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# EXISTENCE AND NON-EXISTENCE RESULTS FOR A NONLINEAR HEAT EQUATION

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Abstract. In this study, we consider the nonlinear heat equation

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) + u(x,t)^p \quad \text{in } \Omega \times (0,T), \\ Bu(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered}$$

with Dirichlet and mixed boundary conditions, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $p = 1 + 2/n$ is the critical exponent. For an initial condition $u_0 \in L^1$, we prove the non-existence of local solution in $L^1$ for the mixed boundary condition. Our proof is based on comparison principle for Dirichlet and mixed boundary value problems. We also establish the global existence in $L^{1+\epsilon}$ to the Dirichlet problem, for any fixed $\epsilon > 0$ with $\|u_0\|_{1+\epsilon}$ sufficiently small.

## 1. Introduction

In this paper we study the existence and non-existence results for the initial and boundary value problems of the form

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) + |u(x,t)|^{p-1}u(x,t) \quad \text{in } \Omega \times (0,T), \\ Bu(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $B$ denotes the corresponding boundary condition and $p$ is the critical exponent, which will be specified later.

Throughout this paper we use the following definition for the solution.

**Definition 1.1.** A function $u = u(x,t)$ is a mild solution of initial and boundary value problem (1.1) in $\bar{\Omega} \times [0,T]$, $T$ being a positive number, if and only if $u \in C([0,T], L^q(\Omega))$ satisfies the integral equation

$$u(x,t) = \int_\Omega K(x,y,t)u_0(y)dy + \int_0^t\int_\Omega K(x,y,t-s)|u(y,s)|^{p-1}u(y,s)\,dy\,ds, \tag{1.2}$$

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where $u_0 \in L^q(\Omega)$ and $K$ denotes the heat kernel for the linear heat equation with Dirichlet boundary condition.

It is well known that if the initial condition $u_0 \in L^\infty(\Omega)$, there exists a unique solution of (1.1) on $[0, T_{\max})$. However the question of local existence and uniqueness was interesting when $u_0 \notin L^\infty(\Omega)$; i.e., $u_0 \in L^q(\Omega)$ where $1 \le q < \infty$.

This type of initial and boundary value problem was studied by many authors. First Fujita [4, 5] studied this problem for classical solutions and established the following results for the Cauchy problem

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) + u^p(x,t) \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,t) = u_0(x) \quad \text{on } \mathbb{R}^n, \end{gathered} \tag{1.3}$$

where $u_0(x) \ge 0$.

(i) If $n(p-1)/2 < 1$, no non-negative global solution exists for any non-trivial initial data $u_0 \in L^1$. That is, every positive solution to this initial value problem blows up in a finite time.
(ii) If $n(p-1)/2 > 1$, global solution do exist for small initial data. To be precise, for any $k > 0$, $\delta$ can be chosen such that problem (1.3) has a global solution whenever $0 \le u_0(x) \le \delta e^{-k|x|^2}$.

Later Hayakawa [8] showed that the critical case for the Cauchy problem (1.3), that is $n(p-1)/2 = 1$, also belongs to blow up case for $n = 1, 2$. After Hayakawa the same result was proved by Kobayashi, Sirao and Tanaka [9] for general $n$. Also Weissler [12] obtained that for $n(p-1)/2 \le 1$ with $1 \le p < \infty$, non-negative $L^p$ solutions to the Cauchy problem blow up in $L^p$ norm in finite time. However in the case $n(p-1)/2 > 1$, he obtained sufficient conditions for the initial data $u_0$ which guarantee the existence of global solution.

The initial and boundary value problem was considered by Weissler [13, 14] who obtained some important local existence and uniqueness results in $C([0,T], L^q(\Omega))$. In [13], he considered the case $q > n(p-1)/2$ and $q > p$ and proved the local existence and uniqueness in $C([0,T], L^q(\Omega))$ for which the same argument works for $q > n(p-1)/2$ and $q = p$. Later local existence in $C([0,T], L^q(\Omega))$ with $q \ge n(p-1)/2$ and $q > 1$ or $q > n(p-1)/2$ and $q \ge 1$ was proved in [14]. In the same study he also proved a uniqueness result in a smaller class which was improved in [2]. If we notice, while the applications of the abstract results in [13] and [14] are given on a bounded domain, they work equally well on all of $\mathbb{R}^n$.

The first non-uniqueness result for this problem was given by Haraux-Weissler in [7]. Further non-uniqueness results which were motivated by [7] came in [1] and [10]. A unique solution to the initial and boundary value problem (1.1) in $L^{p_1}(0,T; L^{p_2})$ was constructed by Giga [6]. In his work, the main relation between $p_1$ and $p_2$ was $\frac{1}{p_2} = \frac{n}{2}(\frac{1}{r} - \frac{1}{p_1})$, $p_1 > r$, provided that the initial data $u_0 \in L^r$ with $r = n(p-1)/2 > 1$.

As we stated above, it is well known that if $u_0 \in L^\infty(\Omega)$, there exists a unique solution defined on a maximal interval $[0,T_{\max})$ and Brezis and Cazenave [2] considered the case if $u_0 \notin L^\infty(\Omega)$; i.e., $u_0 \in L^q(\Omega)$ for some $1 \le q < \infty$. And they studied two cases.

(i) If $q > n(p-1)/2$ (resp. $q = n(p-1)/2$) and $q \ge 1$ (resp. $q > 1$) with $n \ge 1$, they obtained existence and uniqueness of a local solution in

$C([0,T],L^q(\Omega))$ which is a classical solution of the initial boundary value problem on $\Omega\times(0,T)$.

(ii) If $q<n(p-1)/2$, they showed that there exists no local solution in any reasonable sense for some initial data $u_0\in L^q(\Omega)$.

**Remark 1.2.** The quantity $q=n(p-1)/2$ plays a critical role for this initial and boundary value problem. To see that $q=n(p-1)/2$ is the critical exponent, we observe the following dilation argument. If $v$ is the solution of

$$\begin{gathered} v_t = v_{xx} + v^p,\\ v(x,0)=g(x), \end{gathered}$$

then $u(x,t)=k^s v(kx,k^2t)$ is the solution of

$$\begin{gathered} u_t = u_{xx} + u^p,\\ u(x,0)=k^s g(kx). \end{gathered}$$

By using the equation $u_t=u_{xx}+u^p$ with $u(x,t)=k^s v(kx,k^2t)$, we get

$$k^{s+2}v_t = k^{s+2}v_{xx}+k^{sp}v^p$$

which gives the condition $s=2/(p-1)$. Also note that,

$$\|u(0)\|_q^q=\int u^q(x,0)dx=\int k^{sq-n}v^q(y,0)dy=\|v(0)\|_q{}^q$$

when $s=n/q$. So, combining these two conditions for $s$, we have $q=n(p-1)/2$, which is the critical exponent.

The question of existence and uniqueness of a local solution for the doubly critical case which is $q=n(p-1)/2$ and $q=1$, which was proposed by Brezis and Cazenave [2], was wide open even for $n=1$,($p=3$). In [3], we proved that for $u_0=\sum_{j=1}^{\infty}\frac{1}{j^2}e^{j^{24}}e^{-(e^{2j^{24}})x^2}$, Dirichlet problem has no non-negative local solution in $L^1(-1,1)$. Moreover we generalized the non-existence result of local solution for $n$-dimensional case with the critical exponent $p=1+\frac{2}{n}$. More general nonlinearity was also considered for Dirichlet boundary value problem.

This paper is organized as follows: In Section 2, we summarize the non-existence results of local solution for both cauchy and the Dirichlet problems for doubly critical case $q=n(p-1)/2$ and $q=1$, first for one dimensional case and then $n$-dimensional case.

In Section 3, we consider the same problem with mixed boundary conditions; that means having $u$ and the normal derivative of $u$ in the boundary condition. We prove the non-existence of local solution for the mixed boundary conditions by using the comparison argument for the kernels with Dirichlet and mixed boundary conditions.

In Section 4, we give the sufficient conditions for $q$ and the initial data $u_0$ to guarantee the existence of the global solution to the Dirichlet problem. In particular, we prove the global existence in $L^{1+\epsilon}$ with $\|u_0\|_{1+\epsilon}$ sufficiently small and $\epsilon>0$, for the critical exponent $p=3$.

## 2. Non-existence of a Local Solution

In the following two sections, we summarize the non-existence of local solution for both cauchy and Dirichlet problems.

2.1. **Non-existence of Local Solutions for the Cauchy Problem.** To motivate the proof of the non-existence of local $L^1$ solution for the Dirichlet problem, we first prove the non-existence of local $L^1$ solution for the Cauchy problem for the one dimensional case,

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + u^3(x,t) \quad \text{in } \mathbb{R}\times(0,T), \\ u(x,t) = u_0(x) \quad \text{on } \mathbb{R}, \end{gathered} \tag{2.1}$$

for some $u_0 \in L^1(\mathbb{R})$. First we choose a particular initial data $u_0(x) = ke^{-k^2x^2} \in L^1(\mathbb{R})$ and observe the following result. After this observation, we prove in Theorem 2.1. that the Cauchy problem has no local $L^1$ solution for initial data of the form $u_0 = \sum_k c_k k e^{-k^2x^2}$ where $c_k \geq 0$ will be determined later.

We denote the solution of the Cauchy problem (2.1) by $u_c$. For the Cauchy problem (2.1) with the initial data $u_0(x) = ke^{-k^2x^2} \in L^1(\mathbb{R})$, using that

$$\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} e^{-ay^2}\,dy = \frac{e^{\frac{-ax^2}{1+4at}}}{\sqrt{1+4at}},$$

the linear solution is

$$u_c^L(x,t) = \frac{k}{\sqrt{1+4k^2t}} e^{\frac{-k^2x^2}{1+4k^2t}}.$$

So the solution to the Cauchy problem (2.1) is

$$u_c(x,t) = u_c^L(x,t) + \int_0^t \int_{\mathbb{R}} K_c(x,y,t-s)(u_c(y,s))^3\,dy\,ds,$$

where the Cauchy kernel is

$$K_c(x,y,t) = \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}}.$$

Note that

$$(u_c^L(y,s))^3 = \frac{k^3}{(1+4k^2s)^{3/2}} e^{\frac{-3k^2y^2}{1+4k^2s}}$$

and $\int_{\mathbb{R}} K_c(x,y,t-s)dx = 1$. Since $u_c^L(x,t) > 0$ and $u_c(x,t) > u_c^L(x,t)$, we have

$$u_c(x,t) > \int_0^t \int_{\mathbb{R}} K_c(x,y,t-s)(u_c^L(y,s))^3\,dy\,ds.$$

Then for any $\delta > 0$,

$$\begin{aligned} \|u_c\|_{L^1(\mathbb{R}\times(0,\delta))} &= \int_0^\delta \int_{\mathbb{R}} u_c(x,t)\,dx\,dt \\ &> \int_0^\delta \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} K_c(x,y,t-s)(u_c^L(y,s))^3\,dy\,ds\,dx\,dt \\ &= \int_0^\delta \int_0^t \int_{\mathbb{R}} (u_c^L(y,s))^3 \int_{\mathbb{R}} K_c(x,y,t-s)dx\,dy\;ds\,dt \\ &= \int_0^\delta \int_0^t \int_{\mathbb{R}} \frac{k^3 e^{\frac{-3k^2y^2}{1+4k^2s}}}{(1+4k^2s)^{3/2}}\,dy\,ds\,dt \\ &= \sqrt{\frac{\pi}{3}} \int_0^\delta \int_0^t \frac{k^2}{(1+4k^2s)}\,ds\,dt = \sqrt{\frac{\pi}{3}} \int_0^\delta \frac{1}{4}\ln(1+4k^2t)dt. \end{aligned}$$

We can use the above estimate to establish the following theorem for the Cauchy problem.

**Theorem 2.1.** *For some initial data $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$, the Cauchy problem* (2.1) *has no non-negative local mild solution in $C([0,\delta], L^1(\mathbb{R}))$ for all $\delta > 0$.*

*Proof.* Let $u_0 = \sum_k c_k k e^{-k^2x^2}$ where $c_k \geq 0$ will be determined later. The linear solution to the Cauchy problem (2.1) is

$$u_c^L(x,t) = \sum_k \frac{c_k k}{\sqrt{1+4k^2t}} e^{\frac{-k^2x^2}{1+4k^2t}}.$$

Note that

$$(u_c^L(y,s))^3 \geq \sum_k \frac{{c_k}^3 k^3}{(1+4k^2s)^{3/2}} e^{\frac{-3k^2y^2}{1+4k^2s}}.$$

Since the solution of the problem satisfies

$$u_c(x,t) > \int_0^t \int_{\mathbb{R}} K_c(x,y,t-s)(u_c^L(y,s))^3 \, dy \, ds,$$

we have that for any $\delta > 0$,

$$\begin{aligned}
\|u_c\|_{L^1(\mathbb{R}\times(0,\delta))} &= \int_0^\delta \int_{\mathbb{R}} u_c(x,t) \, dx \, dt \\
&> \int_0^\delta \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} K_c(x,y,t-s)(u_c^L(y,s))^3 \, dy \, ds \, dx \, dt \\
&= \int_0^\delta \int_0^t \int_{\mathbb{R}} (u_c^L(y,s))^3 \int_{\mathbb{R}} K_c(x,y,t-s) dx \, dy \; ds \, dt \\
&\geq \int_0^\delta \int_0^t \int_{\mathbb{R}} \sum_k \frac{c_k^3 k^3}{(1+4k^2s)^{3/2}} e^{\frac{-3k^2y^2}{1+4k^2s}} \, dy \; ds \, dt \\
&= \sum_k C \int_0^\delta \int_0^t \frac{c_k^3 k^2}{(1+4k^2s)} \, ds \, dt \\
&= \sum_k C \int_0^\delta \frac{1}{4} c_k^3 \ln(1+4k^2t) dt \\
&\geq C \sum_k \int_{\delta/2}^\delta c_k^3 \ln(1+4k^2t) dt \\
&\geq C \sum_k \delta c_k^3 \ln(1+2k^2\delta).
\end{aligned}$$

Now we choose $c_k$'s such that

(i) $\sum_k c_k^3 \delta \ln(1+2k^2\delta) = \infty$; and

(ii) $\sum_k c_k < \infty$ so that $u_0 \in L^1$.

We will find a sequence $\{k_j\}_{j=1}^\infty$ such that $c_k \neq 0$ only when $k = k_j$. For $k$ big enough, we have

$$c_k^3 \delta \ln(1+2k^2\delta) \geq c_k^3 \sqrt{\ln k}.$$

So if we set $c_k^3 \sqrt{\ln k} = (\ln k)^{1/4}$ for some $k = k_j \to \infty$ as $j \to \infty$; that is, if we choose $c_k = \frac{1}{(\ln k)^{1/12}}$, for $k = k_j, j = 1, 2, \cdots$, condition (i) will be satisfied for all $\delta > 0$.

For (ii), we set $\frac{1}{(\ln k)^{1/12}} = \frac{1}{j^2}$ which implies that $k = e^{j^{24}} = k_j, j = 1, 2, \cdots$. Then for

$$u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{j^{24}} e^{-(e^{2j^{24}})x^2},$$

the Cauchy problem (2.1) has no local solution in $L^1$. □

2.2. **Non-existence of Local Solutions for the Dirichlet Problem.** In this section we will prove that there is no local $L^1$ solution to the one dimensional Dirichlet problem

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + u^3(x,t) \quad \text{in } (-1,1)\times(0,T) \\ u(\pm 1, t) = 0, \\ u(x,0) = u_0(x) \quad \text{in } (-1,1), \end{gathered} \tag{2.2}$$

for some initial data $u_0 \in L^1$. First by using the lower solution argument with a cutoff function we prove some estimates for a family of particular initial data and then we will construct the initial data $u_0 \in L^1(-1,1)$ for which there is no local $L^1$ solution to the Dirichlet problem (2.2).

Throughout this section we denote the solution of the Dirichlet problem (2.2) by $u_d$. Let $u_0(x) = ke^{-k^2x^2}$, then by Fourier series, the linear solution to the problem (2.2) is

$$u_d^L(x,t) = \int_{-1}^{1} K_d(x,y,t) k e^{-k^2y^2}\, dy$$

where

$$K_d(x,y,t) = \sum_{j=1}^{\infty} e^{-j^2\pi^2 t} \sin j\pi\Big(\frac{1+y}{2}\Big) \sin j\pi\Big(\frac{1+x}{2}\Big)$$

is the Dirichlet kernel. Then the solution of (2.2) is

$$u_d(x,t) = u_d^L(x,t) + \int_0^t \int_{-1}^{1} K_d(x,y,t-s) u_d^3(y,s)\, dy\, ds.$$

The difficulty for the Dirichlet problem is that we do not have a good explicit formula for the solution in a bounded domain as we have for the Cauchy problem. In the proof for the Cauchy problem, $\int_R K_c(x,y,t-s)dx = 1$ is used to simplify the argument. For the Dirichlet problem we do not have that but we know that $\int_\Omega K_d(x,y,t)dx$ decays exponentially for $t > 0$.

The idea behind the proof of the non-existence of local solution for the Dirichlet problem is to use the non-existence of local solution for the Cauchy problem (which was proved in the previous section). To do this, we first construct a cutoff function and estimate the Dirichlet kernel $K_d$ in terms of the Cauchy kernel $K_c$. Then using this estimate we find a lower bound for the linear solution of the Dirichlet problem $u_d^L$ in terms of the linear solution of the Cauchy problem $u_c^L$. Combining the estimates for $K_d$ and $u_d^L$, we can estimate $\|u_d\|_{L^1}$ and obtain the non-existence result of local solution for the Dirichlet problem.

In the first lemma below, we estimate the Dirichlet kernel $K_d$ in terms of the Cauchy kernel $K_c$ by constructing a cutoff function in a concentrated domain.

**Lemma 2.2.** *For $|x-y| \le \frac{1}{\sqrt{2}}$ and $|x| \le \frac{1}{4}$,*

$$K_d(x,y,t) \ge e^{-12t}\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big)K_c(x,y,t).$$

*Proof.* The crux of the proof of this lemma is to construct a cutoff function $g$ so that the function

$$w(x,y,t) = gK_c(x,y,t) - K_d(x,y,t)$$

will satisfy the following conditions:

(i) $L(w) \le 0$ where $L(w) = w_t - w_{xx}$,

(ii) $w \le 0$ on the boundary $|x-y| = \frac{1}{\sqrt{2}}$ with $|x| \le \frac{1}{4}$ and at $t = 0$.

Since $L(K_d) = 0$, we want $w$ to satisfy

$$L(w) = L(gK_c - K_d) = L(gK_c) = L(g)K_c - 2\nabla K_c.\nabla g \le 0.$$

Note that $K_c(x,y,t) = \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}}$ implies

$$\nabla K_c = \frac{-2(x-y)}{4t}\frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} = \frac{-(x-y)}{2t}K_c.$$

So $g$ must satisfy

$$L(w) = L(g)K_c + \frac{(x-y)}{t}K_c \cdot \nabla g \le 0;$$

that is,

$$g_t - g_{xx} + \frac{(x-y)}{t}g_x \le 0$$

in the one-dimensional case. To satisfy this inequality and to have zero on the boundary (i.e., when $y = x \pm \frac{1}{\sqrt{2}}$), we choose the cutoff function as

$$g(x,y,t) = e^{-\alpha t}\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big)$$

where $\alpha$ will be determined later. So we want

$$\begin{aligned} &g_t - g_{xx} + \frac{(x-y)}{t}g_x \\ &= -\alpha\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big) - \Big(\frac{6(x-y)^2-2}{(1+(x-y)^2)^3}\Big) - \frac{2(x-y)^2}{t(1-(x-y)^2)^2} \le 0. \end{aligned}$$

If $|x-y| \ge \frac{1}{\sqrt{3}}$, then $g_t - g_{xx} + \frac{(x-y)}{t}g_x \le 0$ for any $\alpha \ge 0$. If $|x-y| < \frac{1}{\sqrt{3}}$, we need to find $\alpha$ so that this expression will be negative. So, if we choose

$$\alpha \ge \frac{\frac{2-6(x-y)^2}{(1+(x-y)^2)^3}}{\big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\big)};$$

i.e., if $\alpha \ge 12$, then we have

$$g_t - g_{xx} + \frac{(x-y)}{t}g_x \le 0.$$

Set $\alpha = 12$. Then the cutoff function is

$$g(x,y,t) = e^{-12t}\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big).$$

So $L(w) \le 0$ and (i) is satisfied.

Now we show that condition (ii) will be satisfied by $w$ with this cutoff function; i.e., we will show that $w \le 0$ on the boundary and at $t = 0$. Since $g(x, x \pm \frac{1}{\sqrt{2}}, t) = 0$, $w \le 0$ on the boundary $|x - y| = \frac{1}{\sqrt{2}}$ and $|x| \le \frac{1}{4}$. To show that $w \le 0$ at $t = 0$, it is sufficient to show that

$$\lim_{t\to 0} \int_{x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} w(x,y,t)h(y)dy \le 0 \quad \text{for every } h \ge 0.$$

Since

$$\begin{aligned}
\lim_{t\to 0} \int_{x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} w(x,y,t)h(y)dy &= \int_{x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} (g(x,y,t)K_c(x,y,t) - K_d(x,y,t))h(y)dy \\
&\le Ch(x) - h(x)
\end{aligned}$$

where $g \le C \le 1$, we have $w \le 0$ at $t = 0$. Hence, by the maximum principle, (i) and (ii) imply that $w \le 0$ inside the domain, which implies

$$K_d(x,y,t) \ge e^{-12t}\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big)K_c(x,y,t)$$

when $|x - y| \le \frac{1}{\sqrt{2}}$ and $|x| \le \frac{1}{4}$. □

In the following lemma we use the estimate for $K_d$ to get an estimate for $u_d^L$ which is the linear solution of the Dirichlet problem.

**Lemma 2.3.** *For $|x| \le \frac{1}{4}$, $u_d^L(x,t) \ge c_1 e^{-12t} u_c^L(x,t)$ with $u_d(x,0) = ke^{-k^2x^2}$, where $c_1$ is a uniform constant independent of $k$.*

*Proof.* To show this estimate for the linear solution of the Dirichlet problem, the main idea is to use the crucial estimate for the Dirichlet kernel in terms of the Cauchy kernel that we proved in Lemma 2.2. The linear solution for the Dirichlet problem is

$$u_d^L(x,t) = \int_{-1}^{1} K_d(x,y,t)ke^{-k^2y^2}dy.$$

Using the estimate of Lemma 2.2. for $K_d(x,y,t)$, we have

$$\begin{aligned}
u_d^L(x,t) &= \int_{-1}^{1} K_d(x,y,t)ke^{-k^2y^2}dy \\
&\ge \int_{x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} K_d(x,y,t)ke^{-k^2y^2}dy \quad (\text{since } |x-y| \le \tfrac{1}{\sqrt{2}}) \\
&\ge \int_{x-\frac{1}{\sqrt{2}}}^{x+\frac{1}{\sqrt{2}}} e^{-12t}\Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big)K_c(x,y,t)ke^{-k^2y^2}dy \\
&\ge \int_{-\frac{1}{4}}^{\frac{1}{4}} ce^{-12t}K_c(x,y,t)ke^{-k^2y^2}dy \\
&\quad (\text{since } |x| \le \frac{1}{4} \text{ implies } \Big(\frac{1}{1+(x-y)^2} - \frac{1}{1+\frac{1}{2}}\Big) \ge c) \\
&= ce^{-12t}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}ke^{-k^2y^2}dy
\end{aligned}$$

$$
\begin{aligned}
&= ce^{-12t}\frac{e^{\frac{-x^2}{4t}}}{\sqrt{4\pi t}}\int_{\frac{-k}{4}}^{\frac{k}{4}} e^{\frac{xs}{2kt}-(\frac{1+4k^2t}{4k^2t})s^2}ds \quad (\text{where } s=ky)\\
&= ce^{-12t}\frac{e^{\frac{-x^2}{4t}+\frac{k^2x^2}{4k^2t(1+4k^2t)}}}{\sqrt{4\pi t}}\int_{\frac{-k}{4}}^{\frac{k}{4}} e^{-(\frac{1+4k^2t}{4k^2t})(s-\frac{kx}{1+4k^2t})^2}ds\\
&= ce^{-12t}\frac{e^{\frac{-k^2x^2}{1+4k^2t}}}{\sqrt{4\pi t}}\frac{\sqrt{4k^2t}}{\sqrt{1+4k^2t}}\int_{u_1}^{u_2} e^{-u^2}\,du
\end{aligned}
$$

where $u=\sqrt{\frac{1+4k^2t}{4k^2t}}(s-\frac{kx}{1+4k^2t})$, $u_1=\sqrt{\frac{1+4k^2t}{4k^2t}}(-\frac{k}{4}-\frac{kx}{1+4k^2t})$ and $u_2=\sqrt{\frac{1+4k^2t}{4k^2t}}(\frac{k}{4}-\frac{kx}{1+4k^2t})$. The above expression is greater than or equal to

$$
\begin{aligned}
&\begin{cases}
ce^{-12t}\frac{ke^{\frac{-k^2x^2}{1+4k^2t}}}{\sqrt{\pi}\sqrt{1+4k^2t}}\int_0^{u_2} e^{-u^2}du & \text{if } -\frac14\le x<0,\\
ce^{-12t}\frac{ke^{\frac{-k^2x^2}{1+4k^2t}}}{\sqrt{\pi}\sqrt{1+4k^2t}}\int_{u_1}^{0} e^{-u^2}du & \text{if } 0\le x\le\frac14
\end{cases}\\
&\ge \frac{c}{\sqrt{\pi}}e^{-12t}u_c^L(x,t)c'\\
&\ge c_1e^{-12t}u_c^L(x,t),
\end{aligned}
$$

where $c_1=c'\frac{c}{\sqrt{\pi}}$, $c'=\min(c_1{}',c_2{}')$, $\int_{u_1}^0 e^{-u^2}du\ge c_1{}'$, and $\int_0^{u_2}e^{-u^2}du\ge c_2{}'$. $\square$

Now using the estimates in Lemmas 2.2 and 2.3, we prove that for some initial data $u_0\in L^1$ there is no local solution to the Dirichlet problem (2.2) in $L^1$.

**Lemma 2.4.** *For any fixed $\delta>0$, we have $\|u_d\|_{L^1((-1,1)\times(0,\delta))}=\infty$ with $u_0=\sum_k c_kke^{-k^2x^2}$ where the $c_k$'s will be determined later.*

*Proof.* For the solution of the Dirichlet problem, for any fixed $\delta>0$,

$$
\begin{aligned}
\|u_d\|_{L^1((-1,1)\times(0,\delta))} &= \int_0^\delta\int_{-1}^1 u_d(x,t)\,dx\,dt\\
&> \int_0^\delta\int_{-1}^1\int_0^t\int_{-1}^1 K_d(x,y,t-s)(u_d^L(y,s))^3\,dy\,ds\,dx\,dt\\
&> \int_0^\delta\int_{-\frac14}^{\frac14}\int_0^t\int_{-\frac14}^{\frac14} K_d(x,y,t-s)(u_d^L(y,s))^3\,dy\,ds\,dx\,dt\\
&= \int_0^\delta\int_0^t\int_{-\frac14}^{\frac14}(u_d^L(y,s))^3\int_{-\frac14}^{\frac14} K_d(x,y,t-s)\,dx\,dy\;ds\,dt.
\end{aligned}
$$

Then using Lemma 2.2, we get

$$
\begin{aligned}
\int_{-\frac14}^{\frac14} K_d(x,y,t-s)dx &\ge \int_{\frac{-1}{4}}^{\frac14} ce^{-12(t-s)}K_c(x,y,t-s)dx\\
&\ge ce^{-12(t-s)}\int_{\frac{-1}{4}}^{\frac14}\frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}dx\\
&= \frac{ce^{-12(t-s)}}{\sqrt{\pi}}\int_{\frac{-\frac14-y}{\sqrt{4(t-s)}}}^{\frac{\frac14-y}{\sqrt{4(t-s)}}} e^{-u^2}du \quad (\text{where } u=\tfrac{x-y}{\sqrt{4(t-s)}})
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{ce^{-12(t-s)}}{\sqrt{\pi}} \int_0^{\frac{\frac{1}{4}-y}{\sqrt{4(t-s)}}} e^{-u^2}\,du \\
&\geq Ce^{-12(t-s)}
\end{aligned}
$$

for some constant $C$. Using Lemma 2.3 for $u_d^L$ and the fact that

$$
(u_c^L(y,s))^3 \geq \sum_k \frac{c_k^3 k^3 e^{-\frac{3k^2y^2}{1+4k^2s}}}{(1+4k^2s)^{3/2}},
$$

we have

$$
\begin{aligned}
\|u_d\|_{L^1((-1,1)\times(0,\delta))} &\geq \int_0^\delta \int_0^t \int_{-\frac{1}{4}}^{\frac{1}{4}} (u_d^L(y,s))^3 \int_{-\frac{1}{4}}^{\frac{1}{4}} K_d(x,y,t-s)dx\,dy\;ds\,dt \\
&\geq \int_0^\delta \int_0^t \int_{-\frac{1}{4}}^{\frac{1}{4}} (u_d^L(y,s))^3 (Ce^{-12(t-s)})\,dy\;ds\,dt \\
&\geq \int_0^\delta \int_0^t \int_{\frac{-1}{4}}^{\frac{1}{4}} Ce^{-12(t-s)}(c_1e^{-12s}u_c^L(y,s))^3\,dy\;ds\,dt \\
&= \int_0^\delta \int_0^t Cc_1e^{-12t-24s} \int_{\frac{-1}{4}}^{\frac{1}{4}} (u_c^L(y,s))^3\,dy\;ds\,dt \\
&\geq \int_0^\delta \int_0^t Cc_1e^{-12t-24s} \int_{\frac{-1}{4}}^{\frac{1}{4}} \sum_k \frac{c_k^3k^3e^{\frac{-3k^2y^2}{1+4k^2s}}}{(1+4k^2s)^{3/2}}\,dy\;ds\,dt \\
&\geq \sum_k C' \int_0^\delta \int_0^t \frac{c_k^3k^2}{1+4k^2s}\,ds\,dt \\
&= \sum_k C' \int_0^\delta \frac{1}{4}c_k^3\ln(1+4k^2t)dt,
\end{aligned}
$$

for some constant $C'$. $\square$

Now as we discussed for the Cauchy problem in previous section, we choose $c_k = \frac{1}{(\ln k)^{1/12}}$ which will imply that $k = e^{j^{24}} = k_j$, for $j = 1, 2, \cdots$. So for

$$
u_0(x) = \sum_{j=1}^\infty \frac{1}{j^2}e^{j^{24}}e^{-(e^{2j^{24}})x^2},
$$

using Lemmas 2.2, 2.3 and 2.4 we can prove the following theorem.

**Theorem 2.5.** *For $u_0 = \sum_{j=1}^\infty \frac{1}{j^2}e^{j^{24}}e^{-(e^{2j^{24}})x^2}$, the Dirichlet problem (2.2) has no non-negative local mild solution in $L^1(-1,1)$.*

We have also generalized this result to $n$-dimensional Dirichlet problem which is

$$
\begin{gathered}
u_t(x,t) = \Delta u(x,t) + |u(x,t)|^{\frac{2}{n}}u(x,t) \quad \text{in } \Omega\times(0,T), \\
u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\
u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{gathered}
\tag{2.3}
$$

where $\Omega = (-1,1)\times\cdots\times(-1,1)\subset\mathbb{R}^n$ and obtained the following result.

**Theorem 2.6.** *For* $u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{nj^{\frac{16+8n}{n}}} e^{-(e^{2j^{\frac{16+8n}{n}}})|x|^2} \in L^1$ *, the* $n$ *- dimensional Dirichlet problem has no non-negative local mild solution in* $L^1((-1,1) \times \cdots \times (-1,1))$.

We have also considered the general nonlinearity for Dirichlet boundary value problem in [3] and proved the following result.

**Corollary 2.7.** *The Dirichlet problem*

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) + f(u) \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.4}$$

*where* $\Omega \subset \mathbb{R}^n$ *is any smooth bounded domain and* $f(u) \geq |u|^{\frac{2}{n}+1}$*, has no non-negative local mild solution for some* $u_0 \geq 0$*, in* $L^1$.

## 3. Non-existence of Local Solutions for the Mixed Boundary Condition

In this section, we study the same problem with the mixed boundary conditions; i.e., considering the boundary conditions in terms of $u$ and the normal derivative of $u$, which is denoted by $\frac{\partial u}{\partial n}$. Let $\tilde{u}$ be the solution of the one-dimensional mixed boundary value problem,

$$\begin{gathered} \tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \tilde{u}^3(x,t) \quad \text{in } (-1,1) \times (0,T), \\ \frac{\partial \tilde{u}}{\partial n}(x,t) + \beta \tilde{u}(x,t) = 0 \quad x = \pm 1 \quad \text{in } (0,T), \\ \tilde{u}(x,0) = u_0 \quad \text{in } (-1,1) \end{gathered} \tag{3.1}$$

where $\beta \geq 0$ is a constant.

First we prove the non-existence of local solution for the one-dimensional mixed boundary value problem (3.1) and then we generalize this result to the $n$ - dimensional case. The main idea of the proof is to use the comparison principle for the kernels of heat semigroups with Dirichlet and mixed boundary conditions.

Recall that one-dimensional Dirichlet problem is

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + u^3(x,t) \quad \text{in } (-1,1) \times (0,T), \\ u(\pm 1, t) = 0, \\ u(x,0) = u_0(x) \quad \text{in } (-1,1). \end{gathered} \tag{3.2}$$

where $u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{j^{24}} e^{-(e^{2j^{24}})x^2}$ for which there is no local solution.

Let $K(t)$ and $\tilde{K}(t)$ be the heat kernels on $(-1,1)$ with Dirichlet and mixed boundary conditions respectively. Now we claim that $\tilde{K}(t) \geq K(t)$ on $(-1,1)$. In fact, let $u_0$ be any positive initial data, and let $u$ and $\tilde{u}$ be the solution to linear heat equation with Dirichlet and mixed boundary conditions respectively. Hence,

$$\begin{gathered} (\tilde{u} - u)_t = (\tilde{u} - u)_{xx} \quad \text{in } (-1,1) \times (0,T), \\ \frac{\partial(\tilde{u} - u)}{\partial n} + \beta(\tilde{u} - u) = -\frac{\partial u}{\partial n} \geq 0, \quad x = \pm 1,\ t \in (0,T), \\ (\tilde{u} - u)(x,0) = 0 \quad x \in (-1,1), \end{gathered} \tag{3.3}$$

and by the maximum principle, we have $\tilde{u} \geq u$; i.e., $\tilde{K}(t) \geq K(t)$ on $(-1,1)$.

Now suppose for the same initial data $u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{j^{24}} e^{-(e^{2j^{24}})x^2}$ problem (3.1) has a solution $\tilde{u} \in C([0.T], L^1(-1,1))$, then

$$\tilde{u}(t) = \tilde{K}(t)u_0 + \tilde{H}(\tilde{u})(t) \geq K(t)u_0 + H(\tilde{u})(t),$$

where $H(u)(t) = \int_0^t K(t-s)u^3(s)ds$ and $\tilde{H}(u)(t) = \int_0^t \tilde{K}(t-s)u^3(s)ds$. Let $\tilde{u}(t) = u_1(t)$, and define by iteration,

$$u_{k+1}(t) = K(t)u_0 + H(u_k)(t).$$

It follows by induction that

$$K(t)u_0 \leq u_{k+1}(t) \leq u_k(t).$$

Thus, the $u_k(t)$ converge $v(t)$ (by monotone convergence theorem) which must be a solution of the integral equation $v(t) = K(t)u_0 + H(v)(t)$, which is a contradiction that proves the following theorem for the one dimensional mixed boundary value problem.

**Theorem 3.1.** *For $u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{j^{24}} e^{-(e^{2j^{24}})x^2}$ there is no local $L^1$ solution to the one-dimensional mixed boundary value problem* (3.1)*.*

Since we can generalize the non-existence result of local solution to the $n$-dimensional Dirichlet problem, we can also obtain the non-existence of local solution for the $n$-dimensional mixed boundary value problem by the similar comparison argument as above.

**Theorem 3.2.** *For the $n$-dimensional mixed boundary value problem,*

$$\begin{gathered} \tilde{u}_t(x,t) = \Delta\tilde{u}(x,t) + |\tilde{u}(x,t)|^{\frac{2}{n}}\tilde{u}(x,t) \quad \text{in } (0,T)\times\Omega, \\ \frac{\partial\tilde{u}}{\partial n} + \beta\tilde{u} = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \tilde{u}(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{3.4}$$

*where $\Omega \subset \mathbb{R}^n$ and $u_0(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} e^{nj^{\frac{16+8n}{n}}} e^{-(e^{2j^{\frac{16+8n}{n}}})|x|^2} \in L^1$, there is no non-negative local solution in $L^1$.*

In the previous sections we give the non-existence results of local solution for some initial data $u_0 \in L^1$ for the Dirichlet and mixed boundary value problems. However in the next section, we establish the existence of global solution for some $u_0 \in L^{1+\epsilon}$ sufficiently small.

## 4. Existence of a Global Solution for Small Initial Data

In this section, we give sufficient conditions on $q$ and $u_0$ for the existence of global solution to the Dirichlet problem

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + u^3(x,t) \quad \text{in } (-1,1)\times(0,T) \\ u(\pm1,t) = 0 \quad \text{in } (0,T) \\ u(x,0) = u_0 \quad \text{in } (-1,1). \end{gathered} \tag{4.1}$$

where $u_0 \in L^q(-1,1)$. Mainly we will prove that if the initial condition $u_0 \in L^{1+\epsilon}$ for any fixed $\epsilon > 0$, then the Dirichlet problem (4.1) has a global solution in $L^{1+\epsilon}$ for $\|u_0\|_{1+\epsilon}$ sufficiently small.

For notational simplicity, we will use the following form of the solution throughout this section;

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}u^3(s)ds$$

where $e^{t\Delta}u_0$ denotes the linear solution of problem (4.1). Before proving the global existence result, we give some interpolation inequalities for the linear solution of (4.1).

**Remark 4.1.** Let $1 < p < 2$. Then it is easy to see that for all $t \geq 0$, $e^{t\Delta} : L^1 \to L^1$ with norm $M_1 \leq 1$; i.e.,

$$\|e^{t\Delta}\phi\|_1 \leq \|\phi\|_1,$$

and $e^{t\Delta} : L^2 \to L^2$ with norm $M_2 = e^{t\gamma_1}$ where $\gamma_1$ is the first (negative) eigenvalue of the Laplacian with vanishing Dirichlet boundary condition; i.e.,

$$\|e^{t\Delta}\phi\|_2 \leq e^{t\gamma_1}\|\phi\|_2.$$

Now we recall the following interpolation theorem.

**Theorem 4.2** (Nirenberg [11])**.** *Let $T(t)$ be a continuous linear mapping of $L^p$ into $L^p$ with norm $M_1$ and $L^q$ into $L^q$ with norm $M_2$. Then $T(t)$ is a continuous mapping of $L^r$ into $L^r$ with the norm $M \leq {M_1}^{\lambda}{M_2}^{1-\lambda}$ where $1 \leq p \leq r \leq q \leq \infty$ and $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$.*

By this theorem, we conclude that $e^{t\Delta} : L^p \to L^p$ with the norm $M \leq {M_1}^{\lambda}{M_2}^{1-\lambda}$ with $\lambda = \frac{2-p}{p}$. Hence

$$\|e^{t\Delta}\phi\|_p \leq e^{t\gamma_1\frac{2p-2}{p}}\|\phi\|_p \quad \text{for} \quad 1 < p < 2.$$

For $p \geq 2$, we can get a similar estimate by the following interpolation argument. $e^{t\Delta} : L^2 \to L^2$ with norm $M_1 = e^{t\gamma_1}$ as above. Note that for all $t \geq 0$, $e^{t\Delta} : L^\infty \to L^\infty$ with norm $M_2 \leq 1$ by the maximum principle; i.e.,

$$\|e^{t\Delta}\phi\|_\infty \leq \|\phi\|_\infty.$$

Then again by the interpolation theorem [11] of Nirenberg, we have $e^{t\Delta} : L^p \to L^p$ with the norm $M \leq {M_1}^{\lambda}{M_2}^{1-\lambda}$ with $\lambda = \frac{2}{p}$; i.e.,

$$\|e^{t\Delta}\phi\|_p \leq e^{t\gamma_1\frac{2}{p}}\|\phi\|_p \quad \forall p \in [2, \infty).$$

First by the following lemma, we get the estimate on $\|u(s)\|_\infty$ in terms of $\|u_0\|_{1+\epsilon}$, which is a crucial estimate to prove the global existence theorem.

**Lemma 4.3.** *$\|u(s)\|_\infty \leq G(M,T)\|u_0\|_{1+\epsilon}s^{-\frac{1}{2(1+\epsilon)}}$, where $G(M,T) : R^+ \times R^+ \to R^+$ is an increasing function of $M$ and $T$ respectively when $\|u(s)\|_{1+\epsilon} \leq M + 1$ on $[0,T]$.*

For the proof of the above lemma, we use the following local existence theorem by Brezis and Cazenave (Theorem 1 in [2]).

**Theorem 4.4.** *Assume $q > n(p-1)/2$ (resp. $q = n(p-1)/2$) and $q \geq 1$ (resp. $q > 1$), $n \geq 1$. Given any $u_0 \in L^q$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^q)$ with $u(0) = u_0$, which is a classical solution of* (4.1)*. Moreover, we have smoothing effect and continuous dependence; namely,*

$$\|u - v\|_{L^q} + t^{n/2q}\|u - v\|_{L^\infty} \leq C\|u_0 - v_0\|_{L^q}.$$

*for all $t \in (0,T]$ where $T = min(T(u_0), T(v_0))$ and $C$ can be estimated in terms of $\|u_0\|_{L^q}$ and $\|v_0\|_{L^q}$.*

*Proof of Lemma 4.3.* By replacing $q = 1+\epsilon$ and $p = 3$ in this theorem, we get an $L^\infty$ estimate for the solution $u$ of problem (4.1),

$$\|u(s)\|_\infty \le G(M,T)\|u_0\|_{1+\epsilon} s^{-\frac{1}{2(1+\epsilon)}}$$

where $G(M,T) = e^{c_1(M+1)^{\frac{6+6\epsilon}{2+3\epsilon}} T^{\frac{3\epsilon}{2+3\epsilon}}}$ and $c_1 = c_1(\epsilon)$. □

Next we prove the following global existence theorem.

**Theorem 4.5.** *For any fixed $\epsilon > 0$, there is a $\delta > 0$ such that $\|u_0\|_{1+\epsilon} \le \delta$ implies that $u(t)$ globally exists and*

$$\|u(t)\|_{1+\epsilon} \le 2\delta \quad \textit{for} \quad 0 \le t \le \infty.$$

*Proof.* Let us choose $\delta > 0$ such that $\frac{1+\epsilon}{\epsilon}(G(2\delta,1)\delta)^2 < -\gamma$ and $e^{\frac{1+\epsilon}{\epsilon}(G(2\delta,1)\delta)^2} < 2$. By the local existence theorem that is stated in Lemma 4.3, there exists $T_1 > 0$ such that $u(t)$ exists on $[0,T_1]$ and $\|u(t)\|_{1+\epsilon} \le 2\delta$ for $t \in [0,T_1]$. Now we will prove the following claims to prove the global existence result.
**Claim 1.** $\|u(t)\|_{1+\epsilon} < 2\delta$ for $0 \le t \le 1$.
**Claim 2.** $\|u(1)\|_{1+\epsilon} \le \delta$.

*Proof of Claim 1.* Suppose Claim 1 is not true. Then set

$$\tilde{T} = \min\{t > 0 | \|u(t)\|_{1+\epsilon} \ge 2\delta\}.$$

We have $\|u(\tilde{T})\|_{1+\epsilon} = 2\delta$ and $\|u(t)\|_{1+\epsilon} < 2\delta$ for all $0 \le t < \tilde{T}$ with $0 < \tilde{T} \le 1$. Consequently, for any $t \in [0,\tilde{T}]$, the Remark 4.1 in this section indicates that there is a $\gamma < 0$ ($\gamma = \gamma_1 \frac{2\epsilon}{1+\epsilon}$) such that

$$\begin{aligned}
\|u(t)\|_{1+\epsilon} &\le e^{\gamma t}\|u_0\|_{1+\epsilon} + \int_0^t \|e^{(t-s)\Delta}u^3(s)\|_{1+\epsilon} ds \\
&\le e^{\gamma t}\|u_0\|_{1+\epsilon} + \int_0^t e^{\gamma(t-s)}\|u(s)\|_{1+\epsilon}\|u(s)\|_\infty^2 ds \\
&\le e^{\gamma t}\|u_0\|_{1+\epsilon} + \int_0^t e^{\gamma(t-s)}\|u(s)\|_{1+\epsilon} B s^{-\frac{1}{1+\epsilon}} ds.
\end{aligned}$$

where $B = (G(2\delta,\tilde{T}))^2\|u_0\|_{1+\epsilon}^2$ for notational simplicity. Then

$$e^{-\gamma t}\|u(t)\|_{1+\epsilon} \le \|u_0\|_{1+\epsilon} + B\int_0^t e^{-\gamma s}\|u(s)\|_{1+\epsilon} s^{-\frac{1}{1+\epsilon}} ds := H(t),$$

where, we denote the right-hand side of the above inequality by $H(t)$. Then

$$H'(t) = Be^{-\gamma t}\|u(t)\|_{1+\epsilon} t^{-\frac{1}{1+\epsilon}} \le BH(t)t^{-\frac{1}{1+\epsilon}}.$$

Hence $\frac{H'(t)}{H(t)} \le Bt^{-\frac{1}{1+\epsilon}}$. Integrating from 0 to $t$ yields

$$\ln\frac{H(t)}{H(0)} \le \frac{1+\epsilon}{\epsilon} B t^{\frac{\epsilon}{1+\epsilon}} \quad \text{and} \quad H(t) \le H(0)e^{\frac{1+\epsilon}{\epsilon}Bt^{\frac{\epsilon}{1+\epsilon}}},$$

which implies

$$\|u(t)\|_{1+\epsilon} \le e^{\gamma t}H(t) \le \|u_0\|_{1+\epsilon} e^{\gamma t} e^{\frac{1+\epsilon}{\epsilon}Bt^{\frac{\epsilon}{1+\epsilon}}}. \tag{4.2}$$

Note that for $t \in [0, \tilde{T}] \subset [0, 1]$,

$$e^{\gamma t} e^{\frac{1+\epsilon}{\epsilon} B t^{\frac{\epsilon}{1+\epsilon}}} \leq e^{\frac{1+\epsilon}{\epsilon} B} \leq e^{\frac{1+\epsilon}{\epsilon} (G(2\delta, 1))^2 \delta^2} < 2,$$

by the choice of $\delta$. Then by (4.2), $\|u(\tilde{T})\|_{1+\epsilon} < 2\delta$ which is contrary to the fact that $\|u(\tilde{T})\|_{1+\epsilon} = 2\delta$. So the proof of Claim 1 is completed.

*Proof of Claim 2.* After proving Claim 1, Claim 2 follows immediately by the inequality (4.2) since

$$\|u(1)\|_{1+\epsilon} \leq \delta e^{\gamma + \frac{1+\epsilon}{\epsilon} B} \leq \delta$$

by the choice of $\delta$.

Finally, the existence of a global solution will follow by a simple induction argument. By Claim 1 and 2, we can easily conclude that $\|u(j)\|_{1+\epsilon} \leq \delta$ for each $j = 0, 1, \dots$. Using $u(j)$ as new initial data, we can solve the problem for $t \in [j, j+1]$, and we get $\|u(t)\|_{1+\epsilon} \leq 2\delta$ for all $t \in [j, j+1]$ by Claim 2. $\square$

As a conclusion, we can state the generalized global existence result as follows.

**Corollary 4.6.** *For $q > 1$, problem (4.1) has a global solution in $L^q$ for all $u_0 \in L^q$ with $\|u_0\|_q$ sufficiently small.*

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