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ON THE EXACT MULTIPLICITY OF SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS VIA COMPUTING THE DIRECTION OF BIFURCATIONS

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Dedicated to Louis Nirenberg on his 80-th birthday

Abstract. We consider positive solutions of the Dirichlet problem

$$\begin{gathered} u''(x)+\lambda f(u(x))=0 \quad \text{in } (-1,1), \\ u(-1)=u(1)=0. \end{gathered}$$

depending on a positive parameter λ. We use two formulas derived in [18] to compute all solutions u where a turn may occur and to compute the direction of the turn. As an application, we consider quintic a polynomial $f(u)$ with positive and distinct roots. For such quintic polynomials we conjecture the exact mutiplicity structure of positive solutions and present computer assisted proofs of such exact bifurcation diagrams for various distributions of the real roots. The limiting behavior of the solutions on these bifurcation branches as $\lambda \to \infty$ and their stabilities are also investigated.

1. Introduction

We study exact bifurcation diagrams and exact multiplicity of the positive solutions to the Dirichlet problem

$$\begin{gathered} u''(x)+\lambda f(u(x))=0 \quad \text{on } (-1,1), \\ u(-1)=u(1)=0, \end{gathered} \tag{1.1}$$

depending on a positive parameter λ. We recall that solutions of (1.1) are even functions, with $u'(x)<0$ for $x>0$, and hence any solution is uniquely identified by $\alpha=u(0)$, see [15]. Actually, even more is true: the value of $u(0)=\alpha$ uniquely identifies both λ and $u(x)$, as follows easily by scaling λ out of (1.1), and using uniqueness for initial value problems, see Dancer [7]. Hence the solution curves of (1.1) can be faithfully depicted by two-dimensional curves on (λ,α) plane. It is customary to refer to these curves as *bifurcation diagrams*. The Figure 1 below gives

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such a bifurcation diagram for a quintic polynomial with real roots at $0.1, 0.2, 0.4$ and 0.5. The shape of any bifurcation diagram is determined by the *turning points*. In [18] a necessary and sufficient condition on α for the solution to be singular and thus a necessary condition for the turning point to occur is given as follows:

$$G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}} \, d\tau - 2 = 0, \tag{1.2}$$

with $F(u) = \int_0^u f(t)\,dt$. This formula can be used to compute numerically all turning points. At the turning points we are interested in the *turning direction*. It is shown in [18] that the curve tuns to the right in (λ, α) plane if

$$D(\alpha) \equiv \int_0^\alpha f''(u) \Big(\int_u^\alpha f(s)\,ds \Big) \Big(\int_0^u \frac{ds}{\big(\int_s^\alpha f(t)\,dt \big)^{3/2}} \Big)^3 du < 0, \tag{1.3}$$

and the turn is to the left if the opposite inequality is true.

In this paper we investigate the open problem of exact multiplicity in case of a quintic function $f(u)$. See [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 19, 21, 20, 24] for other multiplicity results. We consider positive solutions in case $f(u) = -(u - a)(u-b)(u-c)(u-d)(u-e)$, i.e. it is a quintic whose roots are five distinct positive constants $0 < a < b < c < d < e < \infty$

$$\begin{gathered} u'' + \lambda(u-a)(u-b)(u-c)(u-d)(u-e) = 0 \quad \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{gathered} \tag{1.4}$$

and λ is a positive parameter. We wish to understand *exactly* how many solutions this problem has, and how these solutions are connected, if one varies λ.

This problem when $f(u) = -(u-a)(u-b)(u-c)$, a cubic polynomial, was studied by Smoller and Wasserman [22], who attempted to solve the problem in general, and succeeded in solving it for $a = 0$. Later Wang [23] solved the problem under some restriction on a. Both authors used the phase-plane analysis. And as it is explained in [18] that the approach in [22] could not possibly cover the general case for other cubics. Then Korman, Li and Ouyang [15], [17] used bifurcation theory to attack the problem, but again some restrictions were necessary (all of the above mentioned papers covered more general $f(u)$, behaving like cubic). Finally in [18] the problem is completely solved for all such cubic polynomials as follows:

Theorem 1.1. *Under the condition*

$$\int_a^c f(t)\,dt > 0. \tag{1.5}$$

Then there exists a critical λ_0, such that problem (1.1) *with $f(u) = -(u-a)(u-b)(u-c)$, $0 < a < b < c < \infty$, has exactly one positive solution for $0 < \lambda < \lambda_0$, exactly two positive solutions at $\lambda = \lambda_0$, and exactly three positive solutions for $\lambda_0 < \lambda < \infty$. Moreover, all solutions lie on two smooth solution curves. One of the curves, referred to as the lower curve, starts at $(\lambda = 0, u = 0)$, it is increasing in λ, and $\lim_{\lambda\to\infty} u(x,\lambda) = a$ for $x \in (-1,1)$. The upper curve is a parabola-like curve with exactly one turn to the right.*

For the quintic nonlinear problem (1.4) we conjecture the following result.

Conjecture 1.2. *The quintic nonlinearity problem* (1.4) *under the condition that*

$$\int_a^c f(t)\,dt > 0 \quad \textit{and} \quad \int_c^e f(t)\,dt > 0, \tag{1.6}$$

has the following solution structure: There exist two critical $0 < \lambda_1, \lambda_2$*, such that the problem* (1.1) *with* $f(u) = -(u-a)(u-b)(u-c)(u-d)(u-e)$, $0 < a < b < c < \infty$, *has exactly one positive solution for* $0 < \lambda < \min(\lambda_1, \lambda_2)$*, exactly three positive solutions for* $\min(\lambda_1, \lambda_2) < \lambda < \max(\lambda_1, \lambda_2)$*, exactly five positive solutions for* $\max(\lambda_1, \lambda_2) < \lambda < \infty$*. Moreover, all solutions lie on three smooth solution curves. One of the curves, referred to as the first curve* Γ_0*, starts at* $(\lambda = 0, u = 0)$*, it is increasing in* λ*, and* $\lim_{\lambda\to\infty} u(x,\lambda) = a$ *for* $x \in (-1,1)$*. The two other curves are parabola-like curves with exactly one turn to the right. The second curve* Γ_1 *has a turning point at* λ_1 *with* $\lim_{\lambda\to\infty,\Gamma_1^+} u(0,\lambda) = c$ *while* $\lim_{\lambda\to\infty,\Gamma_1^-} u(0,\lambda) = \gamma_1$*. The third curve* Γ_2 *has a turning point at* λ_2 *with* $\lim_{\lambda\to\infty,\Gamma_2^+} u(0,\lambda) = e$ *while* $\lim_{\lambda\to\infty,\Gamma_2^-} u(0,\lambda) = \gamma_2$*. Where* γ_1 *is the unique root of* $\int_a^{\gamma_1} f(t)\,dt = 0$ *in* (b,c) *and* γ_2 *is the unique root of* $\int_c^{\gamma_2} f(t)\,dt = 0$ *in* (d,e)*.*

Conditions (1.6) follow from [12], who showed first that (1.6) was a sufficient condition for the existence of positive solutions for (1.1). In [8] it was showed that (1.6) was also a necessary condition for the existence of positive solution for the problem. We provide a computer assisted proof for several groups of parameters (a,b,c,d,e), thus giving a complete solution to the problem (1.4) for these parameter groups. The following is our result on exact multiplicity where the parameter groups are defined as follows.

$$
\begin{aligned}
P_1 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.4t, 0.5t, t), t > 0\};\\
P_2 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.4t, 0.6t, t), t > 0\};\\
P_3 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.5t, 0.6t, t), t > 0\};\\
P_4 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.5t, 0.7t, t), t > 0\};\\
P_5 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.6t, 0.7t, t), t > 0\};\\
P_6 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.6t, 0.8t, t), t > 0\};\\
P_7 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.1t, 0.2t, 0.7t, 0.8t, t), t > 0\};\\
P_8 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.5t, 0.6t, t), t > 0\};\\
P_9 &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.5t, 0.7t, t), t > 0\};\\
P_{10} &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.6t, 0.7t, t), t > 0\};\\
P_{11} &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.6t, 0.8t, t), t > 0\};\\
P_{12} &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.7t, 0.8t, t), t > 0\};\\
P_{13} &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.2t, 0.3t, 0.8t, 0.9t, t), t > 0\};\\
P_{14} &\equiv \{(a,b,c,d,e) \in \mathbb{R}^5 : (a,b,c,d,e) = (0.3t, 0.4t, 0.7t, 0.8t, t), t > 0\}.
\end{aligned}
\tag{1.7}
$$

Theorem 1.3. *For each parameter group defined above in (1.7)* $P_i, i = 1\dots 14$*, there is an open neighborhood* U_i *of* $P_i, i = 1\dots 14$ *in* $\mathbb{R}^5$ *such that the above conjecture holds for* $(a,b,c,d,e) \in U_i$*,* $i = 1\dots 14$*.*

Our proof of this theorem is based on two numerical computations, which would constitute a "traditional" proof if their results could be analytically justified. Our first computation shows that for each of the above parameter groups, ranges of possible $\alpha = u(0)$, $G(\alpha)$ in (1.2) is very narrow and close to 0 (including numerical errors), which identify the ranges of $\alpha = u(0)$, where bifurcation might have

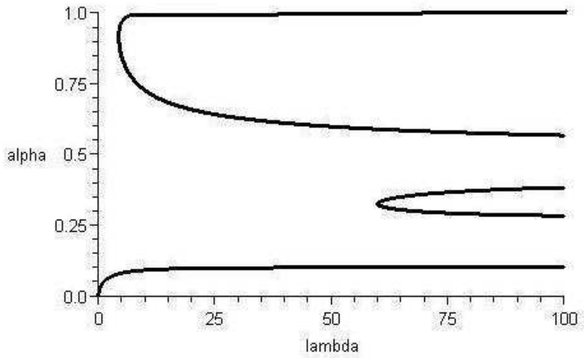

FIGURE 1. Solution with $u(0)$ =alpha vs. λ =lambda, where $a = 0.1$, $b = 0.2$, $c = 0.4$, $d = 0.5$, $e = 1$ with 10000 points

occurred. Our second computation shows that in these ranges of $\alpha = u(0)$ the function $D(\alpha)$ in (1.3) is negative , which means that only turns to the right is possible. These two computations together prove the conjecture for these parameter groups stated in the theorem above.

The paper is organized as follows: In §2 limiting behavior of the solutions on these bifurcation branches and their stabilities are investigated; (1.2) and (1.3) are derived and proved in [18] but for the sake of completeness in 3 we will provide a derivation of (1.3) and in 4 we will provide a derivation of (1.2) following [18].

2. Limiting behavior and the stability of the branches

We know from Theorem 1.3 that all the positive solutions for the quintic problem (1.1) lie on three smooth solution curves for these open sets $G_i, i = 1 \dots 14$. These curves are denoted by Γ_0, which is the curve that start at $(0, 0)$ and it is increasing in λ;the other two curves, which we denoted by Γ_1, and Γ_2 are parabola-like curves, with exactly one turn to the right. We denote Γ_i^+ and Γ_i^- to be the upper and lower branch of Γ_i, respectively for $i = 1, 2$. Although we provide a computer assisted proof for this theorem, we also are able to prove the limiting behavior results of the lower and upper branches of the solution curves inspired by the results in [15].

Theorem 2.1. *The first curve Γ_0, starts at $(\lambda = 0, u = 0)$, it is increasing in λ, and $\lim_{\lambda\to\infty} u(x, \lambda) = a$ for $x \in (-1, 1)$. The second curve Γ_1 has a turning point at λ_1 with $\lim_{\lambda\to\infty,\ \Gamma_1^+} u(0, \lambda) = c$ while $\lim_{\lambda\to\infty,\Gamma_1^-} u(0, \lambda) = \gamma_1$. The third curve Γ_2 has a turning point at λ_2 with $\lim_{\lambda\to\infty,\Gamma_2^+} u(0, \lambda) = e$ while $\lim_{\lambda\to\infty,\ \Gamma_2^-} u(0, \lambda) = \gamma_2$. Where γ_1 is the unique root of $\int_a^{\gamma_1} f(t)\,dt = 0$ in (b, c) and γ_2 is the unique root of $\int_c^{\gamma_2} f(t)\,dt = 0$ in (d, e).*

Proof. We can view Γ_0 as a curve of solution emanating from the solution $(u = 0, \lambda = 0)$ as a consequence of the Implicit Function Theorem. By the standard result in sub-super solution (subsolution $\leq$ supersolution) we have that the solution curve stays below a, and since $f'(u) < 0$ for $u, < a$ by the Implicit Function Theorem we can continue the curve for all $\lambda > 0$. Differentiating (1.1) with respect to λ to obtain that $u_\lambda > 0$, as is was showed in [15]. Hence, we conclude that the curve tends to a as $\lambda \to \infty$.

From our result above we have that at the points (λ_1, u_1), and (λ_2, u_2) the curves of solutions turns to the right in the λu-plane, thus obtaining two parabolas-like curves. First, we will show that Γ_1^+ is increasing for all $\lambda > \lambda_1$. Assume the contrary, and let λ^* be the first value such that $u_\lambda(\lambda^*, x_1) = 0$ for some $x_1 \in (0, 1)$. It is easy to see that x_1 is a minimum point of u_λ, and $u''_\lambda(\lambda^*, x_1) \geq 0$. Differentiating (1.1) with respect to λ and with respect to x:

$$u''_\lambda + f(u) + \lambda f'(u) u_\lambda = 0, \tag{2.1}$$

$$u''_x + \lambda f'(u) u_x = 0. \tag{2.2}$$

From (2.1) and the fact that $u_\lambda(\lambda^*, x_1) = 0$ and that $u''_\lambda(\lambda^*, x_1) \geq 0$, we have $f(u(\lambda^*, x_1)) \leq 0$, which implies that $u(\lambda^*, x_1) \in [a, b]$. Multiplying (2.1) by u_x and (2.2) by u_λ and subtracting and integrating from 0 to x_1 we obtain:

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^{x_1} + \int_0^{x_1} f(u) u_x dx = 0 \tag{2.3}$$

Simplifying (2.3) to obtain,

$$u''(0) u_\lambda(0) + \int_{u(0)}^{u(x_1)} f(u) u_x dx = 0 \tag{2.4}$$

Observe that the first term of (2.4) is less or equal to 0, it follows that the second term must be positive. As we have mentioned that a necessary condition for $u(0)$ is $\int_a^{u(0)} f(u)du > 0$,which implies that $\int_{u(x_1)}^{u(0)} f(u)du > \int_a^{u(0)} f(u)du > 0$. Hence the second term in (2.4) is negative, which is a contradiction, and therefore the upper branch is increasing. Furthermore, the upper branch is bounded by c, in that case as $\lambda \to \infty$ the limit must exists. For $x \in (-1, 1)$ this limit need to be b or c, but u is convex below b and since the upper branch is increasing cannot converge towards b. Hence the $\lim_{\lambda\to\infty} \Gamma_1^+ = c$. Similar argument show the limiting behavior of the upper branch of the third curve.

For the lower branch of Γ_1, first recall that γ_1 is the unique solution of equation $\int_a^{\gamma_1} f(u)du = 0$ in (b, c). From the analysis in [18] it was shown that the bifurcation can only occur in the interval $(\max\{\beta_1, \gamma_1\}, c)$, which implies that Γ_1^- is bounded below by γ_1, where β_1 is the unique root of $f'(\beta_1) = \frac{f(\beta_1)}{\beta_1 - a}$ in (b, c). By a similar argument used for the upper branch case, we will show that the lower branch is decreasing in λ at $x = 0$. By assuming the contrary, let λ_* be the first value so that $u_\lambda(0, \lambda_*) = 0$. By (1.1) we can conclude that $u''_\lambda(0, \lambda_*) < 0$ thus $x = 0$ is not a minimum for $u_\lambda(x, \lambda_*)$, so $u_\lambda(x, \lambda_*) < 0$ for $x > 0$, but closed to 0.

Multiplying (2.1) by u_x and (2.2) by u_λ and subtracting and integrating from 0 to 1 we obtain

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^1 + \int_{u(0)}^0 f(u)du = 0 \tag{2.5}$$

We proved earlier that the integral is negative, and the first term simplify to $u'_\lambda(1)u'(1)$. Hence, $u'_\lambda(1, \lambda_*) < 0$, thus $u_\lambda(x, \lambda_*)$ is positive near 1. Then $u_\lambda(x, \lambda_*)$ must have a zero in $(0, 1)$. Let x_1 be the smallest zero. Now, multiplying (2.1) by

u_x and (2.2) by u_λ and subtracting and integrating from 0 to x_1 we obtain:

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^{x_1} + \int_{u(0)}^{u(x_1)} f(u)du = 0 \tag{2.6}$$

It is easy to observe, by the same arguments that were used before that both terms of the equation are negative, which is a contradiction. Therefore the lower branch Γ_1^- is decreasing in λ at $x = 0$. And similarly the lower branch Γ_2^- can be shown to be decreasing in λ at $x = 0$. □

Theorem 2.2. *Solution on the upper branch of Γ_i are stable, while solution on the lower branch of Γ_i are unstable, for $i = 1, 2$.*

Proof. First it is easy to show that Γ_0 is stable since $f'(u) < 0$ in $(0, a)$. Next assume to the contrary that $u = u(\lambda, x)$ is a solution on the upper branch Γ_1^+ that is unstable. That is, we can find a constant $\mu > 0$, and $w(x) > 0$, such that

$$\begin{gathered} w'' + \lambda f'(u)w = \mu w \quad \text{on } (-1, 1), \\ w(-1) = w(1) = 0 \end{gathered} \tag{2.7}$$

We may assume that $\int_0^1 w^2 dx = 1$. Observe that multiplying (1.1) by u' and integrating over $(0, x)$ we obtain

$$|u'| \ \geq \sqrt{\lambda} K \tag{2.8}$$

for some $K > 0$ when λ is large, for all $x \in (\eta, 1)$, where $u(\eta) = \alpha$ and α is the largest root of $f'(u)$ in (b, c). Recall that $u \to c$, we can find a constant M and $\xi = \xi(\lambda)$ near 0, such that $|u''(\xi)| \leq M$. Next, differentiate (1.1) and multiply by w, and multiply (2.7) by u', and subtracting the equations to obtain,

$$u''_x w - w'' u' = \mu w u' \tag{2.9}$$

Integrating this equation over ξ to 1, we have

$$-u'(1)w(1) - u''(\xi)w(\xi) + u'(\xi)w'(\xi) + \mu \int_\xi^1 wu'dx = 0\,. \tag{2.10}$$

Recall that w is bounded, and $w'' > 0$ on $(0, 1)$, it follows that $w'(\xi) > 0$. Thus we know that the second term in (2.10) is bounded, and the third and fourth term are negative. Assume the first term is positive and small, that is $|w'(1)| = O\big(\frac{1}{\sqrt{\lambda}}\big)$. Since $w(x)$ is convex on $(0, \eta)$, it follows that it must attain its maximum on $(\eta, 1)$, and since $\int_0^1 w^2 dx = 1$ the maximum is at least 1. Changing the variable to $t = 1-x$, we have from (2.7) the following estimate

$$w(t) \leq c\lambda w, w(0) = 0, w'(0) = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad 0 < t < \frac{c_1}{\sqrt{\lambda}} \tag{2.11}$$

For some positive constants c, and c_1. Thus, integrating we obtain

$$w(t) \leq K_1\Big(\sqrt{\lambda}\int_0^t w(s)ds + \frac{1}{\lambda}\Big)\,. \tag{2.12}$$

Therefore, applying the Gronwall inequality we have that $w(t) = O\big(\frac{1}{\lambda}\big)$, which is a contradiction.

Computing the direction of bifurcation

We consider positive solutions of the Dirichlet problem

$$\begin{gathered} u''(x) + f(u(x)) = 0 \quad \text{in } (-1,1), \\ u(-1) = u(1) = 0. \end{gathered} \tag{2.13}$$

It is well known that any solution $u(x)$ is an even function, with a unique point of maximum at $x = 0$, and $u'(x) < 0$ on $(0,1)$. We assume that $u(x)$ is a *singular solution* of (2.1), i.e. the corresponding linearized problem

$$\begin{gathered} w''(x) + f'(u(x))w = 0 \quad \text{on } (-1,1), \\ w(-1) = w(1) = 0 \end{gathered} \tag{2.14}$$

admits a non-zero solution $w(x)$. It is also well known that $w(x)$ may be assumed to be positive, and it is an even function on $(-1,1)$, see [15]. For the singular solutions the following integral is important

$$J \equiv -\int_{-1}^{1} f''(u(x))w^3(x)\,dx = -2\int_0^1 f''(u(x))w^3(x)\,dx. \tag{2.15}$$

If $J \neq 0$ the critical point is *non-degenerate*; i.e. it persists under small perturbations of the equation in (2.1), see [7]. For the problem (1.1) depending on the parameter λ, the sign of J determines the direction of bifurcation at the critical point in the $(\lambda, u(0))$ plane. If $J > 0$ the curve turns to the right, and if $J < 0$ to the left. We derive here a formula for J, which does not require a detailed knowledge of $u(x)$, and any knowledge of $w(x)$. It depends only on the maximal value of the critical solution $u(0) \equiv \alpha$.

Theorem 2.3 ([18]). *At any critical solution $u(x)$, with $u(0) = \alpha$,*

$$J = -c\int_0^{\alpha} f''(u)\Big(\int_u^{\alpha} f(s)\,ds\Big)\Big(\int_0^u \frac{ds}{\big(\int_s^{\alpha} f(t)\,dt\big)^{3/2}}\Big)^3\,du = -cD(\alpha),$$

where $c = \frac{1}{4\sqrt{2}}u'^3(1)w'^3(1) > 0$.

Proof. Differentiating equation (2.1),

$$u''(x) + f'(u(x))u'(x) = 0 \quad \text{in } (-1,1). \tag{2.16}$$

Using this equation and (2.2), we conclude that the function $u''(x)w(x) - u'(x)w'(x)$ is constant, and hence

$$u''(x)w(x) - u'(x)w'(x) = -C, \quad \text{where } C = u'(1)w'(1) > 0. \tag{2.17}$$

We rewrite this as

$$\Big(\frac{w}{u'}\Big)' = \frac{C}{u'^2},$$

and then integrate, concluding that

$$w(x) = -Cu'(x)\int_x^1 \frac{1}{u'^2(t)}\,dt. \tag{2.18}$$

This formula will allow us to exclude $w(x)$ in J. (Observe that $\int_x^1 \frac{1}{u'^2(t)}\,dt$ tends to infinity as $x \to 0$, while $u'(x)$ tends to zero. Hence both terms ought to be kept together in numerical computations.) Using (2.8) in (2.3)

$$J = 2C^3\int_0^1 f''(u(x))u'^3(x)\Big(\int_x^1 \frac{1}{u'^2(t)}\,dt\Big)^3\,dx. \tag{2.19}$$

We now wish to exclude $u'(x)$ from (2.9). Since the energy $\frac{u'^2}{2}(x)+F(u(x))$ is constant,
$$\frac{u'^2}{2}(x)+F(u(x))=F(u(0))=F(\alpha). \tag{2.20}$$
On the interval $(0,1)$ we express
$$u'(x)=-\sqrt{2}\sqrt{F(\alpha)-F(u(x))}. \tag{2.21}$$
We use this formula in the integral $\int_x^1 \frac{1}{u'(t)^2}\,dt$, and then we make a change of variables $t\to s$, by letting $s=u(t)$. We have
$$\int_x^1 \frac{1}{u'^2(t)}\,dt=-\frac{1}{2^{3/2}}\int_x^1 \frac{u'(t)\,dt}{[F(\alpha)-F(u(t))]^{3/2}}=-\frac{1}{2^{3/2}}\int_{u(x)}^0 \frac{1}{[F(\alpha)-F(s)]^{3/2}}\,ds.$$
We then have
$$J=c\int_0^1 f''(u(x))u'(x)\,[F(\alpha)-F(u(x))]\Big(\int_0^{u(x)}\frac{1}{[F(\alpha)-F(s)]^{3/2}}\,ds\Big)^3\,dx,$$
with $c=\frac{1}{4\sqrt{2}}C^3$. Finally, replacing $F(\alpha)-F(u)$ by $\int_u^\alpha f(s)\,ds$, making a change of variables $u=u(x)$, and writing α for $u(0)$, we obtain (2.4). $\square$

Computing the bifurcation points

In the previous section we computed the direction of turn, assuming that bifurcation occurs at $u(0)=\alpha$. We now provide a way to determine all possible α's at which bifurcation may occur, i.e. the corresponding solution of (2.1) is singular.

Theorem 2.4 ([18]). *A solution of the problem* (2.1) *with the maximal value* $\alpha=u(0)$ *is singular if and only if*
$$G(\alpha)\equiv F(\alpha)^{1/2}\int_0^\alpha \frac{f(\alpha)-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau-2=0. \tag{2.22}$$

Proof. We need to show that the problem (2.2) has a non-trivial solution. As follows by (2.8) (or by direct verification) the function $w(x)=-u'(x)\displaystyle\int_x^1\frac{1}{u'^2(t)}\,dt$ satisfies the equation in (2.2). Also $w(1)=0$. If we also have
$$w'(0)=0 \tag{2.23}$$
then since $u(x)$ is an even function, the same is true for $w(x)$ (by uniqueness for initial value problems), and hence $w(-1)=0$, which gives us a non-trivial solution of (2.2). Conversely, every non-trivial solution of (2.2) is an even function, and hence (2.23) is satisfied.

Using the equation in (2.1), we compute
$$w'(x)=f(u(x))\int_x^1\frac{1}{u'^2(t)}\,dt+\frac{1}{u'(x)}.$$
Using the formula (2.11) from the previous section and the one right below it, we express
$$2^{3/2}w'(x)=\int_0^{u(x)}\frac{f(u(x))}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau-\frac{2}{[F(\alpha)-F(u(x))]^{1/2}}. \tag{2.24}$$

If we try to set here $x = 0$, then both terms on the right are infinite. Instead, we observe that

$$
\begin{aligned}
-\frac{2}{[F(\alpha)-F(u)]^{1/2}} &= -\int_0^u \frac{d}{d\tau}\frac{2}{[F(\alpha)-F(\tau)]^{1/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}} \\
&= -\int_0^u \frac{f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}}.
\end{aligned}
\tag{2.25}
$$

Using (2.25) in (2.24), we obtain

$$
2^{3/2}w'(x) = \int_0^{u(x)} \frac{f(u(x))-f(\tau)}{[F(\alpha)-F(\tau)]^{3/2}}\,d\tau - \frac{2}{F(\alpha)^{1/2}}. \tag{2.26}
$$

The integral on the right is now non-singular as we let $x \to 0$. At $x = 0$ we see that (2.23) is equivalent to (2.22). □

A computer assisted proof

For the quintic nonlinearity problem (1.4), by letting $u = ev$, we may assume that $e = 1$, so that our nonlinearity is $f(u) = -(u-a)(u-b)(u-c)(u-d)(u-1)$, with new a, b, c and d, i.e. we consider

$$
\begin{gathered}
u'' + \lambda(u-a)(u-b)(u-c)(u-d)(u-1) = 0 \quad \text{in } (-1,1), \\
u(-1) = u(1) = 0.
\end{gathered}
\tag{2.27}
$$

This substitution allows us to "compactify" the parameter set, since now $0 < a < b < c < d < 1$. We define the functions $F(u) = \int_0^u f(t)\,dt$. It is well-known ([17, 16]) that for existence of positive solutions it is necessary that $\int_a^c f(t)\,dt, \int_c^1 f(t)\,dt > 0$. Now let γ_1 be the unique root of $\int_a^{\gamma_1} f(t)\,dt = 0$ in (b, c) and γ_2 be the unique root of $\int_c^{\gamma_2} f(t)\,dt = 0$ in $(d, 1)$. Next let β_1 be the unique root of $f'(\beta_1) = \frac{f(\beta_1)}{\beta-a}$ in (b, c) and β_2 be the unique root of $f'(\beta_2) = \frac{f(\beta_2)}{\beta-c}$ in $(d, 1)$ (the point where a straight line through the point $(a, 0)$ touches the graph of $y = f(u)$ for $u \in (a, c)$ and where a straight line through the point $(c, 0)$ touches the graph of $y = f(u)$ for $u \in (c, 1)$.)

We parameterize the solutions of (2.27) by their maximum value $\alpha = u(0)$. For each parameter group in Theorem 1.3 we compute β_i and $\gamma_i, i = 1, 2$. Since the bifurcation could only occur for $\alpha \in (\max(\beta_1, \gamma_1), c)$ and for $\alpha \in (\max(\beta_2,\gamma_2), 1)$, we compute G defined by (1.2) for $\alpha \in (\max(\beta_1, \gamma_1), c) \cup (\max(\beta_2, \gamma_2), 1)$ and produce two ranges of $\alpha = u(0)$ in each case (including numerical errors). Next for these ranges we compute the integral D defined by (1.3). Actually we have computed

$$
II \equiv F(\alpha)^{5/2}\int_0^\alpha f''(u)\Big(\int_u^\alpha f(s)\,ds\Big)\Big(\int_0^u \frac{ds}{\big(\int_s^\alpha f(t)\,dt\big)^{3/2}}\Big)^3 du,
$$

where the extra term $F(\alpha)^{5/2}$ is introduced to make this integral scaling invariant in f. Our computation shows that $II < 0$, which implies that only turns to the right are possible near in these ranges. Since both G and D are continuous in the parameters (a, b, c, d, e) and $D \neq 0$ implies that the zero of D is not degenerate, there exists an open neighborhood U_i for each parameter group $P_i, i = 1 \dots 14$.

As an example, when $(a, b, c, d, e) = (0.1, 0.2, 0.4, 0.5, 1)$, we find that

$$
\max(\beta_1, \gamma_1) = 0.277989142, \quad \max(\beta_2, \gamma_2) = 0.83216494
$$

so that G is evaluated in $(0.277989142, 0.4) \cup (0.83216494, 1)$. We find the ranges of where G is "close" to 0 are $(0.32560, 0.32562) \cup (0.91744, 0.91746)$ and compute the values of D there to be negative in $100's$ and in $10's$. $\square$

References

[1] I. Addou, *Multiplicity results for classes of one-dimensional p-Laplacian boundary-value problems with cubic-like nonlinearities*, Electron. J. Differential Equations 200, No. 52, (2000), 1-26.

[2] K. J. Brown and H. Budin, *On the existence of positive solutions for a class of semilinear elliptic boundary value problems*, SIAM J. Math. Anal. 10, no. 5, 875–883 (1979).

[3] A. Castro, *Uniqueness of positive solutions for a sublinear Dirichlet problem. Nonlinear functional analysis and its applications, Part 1* (Berkeley, Calif., 1983), 243–251, Proc. Sympos. Pure Math., 45, Part 1, Amer. Math. Soc., Providence, RI, 1986.

[4] A. Castro, M. Hassanpour and R. Shivaji, *Uniqueness Of Nonnegative Solutions For A Semipositone Problem With Concave Nonlinearity*, Communications In Partial Differential Equations, 20 no. 11-12, 1927–1936 (1995).

[5] K.C. Chang, *Morse theory on Banach space and its applications to partial differential equations*, Chinese Ann. Math. Ser. B, **4**, no. 3, 381-399 (1983).

[6] K. C. Chang, *Infinite-dimensional Morse Theory and Multiple Solution Problems.* Progress in Nonlinear Differential Equations and their Applications, **6**. Birkhuser Boston, Inc., Boston, MA, 1993.

[7] E. N. Dancer, *On the structure of solutions of an equation in catalysis theory when a parameter is large*, J. Differential Equations, **37**, 404-437 (1980).

[8] E. N. Dancer and K. Schmitt, *On positive solutions of semilinear elliptic equations*, Proc. Amer. Math. Soc., 101, no. 3, 445-452 (1987)

[9] Y.H. Du and Y. Lou, *Proof of a conjecture for the perturbed Gelfand equation from combustion theory*, J. Differential Equations, 173, no. 2, 213–230 (2001).

[10] Y. H. Du and Y. Lou, *S-shaped global bifurcation curve and Hopf bifurcation of positive solutions to a predator-prey model*, J. Differential Equations, 144, no. 2, 390–440 (1998).

[11] A. Galstyan, P.L. Korman and Y. Li, *On the oscillations of the solution curve for a class of semilinear equations*, Journal of Mathematical Analysis and Applications, accepted.

[12] P. Hess, *On multiple positive solutions of nonlinear elliptic eigenvalue problems*, Comm. Partial Differential Equations, 6, no. 8, 951–961 (1981).

[13] X.-J. Hou, P. Korman and Y. Li, *On the direction of pitchfork bifurcation*, Proceeding of the MSU-UAB Conference, Electron. J. Diff. Eqns, submitted.

[14] P. Korman and Y. Li, *On the exactness of an S-shaped bifurcation curve*, Proc. Amer. Math. Soc., **127**, no. 4, 1011-1020, (1999).

[15] P. Korman, Y. Li and T. Ouyang, *Exact multiplicity results for boundary-value problems with nonlinearities generalizing cubic*, Proc. Royal Soc. Edinburgh, Ser. A, **126A**, 599-616 (1996).

[16] P. Korman, Y. Li and T.-C. Ouyang, *An exact multiplicity result for a class of semilinear equations*, Comm. Partial Differential Equations, 22(3,4), 661-684 (1997).

[17] P. Korman, Y. Li and T. Ouyang, *Perturbation of global solution curves for semilinear problems*, Adv. Nonlinear Stud., **3** no. 2, 289-299 (2003).

[18] P. Korman, Y. Li and T. Ouyang, *Computing the location and the direction of bifurcation*, Mathematical Research Letters, accepted.

[19] N. Mizoguchi and T. Suzuki, *Equations of gas combustion: S-shaped bifurcation and mushrooms*, J. Differential Equations, 134, no. 2, 183–215 (1997).

[20] JP Shi and R Shivaji, *Exact Multiplicity Of Solutions For Classes Of Semipositone Problems With Concave-Convex Nonlinearity*, Discrete And Continuous Dynamical Systems, 7 no. 3, 559–571 (2001).

[21] R. Shivaji, *Remarks On An S-Shaped Bifurcation Curve*, Journal Of Mathematical Analysis And Applications, 111, 374–387 (1985)

[22] J. Smoller and A. Wasserman, *Global bifurcation of steady-state solutions*, J. Differential Equations, **39**, 269-290 (1981).

[23] S.-H. Wang, *A correction for a paper by J. Smoller and A. Wasserman*, J. Differential Equations, **77**, 199-202 (1989).

[24] S.-H. Wang, *On S-shaped bifurcation curves*, Nonlinear Anal. TMA, **22**, no. 12, 1475-1485 (1994).

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