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ON THE DIRECTION OF PITCHFORK BIFURCATION

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Abstract. We present an algorithm for computing the direction of pitchfork bifurcation for two-point boundary value problems. The formula is rather involved, but its computational evaluation is quite feasible. As an application, we obtain a multiplicity result.

1. Introduction

We study positive, negative, and sign-changing solutions for the problem

$$\begin{gathered} u''(x) + f(u(x)) - \lambda = 0, \quad \text{for } -1 < x < 1, \\ u(-1) = u(1) = 0. \end{gathered} \tag{1.1}$$

Our principal example will be $f(u) = u^{2k}$, with an integer $k \geq 2$, although our result is considerably more general. For $k = 1$ this problem was exhaustively analyzed in Scovel's Ph.D. thesis [7], and in McKean and Scovel [5]. They used explicit integration via elliptic functions, which means that their method does not work for $k > 1$. Anuradha and Shivaji [1] have studied a related problem. Using the quadrature technique, they showed existence of infinitely many points of bifurcation. Korman [3] has used bifurcation theory to approach the problem (1.1), and in particular the case of $f(u) = u^{2k}$, with $k > 1$. He was able to generalize some, but not all, of the results of McKean and Scovel [5]. One of the difficulties involved the direction of pitchfork bifurcation, which is the subject of the present paper.

Let us briefly review part of what is known for this problem in case of $f(u) = u^{2k}$, see [3] for more details. When $\lambda = 0$, the problem has a unique positive solution. This solution continues for a while when $\lambda > 0$. At a critical $\lambda = \lambda_0$ the positive solution develops zero slope at the boundary, i.e. $u'(-1) = u'(1) = 0$, and a pitchfork bifurcation occurs at $\lambda = \lambda_0$. Namely, we have a symmetric sign-changing solution for $\lambda > \lambda_0$, and a parabola-like family of asymmetric solutions. One of these solutions is negative near the $x = -1$ end and positive on the rest of the interval $(-1, 1)$, while the other one is negative near $x = 1$ end, see Figure 1. The issue is: which way this parabola-like curve of asymmetric solutions bifurcates, is

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it for $\lambda > \lambda_0$, or toward decreasing λ? (The numerical evidence of Ramaswamy [6], and of Korman [2] suggests that the pitchfork opens forward in λ.)

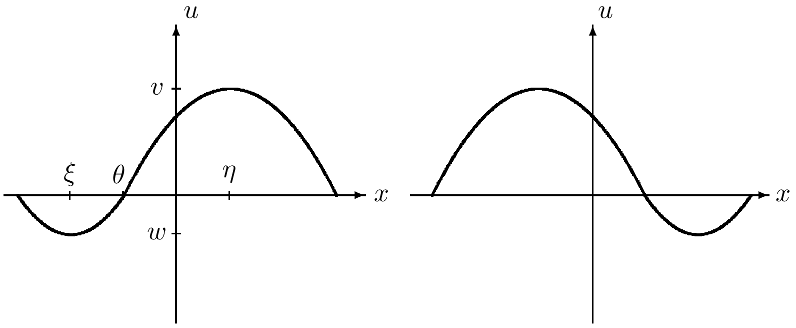

FIGURE 1. Two types of asymmetric solutions

The following elementary example shows why this question is non-trivial. The solution set of the equation

$$x(ax^2 - b\lambda) = 0$$

exhibits a pitchfork in (λ, x) plane, in a neighborhood of the point $(\lambda = 0, x = 0)$, for any non-zero constants a and b. If $a/b > 0$, the pitchfork opens to the right, and if $a/b < 0$ to the left. The same behavior holds in case of more general equations

$$f(x, \lambda) \equiv x(ax^2 - b\lambda) + \cdots = 0,$$

where ... stands for higher order terms at $(0, 0)$ (e.g. $\lambda^2 x^2$, x^4, etc). To obtain the ratio a/b, governing the direction of the pitchfork, one calculates

$$a/b = -\frac{f_{xxx}(0,0)}{6 f_{\lambda x}(0,0)}\,.$$

If one tries the same approach for the equation (1.1), one needs to differentiate that equation three times. The resulting equations has a number of terms, which seems impossible to handle.

We approach the problem by using direct integration. Our algorithm involves integrals that cannot be explicitly evaluated, but their computational evaluation is quite feasible, both in case $f(u) = u^{2k}$, and for more general nonlinearities.

2. The direction of bifurcation

Let us consider the symmetry breaking solution, which is negative near the $x = -1$ end, and positive near $x = 1$. Let us denote by ξ the point of negative minimum, by η the point of positive maximum, and by θ the root of $u(x)$. We also denote $w = u(\xi) < 0$ and $v = u(\eta) > 0$, the minimum and maximum values respectively, see Figure 1. Clearly $\xi = \xi(\lambda)$, $\eta = \eta(\lambda)$, but since solutions of autonomous equations are symmetric with respect to their extremal points, we have

$$\eta - \xi = 1, \quad \text{for all } \lambda. \tag{2.1}$$

(The points ξ and η are midpoints of the intervals $(-1, \theta)$ and $(\theta, 1)$ respectively.) Assume that the symmetry breaking solution bifurcates at $\lambda = \lambda_0$. Then $w = 0$ at $\lambda = \lambda_0$. At other λ's, $\lambda = \lambda(w)$, $w < 0$. If we can show that $\frac{d\lambda}{dw}(0) < 0$ (> 0) then the pitchfork opens forward (backward). (Observe that the function $\lambda(w)$ is defined only for $w \le 0$. Hence $\lambda(w)$ will have a point of minimum at $w = 0$, provided that $\lambda'(0) < 0$.)

Clearly,

$$\frac{1}{2}u'^2(x) + F(u(x)) - \lambda u(x) = c, \tag{2.2}$$

where $F(u) = \int_0^u f(t)\,dt$, and c is a constant. Evaluating the formula (2.2) at $x = \xi$, and then at $x = \eta$, we have $c = F(w) - \lambda w = F(v) - \lambda v$, which implies

$$\lambda = \frac{F(v) - F(w)}{v - w}. \tag{2.3}$$

Also from (2.2) we have on the interval (ξ, η) (where $\frac{du}{dx} > 0$)

$$\frac{du}{dx} = \sqrt{2}\sqrt{F(v) - F(u) - \lambda(v - u)}.$$

Integrating over the unit interval (ξ, η) (see (2.1)),

$$\int_w^v \frac{du}{\sqrt{F(v) - F(u) - \lambda(v - u)}} = \sqrt{2}. \tag{2.4}$$

In formulas (2.3) and (2.4) we regard v and λ as functions of w, i.e. $v = v(w)$, $\lambda = \lambda(w)$, with $w \le 0$. We have $\lambda(0) = \lambda_0$, and $v(0) \equiv v_0 = u_0(0)$, where $(\lambda_0, u_0(x))$ is the point of pitchfork bifurcation. Let us calculate λ_0 and v_0 for our $f(u) = u^{2k}$. Setting $w = 0$ in (2.3), we have

$$\lambda_0 = \frac{F(v_0)}{v_0} = \frac{v_0^{2k}}{2k + 1}. \tag{2.5}$$

Setting $w = 0$ in (2.4), and using (2.3) and (2.5)

$$\int_0^{v_0} \frac{du}{\sqrt{\frac{v_0^{2k}}{2k+1}u - \frac{1}{2k+1}u^{2k+1}}} = \sqrt{2}. \tag{2.6}$$

We evaluate the integral in (2.6) by making a substitution $u = v_0 z$,

$$\begin{aligned}\int_0^{v_0} \frac{du}{\sqrt{\frac{v_0^{2k}}{2k+1}u - \frac{1}{2k+1}u^{2k+1}}} &= v_0^{-k+1/2}\sqrt{2k+1}\int_0^1 \frac{dz}{\sqrt{z - z^{2k+1}}} \\ &\equiv v_0^{-k+1/2}\sqrt{2k+1}\,J(k),\end{aligned}$$

where, using *Mathematica*, we express in terms of the standard gamma function,

$$J(k) \equiv \int_0^1 \frac{dz}{\sqrt{z - z^{2k+1}}} = \frac{2\sqrt{\pi}\Gamma(1 + \frac{1}{4k})}{\Gamma(\frac{1}{2} + \frac{1}{4k})}.$$

Returning to (2.6), we have

$$v_0 = \Big[\frac{(2k+1)J^2(k)}{2}\Big]^{1/(2k-1)}. \tag{2.7}$$

We calculate v_0 from this formula, and then use (2.5) to calculate λ_0. We now turn to the calculation of $\frac{d\lambda}{dw}(0)$. In the formula (2.3) we multiply through by $v - w$, and differentiate with respect to w,

$$\lambda'(w)\,(v(w) - w) + \lambda(w)(v'(w) - 1) = f(v)v'(w) - f(w).$$

We now set $w = 0$. Then $\lambda = \lambda_0$, and $v = v_0$. Since $f(0) = 0$, we obtain

$$\lambda'(0) = \frac{1}{v_0}\left[\lambda_0 + v'(0)\left(f(v_0) - \lambda_0\right)\right]. \tag{2.8}$$

In order to find $v'(0)$, we plug (2.3) into (2.4), obtaining

$$\int_w^v \frac{du}{\sqrt{F(v) - F(u) - \frac{F(v)-F(w)}{v-w}(v-u)}} = \sqrt{2}. \tag{2.9}$$

The integral in (2.9) is improper at both end-points. To regularize it, we use the substitution

$$u = \frac{1}{2}(v-w)\sin\theta + \frac{w}{2} + \frac{v}{2}. \tag{2.10}$$

Then (2.9) takes the form

$$G(v,w) \equiv \int_{-\pi/2}^{\pi/2} H(v,w,\theta)\,d\theta = \sqrt{2}, \tag{2.11}$$

where

$$H(v,w,\theta) = \frac{1}{2}\frac{(v-w)^{3/2}\cos\theta}{\sqrt{(F(v)-F(u))(v-w) - (F(v)-F(w))(v-u)}},$$

and u is given by (2.10). *Mathematica* seems unable to evaluate exactly the integral in (2.11) for general k, however it easily evaluates a very accurate numerical approximation for any particular k. We now differentiate (2.11) with respect to w

$$G_v(v,w)v'(w) + G_w(v,w) = 0,$$

where $G_v = \int_{-\pi/2}^{\pi/2} H_v(v,w,\theta)\,d\theta$, and $G_w = \int_{-\pi/2}^{\pi/2} H_w(v,w,\theta)\,d\theta$. We now set $w = 0$, $v = v_0$, and solve for $v'(0)$,

$$v'(0) = -\frac{G_w(v_0,0)}{G_v(v_0,0)}. \tag{2.12}$$

After calculating $v'(0)$ from (2.12), we are able to calculate $\lambda'(0)$ from (2.8).

Example. Let $k = 2$, i.e. $f(u) = u^4$. Using *Mathematica*, we calculate $\lambda_0 \simeq 6.454$, $v_0 \simeq 2.383$, $v'(0) \simeq -0.542$, and $\lambda'(0) \simeq -3.160$. Conclusion: we have a pitchfork bifurcation at $\lambda_0 \simeq 6.454$, with the pitchfork facing forward in λ.

One can verify that $\lambda'(0) < 0$ for larger k too (the values of $\lambda'(0) < 0$ increase with k, and $\lambda'(0) \simeq -2.003$ at $k = 720$), although at $k = 721$ (and larger k) our program runs into a problem: *Mathematica* is unable to calculate the integral for $G_w(v_0,0)$ to the accuracy it desires. When we had replaced *Mathematica*'s **NIntegrate** command by a "home-made" numerical integration routine, the program worked for larger k too, and the results were similar. However, we state the next result conservatively.

Theorem 2.1. *Consider the problem*

$$u''(x) + u^{2k}(x) - \lambda = 0, \quad \textit{for } -1 < x < 1,\ u(-1) = u(1) = 0, \tag{2.13}$$

with $1 \le k \le 720$. *Compute* $\lambda_0 = \lambda_0(k)$ *by using the formulas* (2.7) *and* (2.5). *Then there is a negative* $\bar\lambda = \bar\lambda(k) < 0$, *so that the problem* (2.13) *has exactly two positive solutions for* $\bar\lambda < \lambda < 0$, *it has exactly one positive and one negative solution on* $(0, \lambda_0)$. *Moreover, there is a* $\lambda_1(k) > \lambda_0$, *so that the problem* (2.13) *has* *four solutions on* (λ_0, λ_1)*, one negative (and symmetric), one sign-changing and symmetric (with* $u(0) > 0$*), and two asymmetric solutions (see Figure 2).*

Proof. It is well known that at $\lambda = 0$ there exists a unique positive solution. This solution is known to be non-degenerate, so that we can continue it for small $\lambda > 0$. Setting $u(x) = \mu v(x)$, with μ determined by the relation $\mu^{2k-1} = \lambda$, we convert the problem (2.13) into

$$v''(x) + \lambda(v^{2k}(x) - 1) = 0, \quad \text{for } -1 < x < 1, \; v(-1) = v(1) = 0. \tag{2.14}$$

With the parameter now in front of the nonlinearity, the results of [4] apply. They imply that we can always continue both positive and sign-changing solutions of (2.14) (and hence of (2.13), and that the curve of positive solutions does not turn for $\lambda > 0$ (for $g(v) \equiv v^{2k} - 1$, we have $vg'(v) > g(v)$ for all $v > 0$). By Korman [2] this curve of positive solutions cannot be continued for all $\lambda > 0$ (the function $g(v) = v^{2k} - 1$ has no "stable" roots, i.e. roots where derivative is negative). By the Sturm's comparison theorem, it is easy to see that positive solutions cannot become unbounded at a finite λ. Hence, solutions on this curve must eventually stop being positive, and the only way this can happen is that $u'(\pm 1) = 0$ at some λ_0 (in view of the symmetry of positive solutions). By P. Korman [3] a pitchfork bifurcation occurs at λ_0, and by the result of the present paper, the pitchfork faces forward in λ. $\square$

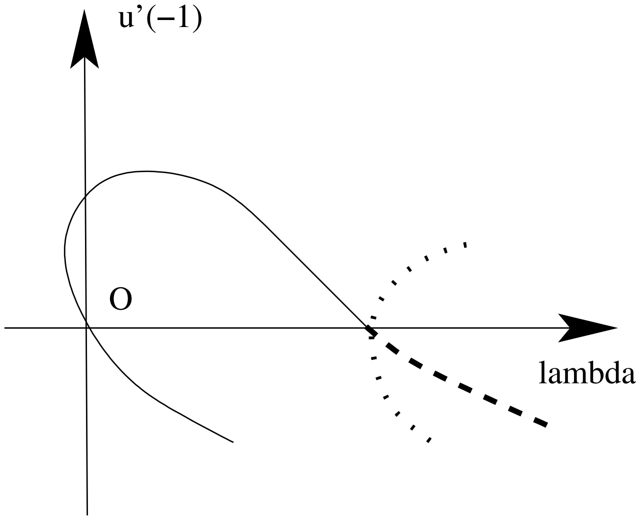

FIGURE 2. Pitchfork bifurcation

Remarks

(1) The bifurcation diagram for the Theorem 2.1 is given in Figure 2, where we draw $u'(-1)$ as a function of λ. In that figure solid lines denote positive and negative solutions, the dashed line denotes sign-changing symmetric solutions, and the doted lines stand for the symmetry breaking solutions.

(2) Our result applies to more general $f(u)$. If one is only interested in local direction of pitchfork bifurcation, one can consider any differentiable $f(u)$, with $f(0) = 0$.
(3) Our computations constitute a *proof* that the pitchfork opens forward, rather than a numerical simulation. We have computed the integrals in (2.12) by using a sophisticated adaptive routine of *Mathematica*. We had $\lambda'(0) < -2$ for all $1 \le k \le 720$. Even assuming a 50% relative error, $\lambda'(0)$ is still negative. *Mathematica*'s relative error is much less than that, and in fact our program quit at $k = 721$, when it could not deliver high accuracy. If someone desires an absolute assurance, one can do error analysis of the integration method, together with computations in exact arithmetics. This would be very time consuming, but straightforward.
(4) It was shown in P. Korman [3] that the problem (2.13) has infinitely many points of pitchfork bifurcation. It follows from our result that they all face forward.

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