_i \phi^2]}{\int_{\mathbb{R}^N} m_i \phi^2}, \ \phi \in D(\mathbb{R}^N) \Big\}. \tag{1.7}$$

The aim of this paper is to study the existence of solutions for the system (1.1). This extends earlier results obtained for the Laplacian operator in a bounded domain (see [17, 18]), for an operator of divergence form in a bounded domain (when each $\rho_{kj,i}$ is independent of j, see [14]), for equations or systems involving Schrödinger operators $-\Delta + q_i$ in $\mathbb{R}^N$ (see [3, 11, 12, 13, 15]). The methods to get the existence of solutions are the Lax-Milgram Theorem for linear systems, applications of the sub and super solutions method, or the bifurcation method. Note that an important tool to obtain positive solutions is the Maximum Principle. We present in this paper a classification of different results for the existence of solutions for the system (1.1). Since these results are refinements of results obtained for Schrödinger operators whose potentials tend to infinity at infinity, we will only express the results for operators of divergence form studied here with some simple sketches of the proofs.

1.2. **Review of results for the scalar case ($i = 1$).** We consider here the following equation, in a variational sense,

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(c_{k,j}\frac{\partial u}{\partial x_j}) + qu = \lambda mu + g \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

We assume the following: The potential q satisfies (H1), the coefficients $c_{k,j}$ satisfy (H2), the weight m satisfies (H3), the constant λ is a real parameter and finally $g \in L^2(\mathbb{R}^N)$. We let L_c be the operator defined by $L_c u := -\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(c_{k,j}\frac{\partial u}{\partial x_j})$ and M the multiplication operator given by the function m. Since the operator $(L_c + q)^{-1}M : (L^2(\mathbb{R}^N), \|\cdot\|_m) \to (L^2(\mathbb{R}^N), \|\cdot\|_m)$ is positive self-adjoint and compact, then its spectrum is discrete and consists of a positive sequence tending to 0. We denote by $\lambda_{c,q}$ the inverse of the first (largest) eigenvalue of the operator $(L_c + q)^{-1}M$.

As in [1, 2, 4, 9, 20], we have the following result.

Proposition 1.1. *The eigenvalue $\lambda_{c,q}$ is simple and there exists an associated eigenfunction $\phi_{c,q}$ which is a strictly positive and continuous function in $\mathbb{R}^N$.*

We recall that the above result is well known for the case of bounded domains or the case where $L_c + q = -\Delta + q$ is a Schrödinger operator in $\mathbb{R}^{\mathbb{N}}$.

Proof. First, we conclude from [25, Theorem 7.1] that every solution $\psi \in L^2(\mathbb{R}^N)$ of the equation $(L_c + q - \lambda_{c,q}m)\psi = 0$ in $L^2(\mathbb{R}^N)$ is a continuous function in $\mathbb{R}^N$.

Next, we will show that $\lambda_{c,q}$ is simple and it has an eigenfunction which is strictly positive. To do this, we follow [1]. Note that $\lambda_{c,q}$ is of finite multiplicity. If in contrast $\lambda_{c,q}$ is not simple, then there exists an eigenfunction, say, ψ, which changes sign. It follows that $\psi^+ := \max(\psi, 0)$ is also an eigenfunction associated with $\lambda_{c,q}$. Let $\Omega_+ := \{x \in \mathbb{R}^N,\ \psi(x) > 0\}$; we denote by λ_+ the inverse of the first eigenvalue of $(L_c + q)^{-1}M$ defined on Ω_+ with the Dirichlet boundary condition. Then $\lambda_{c,q} = \lambda_+$. Since $\Omega_+ \neq \mathbb{R}^N$, ψ and ψ^+ are linearly independent. Hence, by considering all domains Ω_* such that $\Omega_+ \subset \Omega_* \subset \mathbb{R}^N$, we can construct an infinite number of linearly independent eigenfunctions associated with $\lambda_{c,q}$, contradicting the fact that $\lambda_{c,q}$ is of finite multiplicity. Therefore, we have shown that $\lambda_{c,q}$ is simple and every eigenfunction of it does not change sign.

To complete the proof, we consider a non-negative eigenfunction ϕ of $\lambda_{c,q}$ and need to showing that $\phi > 0$. Assuming that there exists $y \in \mathbb{R}^N$ such that $\phi(y) = 0$. Let $R > 0$ and $r > 0$ be such that $B(y, r) \subset B(0, R)$. Using the Harnack Inequality for the operator $L_c + q - \lambda_{c,q}m$ (see [20, Theorem 8.20], [25, Theorem 8.1]), we have $\sup_{B(y,4r)} \phi \leq C \inf_{B(y,r)} \phi = 0$, and we deduce that $\phi = 0$ in $B(y, r)$ (for any $r > 0$). Therefore $\phi \equiv 0$, which is impossible, since ϕ is an eigenfunction. □

We have the following weak Maximum Principle for (1.8).

Theorem 1.1 (The weak Maximum Principle)**.** *Assume that $\lambda < \lambda_{c,q}$, $g \geq 0$ and u is a solution of the equation* (1.8)*. Then $u \geq 0$.*

Proof. The idea is to multiply the equation (1.8) by $u^- = Max(0, -u)$, then to use the characterisation of $\lambda_{c,q}$ (see (1.7)) in order to obtain $(\lambda_{c,q} - \lambda)\int_{\mathbb{R}^N} m(u^-)^2 \leq 0$. Therefore, under the conditions $\lambda_{c,q} - \lambda > 0$ and $m > 0$ we obtain that $u^- = 0$. □

By using the Lax-Milgram Theorem we obtain the existence of a solution for the equation (1.8).

Theorem 1.2. *Assume $\lambda < \lambda_{c,q}$. Then there exists a unique solution $u \in V_q(\mathbb{R}^N)$ for the equation* (1.8)*. Moreover, by the Maximum Principle, if $g \geq 0$, then this solution u satisfies $u \geq 0$.*

Proof. Let $l : (V_q(\mathbb{R}^N))^2 \to \mathbb{R}$ be the quadratic form given by

$$l(u,v) := \int_{\mathbb{R}^N} [\sum_{k,j=1}^{N} c_{k,j} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + quv - \lambda muv] \quad \forall (u,v) \in (V_q(\mathbb{R}^N))^2.$$

Let α be a real number such that $\lambda + \alpha > 0$. Consider the norm

$$\|u\|_{c,q+\alpha m} := \Big(\int_{\mathbb{R}^N} [\sum_{k,j=1}^{N} c_{kj} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + (q+\alpha m)u^2] \Big)^{1/2} \quad u \in V_q(\mathbb{R}^N).$$

It follows from the characterisation (1.7) of $\lambda_{c,q}$ that $l(u,u) \geq \frac{\lambda_{c,q}-\lambda}{\lambda_{c,q}+\alpha} \|u\|^2_{c,q+\alpha m}$. So the continuous bilinear form l is coercive. By the Lax-Milgram theorem, we get the existence of a unique weak solution u for the equation (1.8). Moreover, if $g \geq 0$, by Theorem 1.1, we have $u \geq 0$. □

1.3. **Properties of M-matrices.** We say that a matrix is positive if all its entries are nonnegative and we say that a matrix is positive definite if this matrix is symmetric and if all its principal minors are strictly positive. We recall some results about the M-matrices (see [5, Theorem 2.3, p.134]). Let I be the identity matrix. A matrix $M = sI - B$ is called a non singular M-matrix, if B is a positive matrix and s is a real number such that $s > \rho(B)$, where $\rho(B)$ denotes the spectral radius of B.

Proposition 1.2. *If M is a matrix with nonpositive off-diagonal entries, then the following five conditions are equivalent.*

- (P0) *M is a non singular M-matrix.*
- (P1) *All the principal minors of M are strictly positive.*
- (P2) *M is semi-positive, i.e., there exists $X >> 0$ such that $MX >> 0$. Here $X >> 0$ means that the entries of X are strictly positive.*
- (P3) *M has a positive inverse.*
- (P4) *There exists a diagonal matrix D, $D > 0$ such that ${}^tMD + DM$ is positive definite.*

2. Results for linear systems

In this section, we consider the system (1.1) in the form

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(\rho_{kj,i} \frac{\partial u_i}{\partial x_j}) + q_i u_i = \mu_i m_i u_i + \sum_{j=1; j\neq i}^{n} a_{ij} u_j + f_i \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

$i = 1, \dots, n$. We consider the hypotheses:

- (H4) For all $i, j = 1, \dots, n$, $a_{ij} \in L^\infty(\mathbb{R}^N)$.
- (H5) For all $i = 1, \dots, n$, $f_i \in L^2(\mathbb{R}^N)$.

2.1. **Case of a cooperative system: $a_{ij} \geq 0$ ($\forall i \neq j$).** We obtain here results for the Maximum Principle and the existence of solutions for a linear cooperative system. We use classical methods as in [1, 3, 12, 14, 17, 18]. For a cooperative system we suppose also the following hypothesis

(H6) For all i, j, $i \neq j \Rightarrow a_{ij} \geq 0$.

For each $i \neq j$, since each weight m_i is bounded below by a positive constant, we deduce the existence of positive constants K_{ij} such that $a_{ij} \leq K_{ij}\sqrt{m_i}\sqrt{m_j}$. Note that in the particular case where $m_i = 1$ for each i, we can take $K_{ij} = \|a_{ij}\|_{L^\infty(\mathbb{R}^N)}$. We denote by $L = (l_{ij})$ the $n \times n$-matrix given as follows

$$l_{ii} := \lambda_i - \mu_i \quad \text{and} \quad l_{ij} := -K_{ij} \quad (i \neq j). \tag{2.2}$$

For such a system, we have the following maximum principle.

Theorem 2.1. *Assume* (H1)–(H6) *are satisfied. If the matrix L is a non singular M-matrix, then the cooperative system* (2.1) *satisfies the Maximum Principle.*

Proof. Assume that for all $i = 1, \dots, n$, $f_i \geq 0$. Let $u = (u_1, \dots, u_n)$ be a solution of the system (2.1) and define $u_i^- = \max(0, -u_i)$. Multiplying by u_i^- and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i^-}{\partial x_k} + q_i u_i u_i^-] = \mu_i \int_{\mathbb{R}^N} m_i u_i u_i^- + \sum_{j=1;j\neq i}^n \int_{\mathbb{R}^N} a_{ij} u_j u_i^- + \int_{\mathbb{R}^N} f_i u_i^- .$$

Due to the characterisation of λ_i (see (1.7)), we obtain

$$\lambda_i \int_{\mathbb{R}^N} m_i (u_i^-)^2 \leq \mu_i \int_{\mathbb{R}^N} m_i (u_i^-)^2 + \sum_{j=1;j\neq i}^n K_{ij} (\int_{\mathbb{R}^N} m_i (u_i^-)^2)^{1/2} (\int_{\mathbb{R}^N} m_j (u_j^-)^2)^{1/2}.$$

Thus, if we denote by ${}^tX = (x_1, \dots, x_n)$ where $x_i = (\int_{\mathbb{R}^N} m_i (u_i^-)^2)^{1/2}$, we have $LX \leq 0$. Since L is a non singular M-matrix, we can deduce that $X \leq 0$, so $X = 0$ and therefore $u_i \geq 0$ for each i. $\square$

Existence and uniqueness of a solution is stated as follows.

Theorem 2.2. *Assume* (H1)–(H6) *are satisfied. If the matrix L (given by* (2.2)*) is a non singular M-matrix, then the cooperative system* (2.1) *has a unique solution $u = (u_1, \dots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$. Moreover, due to the Maximum Principle, if $f_i \geq 0$ for all i, then this solution u satisfies $u_i \geq 0$ for $i = 1, \dots, n$.*

Proof. We suppose that L is a non singular M-matrix. Using (P4) (see Proposition 1.2), we introduce D a diagonal positive matrix such that ${}^tLD + DL$ is positive definite. We denote by $d_1, \dots, d_n$ the diagonal entries of the diagonal matrix D. As for one equation, the method is based on the Lax-Milgram theorem.

Let α be a positive number such that for all $1 \leq i \leq n$, $\mu_i + \alpha > 0$. Let $l : (V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N))^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} l(u,v) = \sum_{i=1}^n d_i \int_{\mathbb{R}^N} \Big[&\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_k} \\ &+ (q_i + \alpha m_i) u_i v_i - (\mu_i + \alpha) m_i u_i v_i - \sum_{j=1;j\neq i}^n a_{ij} u_j v_i \Big], \end{aligned}$$

if $u=(u_1,\dots,u_n)\in V_{q_1}(\mathbb{R}^N)\times\dots\times V_{q_n}(\mathbb{R}^N)$, and $v=(v_1,\dots,v_n)\in V_{q_1}(\mathbb{R}^N)\times\dots\times V_{q_n}(\mathbb{R}^N)$. We denote by

$$\|u_i\|_{\rho_i,q_i+\alpha m_i}=\Big(\int_{\mathbb{R}^N}[\sum_{k,j=1}^N\rho_{kj,i}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}+(q_i+\alpha m_i)u_i^2]\Big)^{1/2}. \tag{2.3}$$

By the characterisation of λ_i (see (1.7)) and the Cauchy-Schwartz inequality, we get:

$$\begin{aligned} l(u,u)\ge&\sum_{i=1}^n d_i\frac{\lambda_i-\mu_i}{\lambda_i+\alpha}\|u_i\|^2_{\rho_i,q_i+\alpha m_i}\\ &-\sum_{i,j=1;i\neq j}^n\frac{d_iK_{ij}}{\sqrt{\lambda_i+\alpha}\sqrt{\lambda_j+\alpha}}\|u_i\|_{\rho_i,q_i+\alpha m_i}\|u_j\|_{\rho_j,q_j+\alpha m_j}.\end{aligned}$$

Setting ${}^tX=(x_1,\dots,x_n)$ with $x_i=\frac{\|u_i\|_{\rho_i,q_i+\alpha m_i}}{\sqrt{\lambda_i+\alpha}}$, we get:

$$l(u,u)\ge^t XDLX=\frac{1}{2}\,{}^tX[{}^tLD+DL]X.$$

Since ${}^tLD+DL$ is positive definite, we deduce that l is coercive. Therefore, by the Lax-Milgram Theorem, we get the existence and the uniqueness of a weak solution for the system (2.1). □

2.2. **Case of a non necessarily cooperative system.** We give a very similar result for the existence and uniqueness of a solution for the system (2.1) as the one obtained for a cooperative system. We do not give the proof which is exactly the same as for Theorem 2.2. We note that for all i,j, $i\neq j$ there exists positive constants K'_{ij} such that $|a_{ij}|\le K'_{ij}\sqrt{m_i}\sqrt{m_j}$. We denote by $L'=(l'_{ij})$ the $n\times n$-matrix given by:

$$l'_{ii}:=\lambda_i-\mu_i\quad\text{and}\quad l'_{ij}:=-K'_{ij}\quad(i\neq j). \tag{2.4}$$

Existence and uniqueness of a solution is stated as follows.

Theorem 2.3. *Assume* (H1)–(H5) *are satisfied. If the matrix L' (given by* (2.4)*) is a non singular M-matrix, then the system* (2.1) *has a unique solution in* $V_{q_1}(\mathbb{R}^N)\times\dots\times V_{q_n}(\mathbb{R}^N)$.

3. Results for semilinear systems

In this section, we consider system (1.1) in the form:

$$-\sum_{k,j=1}^N\frac{\partial}{\partial x_k}(\rho_{kj,i}\frac{\partial u_i}{\partial x_j})+q_iu_i=\mu_im_iu_i+\sum_{j=1;j\neq i}^n a_{ij}u_j+f_i(x,u_1,\dots,u_n)\text{ in }\mathbb{R}^N, \tag{3.1}$$

$i=1,\dots,n$.

We consider also the following hypotheses which hold for each $i=1,\dots,n$:

- (H7) f_i is Lipschitz respect to u_i uniformly in x.
- (H8) There exists $\theta_i\in L^2(\mathbb{R}^N)$ such that $0\le f_i(x,u_1,\dots,u_n)\le\theta_i$ for all $u_1\ge0,\dots,u_n\ge0$.
- (H9) There exists $\theta_i\in L^2(\mathbb{R}^N)$ such that $|f_i(x,u_1,\dots,u_n)|\le\theta_i$ for all $u_1,\dots,u_n$.

We obtain two results for the existence of a solution for the system (3.1), which is either cooperative or non cooperative.

3.1. **Case of a cooperative system:** $a_{ij} \geq 0$ **(**$\forall i \neq j$**).** As in [3, 14, 18], we use a sub and super solutions method with a Schauder Fixed Point Theorem to obtain the existence of a positive solution if system (3.1) is cooperative.

Theorem 3.1. *Assume* (H1)–(H4), (H6)–(H8) *are satisfied. If the matrix L (given by* (2.2) *is a non singular M-matrix, then the system* (3.1) *has at least one positive solution in* $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.

Proof. We consider the system

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(\rho_{kj,i}\frac{\partial u_i}{\partial x_j}) + q_i u_i = \mu_i m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + \theta_i \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

$i = 1, \dots, n$. Applying Theorem 2.2 we deduce that the system (3.2) has a unique solution (which is positive by the Maximum Principle) $u^0 = (u^0_1, \dots, u^0_n)$. Moreover, by (H8), u^0 is a super solution of the system (3.1). Note also that $u_0 = (0, \dots, 0)$ is a sub solution of the system (3.1). We denote by $\sigma = [u_0, u^0]$.

To show the existence of positive solutions to system (3.1), we choose a positive real number α to be such that $\mu_i + \alpha > 0$ for all i. Let $T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$ be defined as follows: If $u = (u_1, \dots, u_n)$, then $T(u) := v = (v_1, \dots, v_n)$, v satisfies the equations

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(\rho_{kj,i}\frac{\partial v_i}{\partial x_j}) + (q_i + \alpha m_i)v_i = (\mu_i + \alpha)m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, \dots, u_n) \tag{3.3}$$

in $\mathbb{R}^N$ for each $i = 1, \dots, n$. Note that, by the scalar case, T is well defined and $T(\sigma) \subset \sigma$. As in [3], using the compact embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, we can prove that T is continuous and that $T(\sigma)$ is compact. By the Schauder Fixed Point Theorem, we deduce the existence of $u = (u_1, \dots, u_n) \in \sigma$ such that $T(u) = u$. Equivalently, u is a positive solution of the system (3.1). $\square$

3.2. **Case of a non cooperative semilinear system.** For a non cooperative system, we obtain the following result.

Theorem 3.2. *Assume* (H1)–(H4), (H7), (H9) *are satisfied. If the matrix* L' *(given by* (2.4)*) is a non singular M-matrix, then the system* (3.1) *has at least one solution in* $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.

Proof. We proceed exactly as for Theorem 3.1 by considering a sub and a super solution of the system (3.1) and using the Schauder Fixed Point Theorem. To do so, we consider the following system

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(\rho_{kj,i}\frac{\partial u_i}{\partial x_j}) + q_i u_i = \mu_i m_i u_i + \sum_{j=1;j\neq i}^{n} |a_{ij}|u_j + \theta_i \quad \text{in } \mathbb{R}^N, \tag{3.4}$$

for $i = 1, \dots, n$. Applying Theorem 2.2, we deduce that the system (3.4) has a unique solution (which is positive by the Maximum Principle) $u^0 = (u^0_1, \dots, u^0_n)$. Moreover, by (H8), u^0 is a super solution of the system (3.1). Note also that $-u^0$ is a sub solution of the system (3.1). We denote by $\sigma = [-u^0, u^0]$.

Let α be a positive real number such that $\mu_i + \alpha > 0$ for all i. Let $T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$ be defined as follows: If $u = (u_1, \dots, u_n)$, then $T(u) :=$

$v = (v_1, \dots, v_n)$, where v is determined by the equations:

$$-\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}(\rho_{kj,i}\frac{\partial v_i}{\partial x_j})+(q_i+\alpha m_i)v_i = (\mu_i+\alpha)m_i u_i+\sum_{j=1;j\neq i}^{n} a_{ij}u_j+f_i(x,u_1,\dots,u_n)$$

in $\mathbb{R}^N$ for $i = 1, \dots, n$. Note that, by the scalar case, T is well defined and such that $T(\sigma) \subset \sigma$. As in Theorem 3.1, we can prove that T is continuous and that $T(\sigma)$ is compact. By the Schauder Fixed Point Theorem, we deduce the existence of $u = (u_1, \dots, u_n) \in \sigma$ such that $T(u) = u$. Clearly, u is also a solution of the system (3.1). $\square$

If we relax the hypothesis about each weight m_i i.e. if we do not suppose that each weight m_i is bounded, then for the case where the system (3.1) is non necessarily cooperative we cannot apply the method developed for the Theorem 3.2. Indeed, the operator T would no longer be continuous from $(L^2(\mathbb{R}^N))^n$ to $(L^2(\mathbb{R}^N))^n$; moreover, even if we define T from $(L^{2^*}(\mathbb{R}^N))^n$ to $(L^{2^*}(\mathbb{R}^N))^n$, we would lose the compact embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. So we use, as in [11, 12], an approximation method due to Boccardo, Fleckinger and de Thélin (see [6]). For the following result, we assume that

$$N \geq 3, \quad 0 \leq m_i \in L^{\frac{N}{2}}(\mathbb{R}^N), \quad m_i \neq 0.$$

We recall that $\lambda(m_i)$ has been defined by (1.7). We denote by $L'' = (l''_{ij})$ the $n \times n$-matrix given by:

$$l''_{ii} := \lambda(m_i) - \mu_i \quad \text{and} \quad l''_{ij} := -K'_{ij} \quad (i \neq j),$$

where K'_{ij} is a positive constant such that $|a_{ij}| \leq K'_{ij}\sqrt{m_i}\sqrt{m_j}$.

Theorem 3.3. *Assume* (H1), (H2), (H4), (H7), (H9) *are satisfied. Assume that* $N \geq 3$, $0 \leq m_i \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $m_i \neq 0$. *Assume also that for all* $i = 1, \dots, n$, $m_i \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *If the matrix* L'' *is a non singular M-matrix, then the system* (3.1) *has at least one solution in* $V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$.

Proof. Let $\epsilon \in]0, 1[$ and $B_\epsilon = B(0, 1/\epsilon) = \{x \in \mathbb{R}^N,\ |x| < 1/\epsilon\}$. Let 1_{B_ϵ} be the indicator function of B_ϵ and let α be a positive real such that for i, $\mu_i + \alpha > 0$. Define $\Lambda = 1_{B_\epsilon}$.

Let $T_\epsilon : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$ be defined as follows: If $u = (u_1, \dots, u_n) \in (L^2(\mathbb{R}^N))^n$, then $T_\epsilon(u) := v = (v_1, \dots, v_n)$ where v is determined by the equations:

$$L_{\rho_i}v_i+(q_i+\alpha m_i)v_i = (\mu_i+\alpha)\frac{m_i u_i}{1+\epsilon m_i|u_i|}\Lambda+\sum_{j=1,j\neq i}^{n} a_{ij}\frac{u_j}{1+\epsilon|u_j|}\Lambda+f_i(x,u_1,\dots,u_n) \tag{3.5}$$

in $\mathbb{R}^N$, for $i = 1, \dots, n$. By the scalar case, T_ϵ is well defined and if we denote by $h_i = \max_{j,j\neq i}((\mu_i + \alpha), |a_{ij}|))\frac{1}{\epsilon}\Lambda \in L^2(\mathbb{R}^N)$, the equation $(L_{\rho_i} + q_i + \alpha m_i)\xi_{\epsilon,i} = nh_i + \theta_i$ in $\mathbb{R}^N$ admits a positive solution $\xi_{\epsilon,i}$ in $V_{q_i}(\mathbb{R}^N)$. So $\xi_\epsilon = (\xi_{\epsilon,1}, \dots, \xi_{\epsilon,n})$ is a super solution for (3.5) and $\zeta_\epsilon = (\zeta_{\epsilon,1}, \dots, \zeta_{\epsilon,n}) = -\xi_\epsilon$ is a sub solution for (3.5). Let $\sigma_\epsilon := [\zeta_\epsilon, \xi_\epsilon]$. Then $T_\epsilon(\sigma_\epsilon) \subset \sigma_\epsilon$.

We prove easily that T_ϵ is a continuous function (by the hypotheses (H4), (H7) and using the function $l : \mathbb{R} \to \mathbb{R}$ defined by $l(x) = \frac{x}{1+|x|}$, which is Lipschitz and which satisfies: for all $x, y \in \mathbb{R}$, $|l(x) - l(y)| \leq |x - y|$).

We prove also easily that $T_\epsilon(\sigma_\epsilon)$ is compact (due to the hypotheses (H4) and (H9)). So by the Schauder Fixed Point Theorem, we can deduce the existence of $u_\epsilon = (u_{1,\epsilon}, \dots, u_{n,\epsilon}) \in \sigma_\epsilon$ such that $T_\epsilon(u_\epsilon) = u_\epsilon$.

Note that for i, $(\epsilon u_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i}(\mathbb{R}^N)$.
Indeed, by (3.5) and (H9) we get

$$\begin{aligned} \|\epsilon u_{i,\epsilon}\|^2_{\rho_i,q_i+\alpha m_i} \le & \frac{\mu_i+\alpha}{\lambda(m_i)+\alpha}\|\epsilon u_{i,\epsilon}\|^2_{\rho_i,q_i+\alpha m_i} \\ & + \sum_{j=1,j\neq i}^n K'_{ij}\Big(\int_{\mathbb{R}^N} m_j\Big)^{1/2}\|\epsilon u_{i,\epsilon}\|_{m_i} + \|\theta_i\|_{L^2(\mathbb{R}^N)}\|\epsilon u_{i,\epsilon}\|_{L^2(\mathbb{R}^N)}, \end{aligned}$$

and so we can deduce the existence of a positive constant K, (independent of ϵ) such that $\|\epsilon u_{i,\epsilon}\|_{\rho_i,q_i+\alpha m_i} \le K$.

Since the embedding of $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, we deduce that (for a subsequence) there exists u_i^* such that $\epsilon u_{i,\epsilon} \to u_i^*$ as $\epsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i}(\mathbb{R}^N)$. Using the Dominated Convergence Theorem, due to (H9), we can pass to the limit (see (3.5)) and we can conclude that u_i^* is a weak solution of

$$(L_{\rho_i} + q_i + \alpha m_i)u_i^* = (\mu_i+\alpha)\frac{m_i u_i^*}{1+m_i|u_i^*|} + \sum_{j=1,j\neq i}^n a_{ij}\frac{u_j^*}{1+|u_j^*|} \quad \text{in } \mathbb{R}^N. \tag{3.6}$$

Moreover, if we set ${}^tX = (x_1, \dots, x_n)$ with $x_i = (\int_{\mathbb{R}^N} m_i u_i^{*2})^{1/2}$, using (3.6) we obtain $L''X \le 0$. Since L'' is a non singular M-matrix, we deduce that $X = 0$ i.e. for i, $u_i^* = 0$.

We prove now by contradiction that for i, $(u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i}(\mathbb{R}^N)$, We suppose that (for a subsequence) there exists i_0 such that $\|u_{i_0,\epsilon}\|_{\rho_{i_0},q_{i_0}+\alpha m_{i_0}} \to +\infty$ as $\epsilon \to 0$. Let

$$t_\epsilon = \max_i \|u_{i,\epsilon}\|_{\rho_i,q_i+\alpha m_i} \quad \text{and} \quad z_{i,\epsilon} = \frac{1}{t_\epsilon}u_{i,\epsilon}.$$

Since $(z_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i}(\mathbb{R}^N)$, there exists z_i such that $z_{i,\epsilon} \to z_i$ as $\epsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i}(\mathbb{R}^N)$. We can pass to the limit (see (3.5)) and conclude that z_i is a weak solution of

$$(L_{\rho_i} + q_i + \alpha m_i)z_i = (\mu_i+\alpha)m_i z_i + \sum_{j=1,j\neq i}^n a_{ij}z_j \quad \text{in } \mathbb{R}^N.$$

Then, we can prove that for all i, $z_i = 0$. However, there exists a sequence (ϵ_n) such that there exists i_1, $\|z_{i_1,\epsilon_n}\|_{\rho_{i_1},q_{i_1}+\alpha m_{i_1}} = 1$. But $z_{i_1,\epsilon_n} \to z_{i_1} = 0$ as $n \to +\infty$. So we get a contradiction.

Finally, there exists u_i^0 such that $u_{i,\epsilon} \to u_i^0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i}(\mathbb{R}^N)$. We can pass to the limit (see (3.5)) and to obtain

$$(L_{\rho_i} + q_i + \alpha m_i)u_i^0 = (\mu_i+\alpha)m_i u_i^0 + \sum_{j=1,j\neq i}^n a_{ij}u_j^0 + f_i(u_1^0, \dots, u_n^0) \text{ in } \mathbb{R}^N$$

for all i. This completes the proof. $\square$

3.3. **A bifurcation result.** In this section, we obtain a result for the existence of solutions for the system (1.1) by considering bifurcating solutions from the zero solution. We proceed here as in [10, 13, 15]. In this section, we suppose that the hypotheses (H1)–(H3) as well as (H10) are satisfied, where (H10) reads as follows:

(i) for $i = 1, \dots, n$, $g_i : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$, defined by $g_i(x, y_1, \dots, y_n)$ with $x \in \mathbb{R}^N$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$, satisfies For all $x \in \mathbb{R}^N$, $g_i(x, 0, \dots, 0) = 0$.

(ii) For $i = 1, \dots, n$, g_i is Frechet differentiable with respect to each variable y_j and each derivative $\frac{\partial g_i}{\partial y_j}(x, .)$ is continuous and bounded, uniformly in x.

(iii) For all $i, j = 1, \dots, n$, $\frac{\partial g_i}{\partial y_j}(x, 0, \dots, 0) = 0$.

Note that g_i is Lipschitz in $(y_1, \dots, y_n)$ uniformly in x. We denote by $V = \Pi_{i=1}^n V_{q_i}(\mathbb{R}^N)$ and by $\langle ., . \rangle_V$ the inner product in V (i.e. for $v = (v_1, \dots, v_n) \in V$ and all $w = (w_1, \dots, w_n) \in V$, $\langle v, w \rangle_V = \sum_{i=1}^n \langle v_i, w_i \rangle_{\rho_i, q_i}$ (see (1.2))). We define the operator $T : \mathbb{R}^n \times V \to V$, $T = (T^1, \dots, T^n)$ by:

$$\langle T^i(\mu, u), v_i \rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_k} + q_i u_i v_i - \mu_i m_i u_i v_i - g_i(x, u) v_i]. \tag{3.7}$$

for $i = 1, \dots, n$, $T^i : \mathbb{R}^n \times V \to V_{q_i}(\mathbb{R}^N)$, if $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, $u = (u_1, \dots, u_n) \in V$, for all $v_i \in V_{q_i}(\mathbb{R}^N)$. We proceed in this section exactly as in [15] and so we give only the steps on each proofs.

Proposition 3.1. *The operator T is well defined. Furthermore, for each i, the operator T^i is continuous, Frechet differentiable with continuous derivatives given by: For all $\phi \in V_{q_i}(\mathbb{R}^N)$ and for all $\psi \in V_{q_i}(\mathbb{R}^N)$,*

$$\text{if } j \neq i, \quad T^i_{\mu_j} = 0, \quad \langle T^i_{u_j}(\mu, u)\phi, \psi \rangle_{\rho_i, q_i} = -\int_{\mathbb{R}^N} \frac{\partial g_i}{\partial y_j}(x, u)\phi\psi,$$

$$\text{if } j = i, \quad \langle T^i_{\mu_i}(\mu, u), \phi \rangle_{\rho_i, q_i} = -\int_{\mathbb{R}^N} m_i u_i \phi \text{ and}$$

$$\langle T^i_{u_i}(\mu, u)\phi, \psi \rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial \phi}{\partial x_j} \frac{\partial \psi}{\partial x_k} + q_i \phi\psi - \mu_i m_i \phi\psi - \frac{\partial g_i}{\partial y_i}(x, u)\phi\psi],$$

$$\text{if } j \neq i, \quad T^i_{\mu_j\, u_i} = 0 = T^i_{\mu_i\, u_j},$$

$$\text{if } j = i, \quad \langle T^i_{\mu_i\, u_i}(\mu, u)\phi, \psi \rangle_{\rho_i, q_i} = -\int_{\mathbb{R}^N} m_i \phi\psi \text{ and } T^k_{\mu_i\, u_i} = 0 \text{ if } k \neq i.$$

Proposition 3.2. *The operator $T_u(\lambda, 0)$ is a continuous self-adjoint operator with $\lambda = (\lambda_1, \dots, \lambda_n)$. The kernel of $T_u(\lambda, 0)$, denoted by $N(T_u(\lambda, 0))$ is generated by $\{\Phi_1, \dots, \Phi_n\}$ where for $i = 1, \dots, n$, $\Phi_i = (0, \dots, 0, \phi_i, 0, \dots, 0)$.*

Moreover, if we denote by $R(T_u(\lambda, 0))$ the range of the operator $T_u(\lambda, 0)$, we have:

(1) $\operatorname{codim}(R(T_u(\lambda, 0))) = n$
(2) *for* $i = 1, \dots, n$, $T_{\mu_i\, u}(\lambda, 0)\Phi_i \notin R(T_u(\lambda, 0))$
(3) $\dim(\operatorname{Span}\{T_{\mu_i\, u}(\lambda, 0)\Phi_i,\ 1 \leq i \leq n\}) = n$.

Proof. Since $T_u(\lambda, 0) = (T^1_u(\lambda, 0), \dots, T^n_u(\lambda, 0))$, using (H10)(iii), we get that for $i = 1, \dots, n$, $T^i_u(\lambda, 0)$ is a continuous self-adjoint operator. Therefore, $T_u(\lambda, 0)$ is a continuous self-adjoint operator too.

We have: for all $v = (v_1, \dots, v_n) \in V$,

$v \in N(T_u(\lambda, 0))$

if and only if for all $w \in V$, $\langle T_u(\lambda, 0)v, w\rangle_V = 0$

if and only if for $i = 1, \dots, n$, $v_i \in \mathrm{Span}\{\phi_i\}$

if and only if $v \in \mathrm{Span}\{\Phi_1, \dots, \Phi_n\}$ where $\Phi_i = (0, \dots, 0, \phi_i, 0, \dots, 0)$.

Therefore $\operatorname{codim} R(T_u(\lambda, 0)) = n$.

Now we prove that $T_{\mu_i u}(\lambda, 0)\Phi_i \notin R(T_u(\lambda, 0))$ for all i. Note that we have identified $T_{\mu_i u}(\lambda, 0).(1, \Phi_i)$ with $T_{\mu_i u}(\lambda, 0)\Phi_i$. We have: $\langle T_{\mu_i u}(\lambda, 0)\Phi_i, \Phi_i\rangle_V = -\int_{\mathbb{R}^N} m_i\phi_i^2 \neq 0$. Therefore, $T_{\mu_i u}(\lambda, 0)\Phi_i$ is not orthogonal to Φ_i. Now since $N(T_u(\lambda, 0)) = \mathrm{Span}\{\Phi_1, \dots, \Phi_n\}$ and $R(T_u(\lambda, 0)) = N(T_u(\lambda, 0))^{\perp_V}$, we deduce that $T_{\mu_i u}(\lambda, 0)\Phi_i \notin R(T_u(\lambda, 0))$. Moreover, let $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ be such that: $\sum_{j=1}^n \alpha_j T_{\mu_j u}(\lambda, 0)\Phi_j = 0$. Fix i, $1 \le i \le n$. So that $\langle \sum_{j=1}^n \alpha_j T_{\mu_j u}(\lambda, 0)\Phi_j, \Phi_i\rangle_V = 0$ and

$$\sum_{j=1}^n \alpha_j \langle T^i_{\mu_j u}(\lambda, 0)\Phi_j, \phi_i\rangle_{\rho_i, q_i} = 0.$$

This implies that $-\alpha_i \int_{\mathbb{R}^N} m_i\phi_i^2 = 0$ and thus $\alpha_i = 0$. Consequently, we have that $\dim(\mathrm{Span}\{T_{\mu_i u}(\lambda, 0)\Phi_i,\ 1 \le i \le n\}) = n$, completing the proof. $\square$

Although we cannot apply directly the results obtained in [16], as in [15] we follow the proof of [16, Theorem 1.7] to obtain the following result.

Theorem 3.4. *Assume that the hypotheses* (H1), (H2), (H10) *are satisfied. Then there exist a constant $\epsilon_0 > 0$, a neighbourhood U of $(\lambda, 0)$ (with $\lambda = (\lambda_1, \dots, \lambda_n)$ and $0 = (0, \dots, 0) \in V$) and a continuous function $H : (-\epsilon_0, \epsilon_0) \to U$ such that $T(H(\epsilon)) = 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$.*

Note that $T(H(\epsilon)) = 0$ with $H(\epsilon) = (\mu, u) \in U$ for $\mu = (\mu_1, \dots, \mu_n)$ in a neighbourhood of $\lambda = (\lambda_1, \dots, \lambda_n)$ and $u = (u_1, \dots, u_n)$ in a neighbourhood of $0 = (0, \dots, 0)$ signifies that (μ, u) is a non trivial solution of the system (1.1).

Proof. As in [16], we introduce the following function $h : \mathbb{R} \times \mathbb{R}^n \times V \to V$ be defined as follows: For all $(\alpha, \mu, w) \in \mathbb{R} \times \mathbb{R}^n \times V$,

$$h(\alpha, \mu, w) := \begin{cases} \frac{1}{\alpha} T(\mu, \alpha\Phi_1 + \dots + \alpha\Phi_n + \alpha w) & \text{if } \alpha \neq 0 \\ T_u(\mu, 0)(\Phi_1 + \dots + \Phi_n + w) & \text{if } \alpha = 0 \end{cases} \tag{3.8}$$

Since for $i = 1, \dots, n$, $\Phi_i \in N(T_u(\lambda, 0))$, we deduce that $h(0, \lambda, 0) = 0$.

Let $g : \mathbb{R}^n \times V \to V$ be given by $g(\mu, w) := T_u(\mu, 0)(\Phi_1 + \dots + \Phi_n + w)$ for all $(\mu, w) \in \mathbb{R}^n \times V$. For all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $v = (v_1, \dots, v_n) \in V$ we have that

$$Dg(\mu, w)(\xi, v) = \sum_{i=1}^n \xi_i T_{\mu_i u}(\mu, 0)(\Phi_1 + \dots + \Phi_n + w) + T_u(\mu, 0)v,$$

where Dg is the Frechet derivative of g. It follows that

$$Dg(\lambda, 0)(\xi, v) = \sum_{i=1}^n \xi_i T_{\mu_i u}(\lambda, 0)\Phi_i + T_u(\lambda, 0)v \quad \text{since } T_{\mu_i u}(\lambda, 0)\Phi_j = 0 \quad (j \neq i)$$

for all $\xi \in \mathbb{R}^n$ and $v \in V$.

By Proposition 3.2, we deduce that $Dg(\lambda, 0)$ is a linear homeomorphism from $\mathbb{R}^n \times V$ onto V. Hence, the implicit function theorem implies the existence of a neighbourhood U' of $(\lambda, 0)$, of a constant $\epsilon_0 > 0$ and of a function $K : (-\epsilon_0, \epsilon_0) \to U'$ such that $h(\epsilon, \mu, w) = 0$ (see (3.8)) with $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $K(\epsilon) = (\mu, w) = (K_1\epsilon, K_2\epsilon) \in U'$. Therefore for $\epsilon \in (-\epsilon_0, \epsilon_0)$, $T(K_1\epsilon, \epsilon(\Phi_1 + \cdots + \Phi_n + K_2\epsilon)) = 0$ and if we set $H(\epsilon) := (K_1\epsilon, \epsilon(\Phi_1 + \cdots + \Phi_n + K_2\epsilon))$, then we have that $T(H(\epsilon)) = 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. $\square$

Finally, we study the global nature of the continuum of solutions obtained by bifurcation from the $(\lambda, 0)$ solution in a particular case. As in [13, 15], we follow a method developed in [10] and using the [23, Theorems 1.3 and 1.40], we obtain the following result.

Theorem 3.5. *Assume* (H1), (H2), (H10) *are satisfied. Assume also that for all* i, j, $\lambda_i = \lambda_j$. *We denote by* $\lambda_0 := \lambda_i$ *for all* $i = 1, \dots, n$. *Then, there exists a continuum* C *of non trivial solutions for the system* (1.1) *obtained by bifurcation from the* $(\lambda_0, 0)$ *solution, which is either unbounded or contains a point* $(\mu, 0)$ *where* $\mu \neq \lambda_0$ *is the inverse of an eigenvalue of the operator* $A = (L_1, \dots, L_n)$ *(where* L_i *is defined by for* $u = (u_1, \dots, u_n) \in V$ *and* $\phi \in V_{q_i}(\mathbb{R}^N)$, $\langle L_i u, \phi\rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} m_i u_i \phi$.*) Since* λ_0 *is simple, the continuum* C *has two connected subsets* C^+ *and* C^- *which satisfy also the above alternatives.*

Proof. First, we define an operator S by setting $S(\mu, u) = u - T(\mu, u)$, $S = (S^1, \dots, S^n)$ i.e. for all $\mu \in \mathbb{R}$, for all $u = (u_1, \dots, u_n) \in V$, for all $v_i \in V_{q_i}(\mathbb{R}^N)$,

$$\langle S^i(\mu, u), v_i\rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} [\mu m_i u_i v_i + g_i(x, u) v_i].$$

So $u = (u_1, \dots, u_n)$ is a solution of the system (1.1) if and only if $u = S(\mu, u)$. We write $S^i(\mu, u) = \mu L_i u + H_i u$ where for all $v_i \in V_{q_i}(\mathbb{R}^N)$,

$$\langle L_i u, v_i\rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} m_i u_i v_i \quad \text{and} \quad \langle H_i u, v_i\rangle_{\rho_i, q_i} = \int_{\mathbb{R}^N} g_i(x, u) v_i.$$

So $S(\mu, u) = \mu A u + H u$ with $Au = (L_1 u, \dots, L_n u)$ and $Hu = (H_1 u, \dots, H_n u)$.

To apply the results in [23], we must prove that $S^i : \mathbb{R} \times V \to V_{q_i}(\mathbb{R}^N)$ is continuous and compact, that $L_i : V \to V_{q_i}(\mathbb{R}^N)$ is linear and compact, that $H_i u = O(\|u\|_V)$ for $u = (u_1, \dots, u_n)$ near $0 = (0, \dots, 0)$ uniformly on bounded intervals of μ and that $\frac{1}{\lambda_0}$ is a simple eigenvalue of A (which is true because it is a simple eigenvalue of $(L_{\rho_i} + q_i)^{-1} M_i$.)

Let $((\mu_p, u_p))_p$ be a bounded sequence in $\mathbb{R} \times V$, with $u_p = (u_{1p}, \dots, u_{np})$. Since the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $((\mu_p, u_p))_p$ in $\mathbb{R} \times (L^2(\mathbb{R}^N))^n$. We have:

$$\begin{aligned}
&\|S^i(\mu_p, u_p) - S^i(\mu_m, u_m)\|^2_{\rho_i, q_i} \\
&= (\mu_p - \mu_m) \int_{\mathbb{R}^N} m_i u_{ip} [S^i(\mu_p, u_p) - S^i(\mu_m, u_m)] \\
&\quad + \mu_m \int_{\mathbb{R}^N} m_i (u_{ip} - u_{im}) [S^i(\mu_p, u_p) - S^i(\mu_m, u_m)] \\
&\quad + \int_{\mathbb{R}^N} [g_i(x, u_p) - g_i(x, u_m)][S^i(\mu_p, u_p) - S^i(\mu_m, u_m)].
\end{aligned}$$

We deduce that $(S^i(\mu_p, u_p))_p$ is a Cauchy sequence and therefore a convergent sequence. So S^i is compact for all $i = 1, \dots, n$ and $S = (S^1, \dots, S^n)$ is also compact. Moreover we have for each i that

$$\|L_i u_p - L_i u_m\|^2_{\rho_i, q_i} = \int_{\mathbb{R}^N} m_i (u_{ip} - u_{im})[L_i u_p - L_i u_m]$$

and thus

$$\|L_i u_p - L_i u_m\|_{\rho_i, q_i} \leq \text{const.} \|u_{ip} - u_{im}\|_{L^2(\Omega)}.$$

Therefore $(L_i u_p)_p$ is a Cauchy sequence, so L_i is compact and A is compact too. Finally we have:

$$\|H_i u\|^2_{\rho_i, q_i} = \int_{\mathbb{R}^N} g_i(x, u) H_i u \leq \text{const} \|u\|_V \|H_i u\|_{\rho_i, q_i}.$$

So $H_i u = O(\|u\|_V)$ and therefore $Hu = O(\|u\|_V)$. □

4. Existence of positive solutions in $\mathbb{R}^N$ for a particular case

In this section, we follow a method developed in [22] for the p-Laplacian in a bounded domain of $\mathbb{R}^N$, then in [13] for an equation defined in $\mathbb{R}^N$ and involving a Schrödinger operator with a potential satisfying the hypothesis (H1) and a weight satisfying the hypothesis (H3) and in [15] for a system defined in $\mathbb{R}^N$ and involving Schrödinger operators with potentials satisfying the hypothesis (H1) and weights satisfying the hypothesis (H3). We redefine the system (1.1) for this section.

We write (1.1) in the form

$$(L_{\rho_i} + q_i) u_i = \mu_i m_i u_i + \sum_{j=1; j \neq i}^{n} a_{ij} u_i^p u_j^q + \sum_{j=1; j \neq i}^{n} f_{ij} u_j^{p+q} \text{ in } \mathbb{R}^N, \ i = 1, \dots, n \tag{4.1}$$

where $N = 3, 4$; $\gamma = 2^* = \frac{2N}{N-2} = 6, 4$ and p and q are positive integers be such that $p + q < \gamma$.

We define for sufficiently large numbers $C > 0$ the sets

$$\begin{aligned} X_{q_i, C} = \{\phi \in V_{q_i}(\mathbb{R}^N) :\ & \text{there exists a positive constant s such that} \\ & \phi_i \leq s\phi \leq C \text{ a. e. } \} \end{aligned} \tag{4.2}$$

(the sets are non-empty by the properties of ϕ_i). We impose the following assumptions:

- (H11) For $i, j = 1, \dots, n$, $a_{ij} \in L^\infty(\mathbb{R}^N)$ and $f_{ij} \in L^\infty(\mathbb{R}^N)$.
- (H12) For $i, j = 1, \dots, n$, $f_{ij} \geq 0$ a.e.
- (H13) For $i = 1, \dots, n$, there exists $j_i \in \{1, \dots, n\} - \{i\}$ such that the following items hold:
  1. Denote $\Omega_{i,+} := \{x \in \mathbb{R}^N, a_{ij_i} > 0\}$ and $\Omega_{i,-} := \{x \in \mathbb{R}^N, a_{ij_i} < 0\}$. Then $\text{meas}(\Omega_{i,+}) \neq 0$, $\text{meas}(\Omega_{i,-}) \neq 0$.
  2. For each $k \in \{1, \dots, n\} - \{i, j_i\}$, a_{ik} is a nonnegative function, $a_{ik} = 0$ in D_i where D_i is a measurable subset of $\Omega_{i,-}$ with positive measure.
  3. For each $k \in \{1, \dots, n\}$, $f_{ik} = 0$ in D_i.
- (H14) There exists $\epsilon > 0$ and $l \geq 1$ such that for $i = 1, \dots, n$, $a_{ij_i} \geq -\epsilon m_i$ and $\epsilon < \frac{\mu_i}{p(lC)^{p+q-1}}$
- (H15) For each $i = 1, \dots, n$, there exists a positive constant k_{ij_i} such that $k_{ij_i} \leq \frac{(p+q)}{lq(lC)^{p+q-1}}$ and $a_{ij_j} \geq -k_{ij_i} f_{ij_i} \phi_{j_i}^{p+q-1}$ a.e.

We denote by

$$F_i(u_1,\dots,u_n)=\int_{\mathbb{R}^N}[\sum_{j=1,j\neq i}^{n} a_{ij}u_i^{p+1}u_j^q+(p+1)\sum_{j=1,j\neq i}^{n} f_{ij}u_j^{p+q}u_i] \tag{4.3}$$

for $i=1,\dots,n$ and all $(u_1,\dots,u_n)\in V_{q_1}(\mathbb{R}^N)\times\dots\times V_{q_n}(\mathbb{R}^N)$ and by

$$H_{\mu_i}(v)=\int_{\mathbb{R}^N}[\sum_{k,j=1}^{N}\rho_{kj,i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k}+q_iv^2-\mu_im_iv^2] \tag{4.4}$$

for all $i=1,\dots,n$ and all $v\in V_{q_i}(\mathbb{R}^N)$. Let

$$\lambda_i^*=\sup_{v_i\in V_{q_i}(\mathbb{R}^N),v_i\geq 0}\Big\{\inf_{\phi\in\Phi_{v_i}}\{\frac{\int_{\mathbb{R}^N}\sum_{k,j=1}^N\rho_{kj,i}\frac{\partial v_i}{\partial x_j}\frac{\partial\phi}{\partial x_k}+q_iv_i\phi}{\int_{\mathbb{R}^N}m_iv_i\phi}\}\Big\} \tag{4.5}$$

and

$$\lambda_i^{**}=\sup_{v_i\in X_{q_i,C}}\Big\{\inf_{\phi\in\Phi_{v_i}}\{\frac{\int_{\mathbb{R}^N}\sum_{k,j=1}^N\rho_{kj,i}\frac{\partial v_i}{\partial x_j}\frac{\partial\phi}{\partial x_k}+q_iv_i\phi}{\int_{\mathbb{R}^N}m_iv_i\phi}\}\Big\} \tag{4.6}$$

where

$$\begin{aligned}\Phi_{v_i}:=\big\{&\phi\in D(\mathbb{R}^N),\ \phi\geq 0,\ \text{such that for } j\neq i, \text{ there exists}\\ &v_j\in V_{q_j}(\mathbb{R}^N),\ v_j\geq 0 \text{ and } \frac{\partial F_i}{\partial u_i}(v_1,\dots,v_n)(\phi)\geq 0\big\}\end{aligned} \tag{4.7}$$

and where $\frac{\partial F_i}{\partial u_i}$ denotes the i-th partial derivative of F_i.

Note that the existences of λ_i^* and λ_i^{**} are due to the hypotheses (H4), (H11), (H12) and that $\lambda_i^{**}\leq\lambda_i^*$. Assume the following hypotheses for $i=1,\dots,n$:

(H16) $\lambda_i^{**}<+\infty$.

(H17) $\lambda_i^*<+\infty$.

We proceed exactly as in [15] in this section; so that we will give only the steps of the proofs.

Lemma 4.1. (1) *For $i=1,\dots,n$, and all $\phi\in D(\mathbb{R}^N)$,*

$$\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi)=(p+1)\sum_{j=1,j\neq i}^{n}\int_{\mathbb{R}^N}[a_{ij}u_i^pu_j^q\phi+f_{ij}u_j^{p+q}\phi],$$

$$H'_{\mu_i}(v)(\phi)=2\int_{\mathbb{R}^N}[\sum_{k,j=1}^{N}\rho_{kj,i}\frac{\partial v}{\partial x_j}\frac{\partial\phi}{\partial x_k}+q_iv\phi-\mu_im_iv\phi].$$

(2) *$(u_1,\dots,u_n)\in V_{q_1}(\mathbb{R}^N)\times\dots\times V_{q_n}(\mathbb{R}^N)$ is a supersolution (resp. subsolution) of the system (4.1) if and only if for all $\phi\in D(\mathbb{R}^N)$, $\phi\geq 0$, for $i=1,\dots,n$,*

$$H'_{\mu_i}(u_i)(\phi)\geq\frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi)\ (\ \textit{resp.}\ \leq).$$

(3) *For $i=1,\dots,n$, all $\phi\in D(\mathbb{R}^N)$, for all $t>0$,*

$$\frac{\partial F_i}{\partial u_i}(tu_1,\dots,tu_n)(\phi)=t^{p+q}\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi)\ \textit{and}\ H'_{\mu_i}(tu_i)(\phi)=tH'_{\mu_i}(u_i)(\phi).$$

As in [13, 15, 22], we obtain the following lemma.

Lemma 4.2. *For $i=1,\dots,n$, we have $\lambda_i\leq\lambda_i^{**}$.*

Proof. Suppose (for example) that $\lambda_1 > \lambda_1^{**}$. Because of the characterisation of λ_1 (see (1.6), (1.7) and (4.4), we have $H_{\lambda_1}(\phi_1) = 0$. By the definition of λ_1^{**} (see (4.6)), there exists $\phi \in D(\mathbb{R}^N)$, $\phi \geq 0$, there exists $(v_2, \dots, v_n) \in V_{q_2}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$, $v_i \geq 0$, such that $\frac{\partial F_1}{\partial u_1}(\phi_1, v_2, \dots, v_n)(\phi) \geq 0$ and

$$\frac{\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,1} \frac{\partial \phi_1}{\partial x_j} \frac{\partial \phi}{\partial x_k} + q_1 \phi_1 \phi]}{\int_{\mathbb{R}^N} m_1 \phi_1 \phi} \leq \lambda_1^{**} < \lambda_1.$$

So $H'_{\lambda_1}(\phi_1)(\phi) < 0$. Moreover, we have for all $\eta > 0$,

$$H_{\lambda_1}(\phi_1 + \eta\phi) = H_{\lambda_1}(\phi_1) + \eta H'_{\lambda_1}(\phi_1)(\phi) + \|\eta\phi\| h(\eta\phi) \text{ with } h(\eta\phi) \to 0$$

as $\eta \to 0$. Therefore, for η small enough, we have $H_{\lambda_1}(\phi_1 + \eta\phi) < 0$ contradicting the definition of λ_1. $\square$

We obtain now the main result of this section.

Theorem 4.3. *Assume* (H1)–(H4), (H11)-(H16) *are satisfied. If for* $i = 1, \dots, n$, $\lambda_i + \epsilon(lC)^{p+q-1} < \mu_i < \lambda_i^{**}$, *then the system* (4.1) *has at least one positive solution in* $X_{q_1,C} \times \dots \times X_{q_n,C}$.

Proof. Since for all i, $\mu_i < \lambda_i^{**}$, due to the definition of λ_i^{**} (see (4.6)), we can deduce the existence of $v_i^* \in X_{q_i,C}$ such that for all $\phi \in \Phi_{v_i^*}$, $H'_{\mu_i}(v_i^*)(\phi) > 0$.

We proceed exactly as in [15] to prove that there exists a real $t \in (0, l)$ for which $(tv_1^*, \dots, tv_n^*)$ is a supersolution of the system (4.1). Suppose that for all $t \in (0, l)$, $(tv_1^*, \dots, tv_n^*)$ is not a supersolution of the system (4.1). Then for all $t \in (0, l)$, there exist $i_t \in \{1, \dots, n\}$ and $\psi_{i_t} \geq 0$ such that

$$H'_{\mu_{i_t}}(tv_{i_t}^*)(\psi_{i_t}) < \frac{2}{p+1} \frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*, \dots, tv_n^*)(\psi_{i_t}).$$

Consider the sets

$$\begin{aligned} N_t = \big\{ & i \in \{1, \dots, n\}, \text{ there exists } \psi \in D(\mathbb{R}^N),\ \psi \geq 0, \text{ such that} \\ & H'_{\mu_i}(tv_i^*)(\psi) < \frac{2}{p+1} \frac{\partial F_i}{\partial u_i}(tv_1^*, \dots, tv_n^*)(\psi) \big\} \end{aligned} \tag{4.8}$$

and for $i_t \in N_t$,

$$K_{i_t} = \{\psi \in D(\mathbb{R}^N),\ \psi \geq 0,\ H'_{\mu_{i_t}}(tv_{i_t}^*)(\psi) < \frac{2}{p+1} \frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*, \dots, tv_n^*)(\psi)\}. \tag{4.9}$$

We prove that there exists $t > 0$, $i_t \in N_t$, $\phi \in K_{i_t}$, and $\psi \in K_{i_t}$ which satisfy

$$\frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*, \dots, tv_n^*)(\phi) < 0 \quad \text{and} \quad \frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*, \dots, tv_n^*)(\psi) > 0.$$

So we have

$$H'_{\mu_{i_t}}(v_{i_t}^*)(\phi) < \frac{2}{p+1} t^{p+q-1} \frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*, \dots, v_n^*)(\phi) < 0, \tag{4.10}$$

$$0 < H'_{\mu_{i_t}}(v_{i_t}^*)(\psi) < \frac{2}{p+1} t^{p+q-1} \frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*, \dots, v_n^*)(\psi). \tag{4.11}$$

(Note that $\psi \in \Phi_{v_{i_t}^*}$ (see (4.7).) Since $\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*, \dots, v_n^*)$ is a continuous function, there exists $\alpha \in (0, 1)$ such that

$$\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*, \dots, v_n^*)(\alpha\phi + (1 - \alpha)\psi) = 0.$$

Thus we deduce that $\alpha\phi + (1-\alpha)\psi \in \Phi_{v^*_{i_t}}$ and so $H'_{\mu_{i_t}}(v^*_{i_t})(\alpha\phi + (1-\alpha)\psi) > 0$. But using (4.10) and (4.11) we have

$$\begin{aligned}
0 &< \alpha H'_{\mu_{i_t}}(v^*_{i_t})(\phi) + (1-\alpha)H'_{\mu_{i_t}}(v^*_{i_t})(\psi)\\
&< \frac{2}{p+1}t^{p+q-1}[\alpha\frac{\partial F_{i_t}}{\partial u_{i_t}}(v^*_1,\dots,v^*_n)(\phi) + (1-\alpha)\frac{\partial F_{i_t}}{\partial u_{i_t}}(v^*_1,\dots,v^*_n)(\psi)] = 0
\end{aligned}$$

and we get a contradiction. Therefore there exists $t \in (0,l)$, for which $(tv^*_1,\dots,tv^*_n)$ is a supersolution of the system (4.1). Note that for $i = 1,\dots,n$, $tv^*_i \ge s\phi_i$ if $0 < s \le t$.

Using (H11)–(H13), we can prove that $(s\phi_1,\dots,s\phi_n)$ is a subsolution of system (4.1) with $s > 0$ be such that $s \le t \le l$ and $\frac{1}{l} \le s^{p+q-1}$ (which is possible with $l \ge 1$). Let $\sigma = [s\phi_1, tv^*_1] \times \dots \times [s\phi_n, tv^*_n]$ and the operator T be defined by $T(u_1,\dots,u_n) = (v_1,\dots,v_n)$ with $(v_1,\dots,v_n)$ solution of

$$(L_{\rho_i} + q_i)v_i = \mu_i m_i u_i + \sum_{j=1;j\ne i}^{n} a_{ij}u_i^p u_j^q + \sum_{j=1;j\ne i}^{n} f_{ij}u_j^{p+q} \quad \text{in } \mathbb{R}^N, \tag{4.12}$$

for $i = 1,\dots,n$. We want to prove that $T(\sigma) \subset \sigma$.

Let $(u_1,\dots,u_n) \in \sigma$ and $T(u_1,\dots,u_n) = (v_1,\dots,v_n)$. By (4.12) we can write for $i = 1,\dots,n$,

$$(L_{\rho_i} + q_i)(v_i - s\phi_i) = \mu_i m_i u_i + \sum_{j=1;j\ne i}^{n} a_{ij}u_i^p u_j^q + \sum_{j=1;j\ne i}^{n} f_{ij}u_j^{p+q} - s\lambda_i m_i\phi_i.$$

Since for all k, $u_k \ge s\phi_k$, using (H14) $a_{ij_i} \ge -\epsilon m_i$, we can deduce that

$$(L_{\rho_i} + q_i)(v_i - s\phi_i) \ge [\mu_i - \lambda_i - \epsilon s^{p+q-1}\phi_i^{p-1}\phi_{j_i}^q]m_i s\phi_i. \tag{4.13}$$

But $\phi_i^{p-1}\phi_{j_i}^q \le C^{p+q-1}$, $s^{p+q-1} \le l^{p+q-1}$ and $\lambda_i + \epsilon(lC)^{p+q-1} \le \mu_i$, so we deduce from (4.13) that $(L_{\rho_i} + q_i)(v_i - s\phi_i) \ge 0$. By the Maximum Principle, we obtain that $v_i \ge s\phi_i$ for all $i = 1,\dots,n$. Moreover we have: For all $i = 1,\dots,n$,

$$\begin{aligned}
(L_{\rho_i} + q_i)(tv^*_i - v_i) &\ge \mu_i m_i(tv^*_i - u_i) + \sum_{j=1;j\ne i}^{n} a_{ij}[(tv^*_i)^p(tv^*_j)^q - u_i^p u_j^q]\\
&\quad + \sum_{j=1;j\ne i}^{n} f_{ij}[(tv^*_j)^{p+q} - u_j^{p+q}].
\end{aligned}$$

So we can rewrite this equation in form

$$\begin{aligned}
&(L_{\rho_i} + q_i)(tv^*_i - v_i)\\
&\ge \mu_i m_i(tv^*_i - u_i) + \sum_{j=1;j\ne i}^{n} a_{ij}(tv^*_i - u_i)u_j^q[\sum_{k=0}^{p-1}(tv^*_i)^k u_i^{p-1-k}]\\
&\quad + \sum_{j=1;j\ne i}^{n}(tv^*_j - u_j)[a_{ij}(tv^*_i)^p(\sum_{k=0}^{q-1}(tv^*_j)^k u_j^{q-1-k}) + f_{ij}(\sum_{k=0}^{p+q-1}(tv^*_j)^k u_j^{p+q-1-k})].
\end{aligned}$$

Since $(u_1,\dots,u_n)\in\sigma$, we get

$$\begin{aligned}
&(L_{\rho_i}+q_i)(tv_i^*-v_i)\\
&\geq (tv_i^*-u_i)[\mu_i m_i + a_{ij_i}u_{j_i}^q(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k})]\\
&\quad + (tv_{j_i}^*-u_{j_i})[a_{ij_i}(tv_i^*)^p(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k}) + f_{ij_i}(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k})].
\end{aligned}\tag{4.14}$$

Since $u_{j_i}^q(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k})\leq p(lC)^{p+q-1}$, using (H14) we deduce that

$$\mu_i m_i + a_{ij_i}u_{j_i}^q(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k})\geq 0.\tag{4.15}$$

Similarly, using (H15) and $s^{p+q-1}\geq\frac{1}{l}$, we get

$$\frac{f_{ij_i}(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k})}{(tv_i^*)^p(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k})}\geq\frac{(p+q)f_{ij_i}(s\phi_{j_i})^{p+q-1}}{q(lC)^{p+q-1}}$$

and so

$$a_{ij_i}(tv_i^*)^p(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k}) + f_{ij_i}(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k})\geq 0.\tag{4.16}$$

Therefore, by (4.14), (4.15), (4.16), we obtain $(L_{\rho_i}+q_i)(tv_i^*-v_i)\geq 0$ and so by the Maximum Principle we deduce that $v_i\leq tv_i^*$ for all $i=1,\dots,n$. We conclude that $(v_1,\dots,v_n)\in\sigma$. Finally, note that T is a continuous and compact operator (by the compact embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$).

Therefore the Schauder Fixed Point Theorem, implies the existence of at least one positive solution for the system (4.1). $\square$

To complete this section, we give some conditions which assure the validity of the hypothesis (H17). First, we recall the following lemma (see [22]).

Lemma 4.4. *For $i=1,\dots,n$, for all $u\in V_{q_i}(\mathbb{R}^N)$, $u>0$, for all $\phi\in V_{q_i}(\mathbb{R}^N)$, $\phi\geq 0$, and all $\mu_i\in\mathbb{R}$, $H'_{\mu_i}(u)((\frac{\phi}{u})^\alpha\phi)-H'_{\mu_i}(\phi)((\frac{\phi}{u})^\alpha u)\leq 0$ with $\alpha\in\mathbb{N},\alpha>0$.*

So we get the last theorem of this section.

Theorem 4.5. (i) *Assume* (H1)–(H4), (H11)-(H13) *are satisfied. For $i=1,\dots,n$, if $\Omega_{i,+}=\{x\in\mathbb{R}^N, a_{ij_i}(x)>0\}$ is a nonempty, bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega_{i,+}$, then $\lambda_i^*<+\infty$.*

(ii) *Assume* (H1)–(H4), (H11) *are satisfied.*

(1) *We assume here that for all i,j, $f_{ij}=0$. If there exists $i\in\{1,\dots,n\}$ such that for $j\neq i$, there exists $u_j\geq 0$ which satisfies*

$$F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n)\geq 0,$$

then $\lambda_i^*\leq\lambda_i$ and since $\lambda_i^*\geq\lambda_i$ is always satisfied, then $\lambda_i^*=\lambda_i<+\infty$.

(2) *If there exists $u_1\geq 0,\dots,u_n\geq 0$, such that*

$$F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n)<0,$$

then $\lambda_i<\lambda_i^*$.

Note that the condition in Theorem 4.5(ii)(2) is verified if we assume also that the hypothesis (H13) is satisfied and if we take $u_j \geq 0$ such that $supp\, u_j \subset D_i$.

Proof. (i) For $i = 1, \dots, n$ consider the equation $(L_{\rho_i} + q_i)u = \lambda_i m_i u$ defined in $\Omega_{i,+}$ with Dirichlet condition on $\partial\Omega_{i,+}$. We denote by λ_{i+} the first eigenvalue (which is simple and positive) and by ϕ_{i+} the first eigenfunction associated i.e:

$$\begin{gathered} (L_{\rho_i} + q_i)\phi_{i+} = \lambda_{i+} m_i \phi_{i+} \quad \text{in } \Omega_{i,+}, \\ \phi_{i+} > 0 \quad \text{in } \Omega_{i+}, \\ \phi_{i+} = 0 \quad \text{on } \partial\Omega_{i+}. \end{gathered} \tag{4.17}$$

Since $\operatorname{supp} \phi_{i+} \subset \Omega_{i+}$, by the above lemma, we have

$$\text{for all } u_i \in D(\mathbb{R}^N),\ H'_{\lambda_{i+}}(u_i)((\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}) \leq 0$$

i.e. for all $u_i \in D(\mathbb{R}^N)$, $u_i \geq 0$,

$$\frac{\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial((\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+})}{\partial x_k} + q_i u_i (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}]}{\int_{\mathbb{R}^N} m_i u_i (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}} \leq \lambda_{i+} < +\infty. \tag{4.18}$$

Moreover, for all $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \geq 0$,

$$\begin{aligned} &\frac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)((\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}) \\ &= (p+1) \sum_{j=1; j\neq i}^n \int_{\mathbb{R}^N} [a_{ij} u_i^p u_j^q (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+} + f_{ij} u_j^{p+q} (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}] \geq 0 \end{aligned} \tag{4.19}$$

since $\operatorname{supp} \phi_{i+} \subset \Omega_{i+}$ and by the hypotheses (H11)–(H13). So by (4.18) and (4.19), for all $u_i \in V_{q_i}(\mathbb{R}^N)$, $u_i \geq 0$,

$$\begin{aligned} &\inf_{\phi \in D(\mathbb{R}^N)} \Big\{ \frac{\int_{\mathbb{R}^N} \sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial \phi}{\partial x_k} + q_i u_i \phi}{\int_{\mathbb{R}^N} m_i u_i \phi},\ \phi \geq 0 \text{ such that for } j = 1, \dots, n, j \neq i, \\ &\text{there exists } v_j \in V_{q_j}(\mathbb{R}^N),\ v_j \geq 0 \text{ and } \frac{\partial F_i}{\partial u_i}(v_1, \dots, u_i, \dots, v_n)(\phi) \geq 0 \Big\} \\ &\leq \frac{\int_{\mathbb{R}^N} [\sum_{k,j=1}^N \rho_{kj,i} \frac{\partial u_i}{\partial x_j} \frac{\partial((\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+})}{\partial x_k} + q_i u_i (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}]}{\int_{\mathbb{R}^N} m_i u_i (\frac{\phi_{i+}}{u_i})^\alpha \phi_{i+}} \leq \lambda_{i+} < +\infty. \end{aligned}$$

Therefore, $\lambda_i^* \leq \lambda_{i+} < +\infty$.
(ii) For the first claim, we assume that for i, j, $f_{ij} = 0$. So we have: for all $u_1, \dots, u_n$,

$$F_i(u_1, \dots, u_n) = \sum_{j=1; j\neq i}^n \int_{\mathbb{R}^N} a_{ij} u_i^{p+1} u_j^q \tag{4.20}$$

and for all ϕ,

$$\frac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)(\phi) = (p+1) \sum_{j=1; j\neq i}^n \int_{\mathbb{R}^N} a_{ij} u_i^p u_j^q \phi. \tag{4.21}$$

We suppose that: for $j \neq i$ there exists $u_j \geq 0$ such that

$$F_i(u_1, \dots, u_{i-1}, \phi_i, u_{i+1}, \dots, u_n) \geq 0. \tag{4.22}$$

We want to prove that $\lambda_i^* \le \lambda_i$. We use again the above lemma with $\alpha = p$. We have $H'_{\lambda_i}(\phi_i)((\frac{\phi_i}{u_i})^p u_i) = 0$ for all $u_i > 0$. So for all $u_i \ge 0$, $H'_{\lambda_i}(u_i)((\frac{\phi_i}{u_i})^p \phi_i) \le 0$ i.e.,

$$\frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial u_i}{\partial x_j}\frac{\partial((\frac{\phi_i}{u_i})^p u_i)}{\partial x_k} + q_i u_i(\frac{\phi_i}{u_i})^p\phi_i]}{\int_{\mathbb{R}^N} m_i u_i(\frac{\phi_i}{u_i})^p\phi_i} \le \lambda_i < +\infty. \tag{4.23}$$

Moreover, using (4.20)–(4.22), for all $u_i > 0$, for all $j \ne i$, there exists $u_j \ge 0$,

$$\begin{aligned}
&\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_{i-1},u_i,u_{i+1},\dots,u_n)((\frac{\phi_i}{u_i})^p\phi_i)\\
&= (p+1)\sum_{j=1;j\ne i}^{n}\int_{\mathbb{R}^N} a_{ij}u_i^p u_j^q(\frac{\phi_i}{u_i})^p\phi_i\\
&= (p+1)F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n) \ge 0.
\end{aligned} \tag{4.24}$$

Since

$$\begin{aligned}
&\inf_{\phi\in D(\mathbb{R}^N)}\Big\{\frac{\int_{\mathbb{R}^N}\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial u_i}{\partial x_j}\frac{\partial \phi}{\partial x_k} + q_i u_i\phi}{\int_{\mathbb{R}^N} m_i u_i\phi},\ \phi\ge 0 \text{ such that}\\
&\text{for } j\ne i,\ \text{ there exists } u_j\in V_{q_j}(\mathbb{R}^N),\ u_j\ge 0 \text{ and } \frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi)\ge 0\Big\}\\
&\le \frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial u_i}{\partial x_j}\frac{\partial((\frac{\phi_i}{u_i})^p \phi_i)}{\partial x_k} + q_i u_i(\frac{\phi_i}{u_i})^p\phi_i]}{\int_{\mathbb{R}^N} m_i u_i(\frac{\phi_i}{u_i})^p\phi_i} \le \lambda_i < +\infty,
\end{aligned}$$

by (4.23) and (4.24), we get that $\lambda_i^* \le \lambda_i$ and therefore $\lambda_i^* = \lambda_i$.

For the second claim, we assume that there exists $u_1 \ge 0,\dots,u_n \ge 0$ such that

$$F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n) < 0. \tag{4.25}$$

We denote by

$$\begin{aligned}
\lambda_i^- = \inf_{\phi\in D(\mathbb{R}^N),\,\phi\ge 0}\big\{&\frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k} + q_i|\phi|^2]}{\int_{\mathbb{R}^N} m_i|\phi|^2},\\
&\phi \text{ such that } F_i(u_1,\dots,u_{i-1},\phi,u_{i+1},\dots,u_n) > 0\big\}.
\end{aligned} \tag{4.26}$$

Let

$$W_i = \big\{\phi\in V_{q_i}(\mathbb{R}^N), \phi\ge 0,\ F_i(u_1,\dots,u_{i-1},\phi,u_{i+1},\dots,u_n) > 0\big\}.$$

Since $W_i \subset V_{q_i}(\mathbb{R}^N)$, we have $\lambda_i \le \lambda_i^-$. Since $\phi_i \notin W_i$, by the continuity of the function F_i, we deduce that $\lambda_i < \lambda_i^-$ (see (1.7) and (4.26).

We prove now that $\lambda_i^- \le \lambda_i^*$. As in [15], we prove that there exists $u_i^- \in W_i$,

$$\lambda_i^- = \frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial u_i^-}{\partial x_j}\frac{\partial u_i^-}{\partial x_k} + q_i|u_i^-|^2]}{\int_{\mathbb{R}^N} m_i|u_i^-|^2}. \tag{4.27}$$

After that, we prove that $\lambda_i^- \le \lambda_i^*$. Suppose that $\lambda_i^- > \lambda_i^*$. Then there exists $\phi \in \Phi_{u_i^-}$ such that

$$\frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial u_i^-}{\partial x_j}\frac{\partial \phi}{\partial x_k} + q_i u_i^-\phi]}{\int_{\mathbb{R}^N} m_i u_i^-\phi} \le \lambda_i^* < \lambda_i^-. \tag{4.28}$$

Therefore $H'_{\lambda_i^-}(u_i^-)(\phi) < 0$. Since $F_i(u_1, \dots, u_{i-1}, u_i^-, u_{i+1}, \dots, u_n) > 0$, by continuity we have $F_i(u_1, \dots, u_{i-1}, u_i^- + \eta\phi, u_{i+1}, \dots, u_n) > 0$ for $\eta > 0$ small enough. Moreover, by (4.27) and (4.28) we have $H'_{\lambda_i^-}(u_i^-)(\phi) < 0$ and $H_{\lambda_i^-}(u_i^-) = 0$, we can choose $\eta > 0$ small enough such that $H_{\lambda_i^-}(u_i^- + \eta\phi) < 0$. So we obtain that:

$$\frac{\int_{\mathbb{R}^N}[\sum_{k,j=1}^N \rho_{kj,i}\frac{\partial(u_i^-+\eta\phi)}{\partial x_j}\frac{\partial(u_i^-+\eta\phi)}{\partial x_k} + q_i(u_i^- + \eta\phi)^2]}{\int_{\mathbb{R}^N} m_i(u_i^- + \eta\phi)^2} < \lambda_i^-$$

and this contradicts the definition of λ_i^-. Therefore $\lambda_i^- \le \lambda_i^*$. □

5. Remarks

We give several remarks to our conditions and results.

First, note that all the results presented in this paper are true even in a bounded domain Ω with Dirichlet boundary conditions. Indeed, if we assume that for all i, $q_i \in L^\infty(\Omega)$ and $q_i \ge 0$, if we still define the set $V_{q_i}(\Omega)$ as the completion of $D(\Omega)$ under the norm $\|u\|_{q_i} = \big(\int_\Omega[|\nabla u|^2 + q_i u^2]\big)^{1/2}$, we have $V_{q_i}(\Omega) = H_0^1(\Omega)$. Therefore, we can define the eigenpair (λ_i, ϕ_i) by the Courant-Fischer formulas as above.

Second, the case where for all i, $q_i = 0$ in $\mathbb{R}^N$ is quite different. Indeed, if we define the set $V_{q_i}(\mathbb{R}^N)$ as the completion of $D(\mathbb{R}^N)$ under the norm $\|u\|_{q_i} = \big(\int_{\mathbb{R}^N}|\nabla u|^2\big)^{1/2}$ then $V_{q_i}(\mathbb{R}^N) = D^{1,2}$ and we lose the compactness of the embedding of $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. So this case requires the introduction of other weight-spaces depending on the weights m_i and therefore requires other hypotheses upon the weights m_i. For example, we recall from [10] that the equation

$$\begin{gathered} -\Delta u = \lambda m_i u \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to +\infty \end{gathered}$$

admits a positive principle eigenvalue $\lambda(m_i)$, associated with a positive eigenfunction ϕ_{m_i} such that

$$\lambda(m_i)\int_{\mathbb{R}^N} m_i u^2 \le \int_{\mathbb{R}^N}|\nabla u|^2 \quad (\text{ for all } u \in D^{1,2})$$

under the hypotheses that $N \ge 3$, $0 \le m_i \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $m_i \ne 0$. In [8], for example, there are results on the existence of a positive principle eigenvalue associated with a positive eigenfunction for the equation

$$\begin{gathered} -\Delta u = \lambda m_i u \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to +\infty \end{gathered}$$

It is proved that such an eigenvalue exists if m_i is negative and bounded away from 0 at infinity, or if $N \ge 3$ and $|m_i|$ is sufficiently small at infinity (i.e. $|m_i(x)| \le \frac{const}{(1+|x|^2)^\alpha}$ with $\alpha > 1$) but does not exist if $N = 1$ or $N = 2$ and $\int_{\mathbb{R}^N} m_i > 0$. When $N \ge 3$ and the weight m_i satisfies $|m_i(x)| \le \frac{\text{const}}{(1+|x|^2)^\alpha}$, the variational space is defined by $V = \{u : \mathbb{R}^N \to \mathbb{R}, \int_{\mathbb{R}^N}[|\nabla u|^2 + \frac{u^2}{1+|x|^2}] < +\infty\}$. In this last case, under the hypotheses that $N \ge 3$ and $|m_i(x)| \le \frac{const}{(1+|x|^2)^\alpha}$, our results are still valid with the additional conditions that all the parameters μ_i have to be positive (since the equation)

$$-\Delta u = \lambda m_i u \quad \text{in } \mathbb{R}^N$$
$$u(x) \to 0 \quad \text{as } |x| \to +\infty$$

(can admit negative eigenvalues).

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