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YOUNG MEASURE MINIMIZERS IN THE ASYMPTOTIC ANALYSIS OF THIN FILMS

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Dedicated to Klaus Schmitt on the occasion of his 65th birthday

ABSTRACT. An integral representation for a relaxed functional arising in the membrane theory is obtained in terms of Young measures generated by sequences $\{(\nabla_\alpha u_{\varepsilon_n} | \frac{1}{\varepsilon_n}\nabla_3 u_{\varepsilon_n})\}$ of scaled gradients.

1. INTRODUCTION

Let $\omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary, and consider a thin three dimensional domain $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ filled with an elastic material with elastic energy density W_ε and subject to dead loading body forces of densities $f_\varepsilon \in L^{p'}(\Omega_\varepsilon, \mathbb{R}^3)$ where $1 < p < +\infty$, and p' stands for the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We assume that for fixed $\varepsilon > 0$, in order to reach equilibrium, u_ε seeks to minimize

$$E_\varepsilon(u) := \int_{\Omega_\varepsilon} W_\varepsilon(x, \nabla u(x))\, dx - \int_{\Omega_\varepsilon} f_\varepsilon(x) \cdot u(x)\, dx,$$

among all kinematically admissible fields u.

To study the effective behavior of a very thin film, we consider a sequence $\{\varepsilon_n\}$ of positive real numbers (half-thickness) converging to zero and we recast energy functionals over varying domains Ω_{ε_n} into functionals with a fixed domain of integration $\Omega := \omega \times (-1, 1)$ by means of a reformulation of the problem through a $\frac{1}{\varepsilon_n}$-dilation in the transverse direction x_3. With $x = (x_1, x_2, x_3)$, set

$$\begin{gathered} W^{(\varepsilon_n)}(x, \cdot) := W_{\varepsilon_n}(x_1, x_2, \varepsilon_n x_3; \cdot), \\ f^{(\varepsilon_n)}(x) := f_{\varepsilon_n}(x_1, x_2, \varepsilon_n x_3), \\ v_n(x) := u_{\varepsilon_n}(x_1, x_2, \varepsilon_n x_3). \end{gathered}$$

After an appropriate rescaling, v_n seeks to minimize

$$E^{(\varepsilon_n)}(v) := \int_\Omega W^{(\varepsilon_n)}\left(x, \left(\nabla_\alpha v \Big| \frac{1}{\varepsilon_n}\nabla_3 v\right)(x)\right) dx - \int_\Omega f^{(\varepsilon_n)}(x) \cdot v(x)\, dx,$$

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among all kinematically admissible fields $v = (v_1, v_2, v_3)$ on Ω, where $\nabla_\alpha v$ stands for the 3×2 matrix of partial derivatives $\frac{\partial v_i}{\partial x_\alpha}$, $i \in \{1,2,3\}, \alpha \in \{1,2\}$, $\nabla_3 v$ is the three-dimensional vector of partial derivatives $\frac{\partial v_i}{\partial x_3}$, $i \in \{1,2,3\}$, and $(A|a)$ denotes a 3×3 matrix whose first two columns are those of the 3×2 matrix A and the last column is the vector $a \in \mathbb{R}^3$. We assume that the rescaled energy density $W^{(\varepsilon_n)}$ does not explicitly depend on ε_n. Precisely, $W^{(\varepsilon_n)} = W$ where $W : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ is a Carathéodory integrand (see Definition 2.6) satisfying for some $1 < p < +\infty$ the p-growth and coercivity condition

$$\frac{1}{C}|A|^p - C \le W(x, A) \le C(1 + |A|^p) \tag{1.1}$$

for $\mathcal{L}^3$-a.e $x \in \Omega$ and for all $A \in \mathbb{M}^{3\times 3}$, where $C > 0$ is a real constant and $\mathbb{M}^{3\times 3}$ denotes the space of real 3×3 matrices endowed with the usual Euclidean norm $|A| := \sqrt{\operatorname{tr}(A^T A)}$. Assuming, moreover, that the rescaled body force density $f^{(\varepsilon_n)}$ is independent of n (see e.g. [17]), the study of the effective energy of the limiting system is hinged on the understanding of the asymptotic behavior of the energies

$$I_n(v_n) := \int_\Omega W\left(x, \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right) dx.$$

An extensive literature in this direction (see [2, 6, 7, 11, 12, 13, 17, 23, 24], among others) is usually formulated in the natural mathematical setting of Γ-convergence, and this approach gives rise to the so-called membrane theory. In view of the a priori bound

$$\sup_{n\in\mathbb{N}} \int_\Omega \Big| \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\Big|^p dx < +\infty$$

for energy bounded sequences, and derived from (1.1), in this paper we obtain an integral representation of the relaxed energy functional $\mathcal{W} : W^{1,p}(\omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{W}(v, c) := \inf \Big\{ &\liminf_{n\to+\infty} \int_\Omega W\left(x, \left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right) dx : \varepsilon_n \to 0^+, \\ &v_n \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3), \\ &\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c \text{ weakly in } L^p(\Omega;\mathbb{R}^3)\Big\},\end{aligned} \tag{1.2}$$

in terms of scaled gradient p-Young measures, which are essentially Young measures generated by sequences of scaled gradients $\left\{\left(\nabla_\alpha v_n \big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ (see Definitions 2.1 and 2.5).

Definition 1.1. Let $\Omega := \omega \times (-1, 1)$, where $\omega \subset \mathbb{R}^2$ is an open domain, and let $1 \le p \le +\infty$. A Young measure μ on $\Omega \times \mathbb{R}^9$ is called a *scaled gradient* p-Young measure (scaled gradient Young measure if $p = +\infty$) if there exist sequences $\varepsilon_n \to 0^+$ and $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that

(i) $\{v_n\}$ is weakly (weakly * if $p = +\infty$) convergent in $W^{1,p}(\Omega;\mathbb{R}^3)$,
(ii) $\{\frac{1}{\varepsilon_n}\nabla_3 v_n\}$ is weakly (weakly * if $p = +\infty$) convergent in $L^p(\Omega;\mathbb{R}^3)$,
(iii) $\mathcal{E}_{\left(\nabla_\alpha v_n \big| \frac{1}{\varepsilon_n}\nabla_3 v_n\right)} \rightharpoonup \mu$ weakly * in $C_0(\Omega \times \mathbb{R}^9)'$.

The weak (weak * if $p = +\infty$) limit of v_n in $W^{1,p}(\Omega;\mathbb{R}^3)$ is called an *underlying deformation for* μ while the weak (weak * if $p = +\infty$) limit of $\frac{1}{\varepsilon_n}\nabla_3 v_n$ in $L^p(\Omega;\mathbb{R}^3)$

is called a *Cosserat vector associated to* μ. For $1 \le p < +\infty$, we set

$$Y^p_{v,c} := \Big\{\nu \in Y(\Omega \times \mathbb{R}^9) : \nu \text{ is a scaled gradient } p\text{-Young measure with underlying deformation } v \text{ and associated Cosserat vector } c\Big\}.$$

Our relaxation result is the following.

Theorem 1.2. *Let* $1 < p < +\infty$, $\Omega := \omega \times (-1,1)$, *where* $\omega \subset \mathbb{R}^2$ *is an open, bounded Lipschitz domain, and let* $W : \Omega \times \mathbb{M}^{3\times 3} \to \mathbb{R}$ *be a Carathéodory integrand satisfying* (1.1). *Let* $v \in W^{1,p}(\omega;\mathbb{R}^3)$ *and* $c \in L^p(\Omega;\mathbb{R}^3)$ *be given, and define the relaxed functional* $\mathcal{W} : W^{1,p}(\omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3) \to \mathbb{R}$ *by* (1.2). *Then*

$$\mathcal{W}(v,c) = \inf_{\nu \in Y^p_{v,c}} \int_{\Omega\times\mathbb{M}^{3\times3}} W d\nu. \tag{1.3}$$

Moreover, the infimum on the right hand side is attained, i.e. there exists a scaled gradient p-Young measure $\mu_0 \in Y^p_{v,c}$ *such that*

$$\inf_{\nu \in Y^p_{v,c}} \int_{\Omega\times\mathbb{M}^{3\times3}} W d\nu = \int_{\Omega\times\mathbb{M}^{3\times3}} W d\mu_0. \tag{1.4}$$

A challenging open problem is the identification of algebraic and analytical conditions on parametrized probability measures both necessary and sufficient to guarantee that they belong to $Y^p_{v,c}$. The corresponding program for probability measures generated by gradients bounded in L^p was carried out by Kinderlehrer and Pedregal (see [19, 20]) and subsequently generalized to the realm of $\mathcal{A}$-quasiconvexity by Fonseca and Müller in [15]. In the case of Young measures generated by sequences of scaled gradients, recent progress was made by Bocea and Fonseca [9], where the slightly broader class of *bending Young measures* has been completely characterized.

The crucial ingredient needed to prove Theorem 1.2 is the following decomposition result whose proof may be found in [8].

Theorem 1.3. *Let* $\Omega := \omega \times (-1,1)$, *where* $\omega \subset \mathbb{R}^2$ *is an open, bounded Lipschitz domain, let* $\{\varepsilon_n\}$ *be a sequence of positive real numbers converging to zero, and consider a sequence* $\{v_n\}$ *bounded in* $W^{1,p}(\Omega;\mathbb{R}^3)$ $(1 < p < +\infty)$ *and satisfying*

$$\sup_{n\in\mathbb{N}} \int_\Omega \Big| \Big(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n}\nabla_3 v_n\Big)(x)\Big|^p dx < +\infty.$$

Suppose further that $v_n \rightharpoonup v$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$ *and* $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c$ *in* $L^p(\Omega;\mathbb{R}^3)$. *Then there exists a subsequence* $\{v_{n_k}\}$ *of* $\{v_n\}$ *and a sequence* $\{w_k\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ *such that*

$$\lim_{k\to\infty} \mathcal{L}^3\left(\{x\in\Omega : w_k(x) \ne v_{n_k}(x)\} \cup \{x \in \Omega : \nabla w_k(x) \ne \nabla v_{n_k}(x)\}\right) = 0, \tag{1.5}$$

$$\Big\{\Big(\nabla_\alpha w_k \Big| \frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\Big)\Big\} \text{ is } p\text{-equi-integrable,} \tag{1.6}$$

$$w_k \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3), \tag{1.7}$$

$$\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k \rightharpoonup c \text{ weakly in } L^p(\Omega;\mathbb{R}^3). \tag{1.8}$$

This theorem allows us to decompose (up to a subsequence) $\{(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n)\}$ as a sum of a sequence $\{(\nabla_\alpha w_n|\frac{1}{\varepsilon_n}\nabla_3 w_n)\}$ whose p-th power is equi-integrable and a remainder converging to zero in measure. We may say that $\{(\nabla_\alpha w_n|\frac{1}{\varepsilon_n}\nabla_3 w_n)\}$ carries the oscillations, while the remainder accounts for the concentration effects.

In the next section we present the basic facts about Young measures needed for the proof of our relaxation theorem. The proof of Theorem 1.2 is the subject of Section 3 of the paper.

2. Young measures

The characterization of (oscillatory) limits of nonlinear quantities in the Calculus of Variations has been successfully analyzed in several contexts by means of Young measures. Young measures were originally introduced in Optimal Control Theory by L.C. Young in connection to nonconvex problems, thus providing the appropriate framework for the description of generalized minimizers in the Calculus of Variations (see [29, 30]). Later, Tartar introduced the use of Young measures in the PDE framework (see [26, 27, 28]). For more details regarding the study of Young measures we refer the reader to [3, 4, 5, 16, 19, 20, 21, 22, 25], among others.

In this section we recall the definition and the relevant results about Young measures that we will need in the sequel; we follow closely the approach of Kristensen (see [22]). Let D be an open subset of $\mathbb{R}^l (l \geq 1)$, $C(D)$ be the space of real-valued continuous functions on D and define

$$C_0(D) := \Big\{\varphi \in C(D) : \text{for every } \varepsilon > 0 \text{ there exists a compact set } K \subset D \\ \text{such that } |\varphi(x)| \leq \varepsilon \text{ if } x \in D \setminus K\Big\}.$$

Endowed with the supremum norm, $C_0(D)$ is a separable Banach space. In view of Riesz' Theorem the dual space $C_0(D)'$ can be identified with the space of bounded Radon measures on D with the norm $\|\mu\| := |\mu|(D)$, via the duality pairing

$$\langle \mu, \varphi\rangle = \int_D \varphi(x) \cdot \frac{d\mu}{d|\mu|}(x) d|\mu|(x),$$

where $|\mu|$ stands for the total variation of μ and is a non-negative, finite Radon measure on D.

Definition 2.1. (i) A non-negative Radon measure μ on $\Omega \times \mathbb{R}^d$ with the property

$$\mu(B \times \mathbb{R}^d) = \mathcal{L}^N(B) \quad \text{for all Borel subsets of } \Omega,$$

is called a *Young measure*. The set of Young measures on $\Omega \times \mathbb{R}^d$ is denoted by $Y(\Omega \times \mathbb{R}^d)$.

(ii) A Young measure μ for which there exists a $\mathcal{L}^N$-measurable mapping $V : \Omega \to \mathbb{R}^d$ such that

$$\int_{\Omega\times\mathbb{R}^d} f d\mu = \int_\Omega f(x, V(x))dx, \quad \text{for all } f \in C_0(\Omega \times \mathbb{R}^d),$$

is called an elementary Young measure. We write

$$\mu = \mathcal{E}_V := \int_\Omega \delta_x \otimes \delta_{V(x)} dx,$$

where δ_x and $\delta_{V(x)}$ are the Dirac measures on Ω concentrated at x and on $\mathbb{R}^d$ concentrated at $V(x)$, respectively.

(iii) A product measure $\left(\mathcal{L}^N\lfloor\Omega\right)\otimes\tilde{\mu}$ on $\Omega\times\mathbb{R}^d$, where $\tilde{\mu}$ is a probability measure on $\mathbb{R}^d$, is called a homogeneous Young measure.

Remark 2.2. The definition of Young measures in Definition 2.1 (i) follows that of Berliocchi and Lasry (see [5]). It can be shown (cf. [22]) to be equivalent to the original definition of L.C. Young [29] and the ones used in literature (e.g., [3, 4, 25]).

Proposition 2.3. *Let $\mu \in Y(\Omega \times \mathbb{R}^d)$. Then there exists a mapping $x \mapsto \mu_x$ from Ω into the set of non-negative, finite Radon measures on $\mathbb{R}^d$, such that*

(i) $\mu = \int_\Omega \delta_x \otimes \mu_x dx$, *i.e. for any Borel function $f : \Omega \times \mathbb{R}^d \to [0, +\infty]$ the function $x \mapsto \displaystyle\int_{\mathbb{R}^d} f(x, A)d\mu_x(A)$ is $\mathcal{L}^N$-measurable, and*

$$\int_{\Omega\times\mathbb{R}^d} f d\mu = \int_\Omega \int_{\mathbb{R}^d} f(x, A)d\mu_x(A)dx; \tag{2.1}$$

(ii) $\mu_x(\mathbb{R}^d) = 1$, *for $\mathcal{L}^N$-a.e. $x \in \Omega$.*

Moreover, if $x \mapsto \nu_x$ is another such mapping then $\nu_x = \mu_x$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.

Remark 2.4. Proposition 2.3 is a special case of a result in [10] (Proposition 13, pp. 39-40). See also [1].

Consider a sequence $\{V_n\}$ of measurable mappings of Ω into $\mathbb{R}^d$. The corresponding sequence $\{\mathcal{E}_{V_n}\}$ of elementary Young measures is bounded in $C_0(\Omega \times \mathbb{R}^d)'$ and thus, by virtue of Banach-Alaoglu's Theorem, there exists a subsequence $\{V_{n_k}\}$ and a measure $\mu \in C_0(\Omega \times \mathbb{R}^d)'$ such that

$$\mathcal{E}_{V_{n_k}} \rightharpoonup \mu \text{ weakly} * \text{ in } C_0(\Omega \times \mathbb{R}^d)'. \tag{2.2}$$

A necessary and sufficient condition for μ to be a Young measure is that

$$\lim_{R\to\infty} \sup_{k\in\mathbb{N}} \mathcal{L}^N \left(\{x \in \Omega : |V_{n_k}(x)| \geq R\}\right) = 0, \tag{2.3}$$

or, equivalently (see [18, 21]): There exists a Borel function $g : \mathbb{R}^d \to [0, +\infty]$ such that $\lim_{|A|\to+\infty} g(A) = +\infty$, and

$$\sup_{k\in\mathbb{N}} \int_\Omega g(V_{n_k}(x))dx < +\infty.$$

Definition 2.5. If (2.2) and (2.3) hold, then we say that the Young measure μ is generated by the sequence $\{V_{n_k}\}$.

Definition 2.6. (i) A function $f : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is called a normal integrand if f is Borel measurable and $f(x, \cdot) : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous for every $x \in \Omega$.

(ii) A real-valued function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is called a Carathéodory integrand if both f and $-f$ are normal integrands.

Set $f^- := -\min\{f, 0\}$. The following result is well-known (see [3, 4, 5, 14, 20, 21, 22, 25]).

Lemma 2.7. *Let $\{v_n\}$ be a sequence of measurable mappings from Ω into $\mathbb{R}^d$ which generates the Young measure μ.*

(i) *If $f : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a normal integrand and if $\{f^-(\cdot, v_n)\}$ is equi-integrable then*
$$\int_{\Omega\times\mathbb{R}^d} f d\mu \le \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))dx.$$
Moreover, if f is a Carathéodory integrand then $\{f(\cdot, v_n)\}$ is equi-integrable if and only if
$$\int_{\Omega\times\mathbb{R}^d} f d\mu = \lim_{n\to\infty} \int_\Omega f(x, v_n(x))dx.$$

(ii) *If $\{w_n\}$ is a sequence of measurable mappings from Ω into $\mathbb{R}^d$ such that $v_n - w_n \to 0$ in measure then $\{w_n\}$ also generates μ.*

3. Proof of Theorem 1.2

We will identify $\mathbb{R}^9$ with the space of real 3×3 matrices $\mathbb{M}^{3\times 3}$. To prove that
$$\mathcal{W}(v, c) \ge \inf_{\nu\in Y^p_{v,c}} \int_{\Omega\times\mathbb{M}^{3\times 3}} W d\nu, \tag{3.1}$$
let $\varepsilon_n \to 0^+$ and $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c$ weakly in $L^p(\Omega;\mathbb{R}^3)$. Extract a subsequence (not relabelled) so that
$$\begin{aligned}&\liminf_{n\to\infty}\int_\Omega W\left(x, \left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx\\ &= \lim_{n\to\infty}\int_\Omega W\left(x, \left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx < +\infty,\end{aligned} \tag{3.2}$$
where the last inequality follows by (1.1). By Banach-Alaoglu's Theorem, there exists a subsequence $\left\{\left(\nabla_\alpha v_{n_j}\big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)\right\}$ of $\left\{\left(\nabla_\alpha v_n\big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right\}$ such that
$$\mathcal{E}_{\left(\nabla_\alpha v_{n_j}\big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)} \rightharpoonup \mu \quad \text{weakly * in } C_0(\Omega\times\mathbb{R}^9)',$$
for some Radon measure μ on $\Omega\times\mathbb{R}^9$. Since
$$\sup_{j\in\mathbb{N}}\int_\Omega \left|\left(\nabla_\alpha v_{n_j}\Big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)(x)\right|^p dx < +\infty,$$
it follows that (2.3) holds, with k replaced by j, and V_{n_j} by $\left(\nabla_\alpha v_{n_j}\big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)$. Thus, μ is a Young measure. It is clear that we actually have $\mu \in Y^p_{v,c}$. By (1.1), $W(x, A) + C \ge 0$ for $\mathcal{L}^3$-a.e. $x \in \Omega$ and all $A \in \mathbb{M}^{3\times 3}$. Thus, we can apply Lemma 2.7 (i) (take $f = W + C$) and we obtain
$$\begin{aligned}\int_{\Omega\times\mathbb{M}^{3\times 3}} (W + C)d\mu &\le \liminf_{j\to\infty}\int_\Omega\left(W\left(x, \left(\nabla_\alpha v_{n_j}\Big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)(x)\right) + C\right)dx\\ &= \lim_{j\to\infty}\int_\Omega W\left(x, \left(\nabla_\alpha v_{n_j}\Big|\frac{1}{\varepsilon_{n_j}}\nabla_3 v_{n_j}\right)(x)\right)dx + C\mathcal{L}^3(\Omega)\\ &= \lim_{n\to\infty}\int_\Omega W\left(x, \left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx + C\mathcal{L}^3(\Omega),\end{aligned}$$
where we have used (3.2). By Proposition 2.3 we have
$$\int_{\Omega\times\mathbb{M}^{3\times 3}} C d\mu = \int_\Omega\int_{\mathbb{M}^{3\times 3}} C d\mu_x dx = C\int_\Omega \mu_x(\mathbb{M}^{3\times 3})dx = C\mathcal{L}^3(\Omega),$$

and in view of the previous equation we obtain

$$\int_{\Omega\times\mathbb{M}^{3\times 3}} W\,d\mu \le \liminf_{n\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx.$$

Thus,

$$\inf_{\nu\in Y^p_{v,c}}\int_{\Omega\times\mathbb{M}^{3\times 3}} W\,d\nu \le \liminf_{n\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx.$$

By the arbitrariness of $\{\varepsilon_n\}$ and $\{v_n\}$ satisfying the admissibility conditions we assert (3.1).

Conversely, let $\nu\in Y^p_{v,c}$. There exist sequences $\varepsilon_n\to 0^+$ and $\{v_n\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that $v_n\rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_n}\nabla_3 v_n\rightharpoonup c$ weakly in $L^p(\Omega;\mathbb{R}^3)$, and the sequence $\{(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n)\}$ generates ν.

By Theorem 1.3 there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a sequence $\{w_k\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that (1.5)-(1.8) hold. In view of (1.5), the sequence

$$\left\{\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)-\left(\nabla_\alpha v_{n_k}\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 v_{n_k}\right)\right\}$$

converges to zero in measure and thus, by Lemma 2.7 (ii), $\{(\nabla_\alpha w_k|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k)\}$ also generates ν. Since by (1.1) and (1.6) the sequence $\left\{W\left(\cdot,\left(\nabla_\alpha w_k|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)\right)\right\}$ is equi-integrable, we deduce by (i) of Lemma 2.7 that

$$\int_{\Omega\times\mathbb{M}^{3\times 3}} W\,d\nu = \lim_{k\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)(x)\right)dx \ge \mathcal{W}(v,c),$$

where the last inequality follows by (1.7) and (1.8). Passing to the infimum over all $\nu\in Y^p_{v,c}$ we have that

$$\inf_{\nu\in Y^p_{v,c}}\int_{\Omega\times\mathbb{M}^{3\times 3}} W\,d\nu \ge \mathcal{W}(v,c).$$

Taking into account (3.1), we obtain that (1.3) holds. It remains to prove (1.4).

Claim: There exist sequences $\varepsilon_n\to 0^+$ and $\{v_n\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that

$$\begin{gathered} v_n\rightharpoonup v\quad\text{weakly in } W^{1,p}(\Omega;\mathbb{R}^3),\\ \frac{1}{\varepsilon_n}\nabla_3 v_n\rightharpoonup c\quad\text{weakly in } L^p(\Omega;\mathbb{R}^3),\\ \lim_{n\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx=\mathcal{W}(v,c). \end{gathered}\tag{3.3}$$

Assuming that the claim holds, let μ_0 be the scaled gradient p-Young measure generated by a subsequence (not relabelled) of $\{(\nabla_\alpha v_n|\frac{1}{\varepsilon_n}\nabla_3 v_n)\}$. Let $\{n_k\}\subset\{n\}$ and $\{w_k\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$ be the sequences provided by Theorem 1.3. Taking into account (1.5)-(1.8), (3.3), and making use of Lemma 2.7, we have

$$\begin{aligned}
\mathcal{W}(v,c) &\le \liminf_{k\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha w_k\Big|\frac{1}{\varepsilon_{n_k}}\nabla_3 w_k\right)(x)\right)dx\\
&= \int_{\Omega\times\mathbb{M}^{3\times3}} W d\mu_0\\
&\le \lim_{n\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_n\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)(x)\right)dx\\
&= \mathcal{W}(v,c).
\end{aligned}$$

Thus,

$$\int_{\Omega\times\mathbb{M}^{3\times3}} W d\mu_0 = \mathcal{W}(v,c),$$

and in view of (1.3), we deduce (1.4).

Proof of Claim: For any $n\in\mathbb{N}$, let $\{\varepsilon_{k,n}\}\subset(0,+\infty)$ and $\{v_{k,n}\}\subset W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\lim_{k\to\infty}\varepsilon_{k,n}=0$, $v_{k,n}\rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_{k,n}}\nabla_3 v_{k,n}\rightharpoonup c$ weakly in $L^p(\Omega;\mathbb{R}^3)$ as $k\to\infty$ and, in addition,

$$\mathcal{W}(v,c)\le\liminf_{k\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k,n}\Big|\frac{1}{\varepsilon_{k,n}}\nabla_3 v_{k,n}\right)(x)\right)dx\le\mathcal{W}(v,c)+\frac{1}{n}.$$

Extract an increasing subsequence $\{k(j,n)\}_j$ of $\{k\}$ so that

$$\begin{aligned}
&\liminf_{k\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k,n}\Big|\frac{1}{\varepsilon_{k,n}}\nabla_3 v_{k,n}\right)(x)\right)dx\\
&=\lim_{j\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha v_{k(j,n),n}\Big|\frac{1}{\varepsilon_{k(j,n),n}}\nabla_3 v_{k(j,n),n}\right)(x)\right)dx,
\end{aligned}$$

and put $\overline{\varepsilon}_{j,n}:=\varepsilon_{k(j,n),n}$ and $\overline{v}_{j,n}:=v_{k(j,n),n}$. Thus,

$$\lim_{j\to\infty}\overline{\varepsilon}_{j,n}=0, \tag{3.4}$$

$$\lim_{n\to\infty}\lim_{j\to\infty}\int_\Omega W\left(x,\left(\nabla_\alpha\overline{v}_{j,n}\Big|\frac{1}{\overline{\varepsilon}_{j,n}}\nabla_3\overline{v}_{j,n}\right)(x)\right)dx=\mathcal{W}(v,c), \tag{3.5}$$

$\overline{v}_{j,n}\rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ as $j\to\infty$, and

$$\frac{1}{\overline{\varepsilon}_{j,n}}\nabla_3\overline{v}_{j,n}\rightharpoonup c\quad\text{weakly in } L^p(\Omega;\mathbb{R}^3)\text{ as } j\to\infty.$$

Consider a countable family $\{\varphi_i\}_{i\in\mathbb{N}}$ dense in $L^{p'}(\Omega)$. The weak convergence of $\overline{v}_{j,n}$ to v in $W^{1,p}(\Omega;\mathbb{R}^3)$ and that of $\frac{1}{\overline{\varepsilon}_{j,n}}\nabla_3\overline{v}_{j,n}$ to c in $L^p(\Omega;\mathbb{R}^3)$ imply that for each $i,n\in\mathbb{N}$ we have

$$\lim_{j\to\infty}\int_\Omega\varphi_i(x)\overline{v}_{j,n}(x)dx=\int_\Omega\varphi_i(x)v(x)dx, \tag{3.6}$$

$$\lim_{j\to\infty}\int_\Omega\varphi_i(x)\Big(\frac{1}{\overline{\varepsilon}_{j,n}}\nabla_3\overline{v}_{j,n}(x)\Big)dx=\int_\Omega\varphi_i(x)c(x)dx, \tag{3.7}$$

$$\lim_{j\to\infty}\int_\Omega\varphi_i(x)\nabla\overline{v}_{j,n}(x)dx=\int_\Omega\varphi_i(x)\nabla v(x)dx. \tag{3.8}$$

Taking into account (3.4), (3.5), (3.6), (3.7), and (3.8), a diagonalization process allows us to find an increasing subsequence $\{j(n)\}$ of $\{j\}$ such that, after denoting $\varepsilon_n := \overline{\varepsilon}_{j(n),n}$ and $v_n := \overline{v}_{j(n),n}$ we have

$$\begin{gathered}
\varepsilon_n \to 0^+, \quad v_n \rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega;\mathbb{R}^3), \\
\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup c \quad \text{weakly in } L^p(\Omega;\mathbb{R}^3),
\end{gathered}$$

and (3.3) holds. $\square$

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