Sixth Mississippi State Conference on Differential Equations and Computational Simulations, *Electronic Journal of Differential Equations*, Conference 15 (2007), pp. 11–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

BOUNDARY CONDITIONS FOR THE EINSTEIN-CHRISTOFFEL FORMULATION OF EINSTEIN'S EQUATIONS

DOUGLAS N. ARNOLD, NICOLAE TARFULEA

ABSTRACT. Specifying boundary conditions continues to be a challenge in numerical relativity in order to obtain a long time convergent numerical simulation of Einstein's equations in domains with artificial boundaries. In this paper, we address this problem for the Einstein-Christoffel (EC) symmetric hyperbolic formulation of Einstein's equations linearized around flat spacetime. First, we prescribe simple boundary conditions that make the problem well posed and preserve the constraints. Next, we indicate boundary conditions for a system that extends the linearized EC system by including the momentum constraints and whose solution solves Einstein's equations in a bounded domain.

1. Introduction

In the Arnowitt-Deser-Misner or ADM decomposition, Einstein's equations split into a set of evolution equations and a set of constraint equations (see Section 2), and what one does to construct a solution consists of first specifying the initial data that satisfies the constraints and then applying the evolution equations to compute the solution for later times. The problem of well-posedness in the analytic sense has been intensely studied, with the result that there is a great deal of choice of formulations available for analytic studies (see [15, 48, 29, 6, 32, 4, 7, 8, 1, 11, 13, 14, 20, 21, 23, 27, 40, 41], among others). However, in numerical relativity, one usually solves the Einstein equations in a bounded domain (cubic boxes are commonly used) and the question that arises is what boundary conditions to provide at the artificial boundary. In general, most numerical approaches have been made using carefully chosen initial data that satisfies the constraints. On the other hand, finding appropriate boundary conditions that lead to well-posedness and consistent with constraints is a difficult problem and subject to intense investigations in the recent years. In 1998, Stewart [49] has addressed this subject within Frittelli–Reula formulation [29] linearized around flat space with unit lapse and zero shift in the quarter plane. Both main system and constraints propagate as first order strongly hyperbolic systems. This implies that vanishing values of the constraints at $t = 0$ will propagate along characteristics. One wants the values

2000 *Mathematics Subject Classification.* 35Q75, 35L50, 83C99.
Key words and phrases. General relativity; Einstein equations; boundary conditions.

Published February 28, 2007.

of the incoming constraints at the boundary to vanish. However, one can not just impose them to vanish along the boundaries since the constraints involve derivatives of the fields across the boundary, not just the values of the fields themselves. If the Laplace–Fourier transforms are used, the linearity of the differential equations gives algebraic equations for the transforms of the fields. Stewart deduces boundary conditions for the main system in terms of Laplace–Fourier transforms that preserve the constraints by imposing the incoming modes for the system of constraints to vanish and translating these conditions in terms of Laplace–Fourier transforms of the main system variables. In 1999, a well posed initial-boundary value formulation was given by Friedrich and Nagy [22] in terms of a tetrad-based Einstein-Bianchi formulation. In view of our work which is to be presented here, of particular interest are the more recent investigations regarding special boundary conditions that prevent the influx of constraint violating modes into the computational domain for various hyperbolic formulations of Einstein's equations (see [2, 3, 5, 12, 16, 17, 34, 31, 42, 44, 43, 52, 53], among others). A different approach can be found in [28, 26], where the authors stray away from the general trend of seeking to impose the constraints along the boundary. They argue that the projection of the Einstein equations along the normal to the boundary yields necessary and appropriate boundary conditions for a wide class of equivalent formulations. The ideas and techniques introduced in [28, 26] are further developed and proven to be effective by the same authors in [25]. In principal, they show that the projection of the Einstein tensor along the normal to the boundary relates to the propagation of the constraints for two representations of Einstein's equations with vanishing shift vector, namely, the Arnowitt-Deser-Misner (ADM) formulation [10] and the classical Einstein-Christoffel (EC) formulation [6]. In particular, they obtain a set of boundary conditions for the EC formulation which has the same principal part as one of those presented in [17] and [9]. However, it should be said that, although the projection of the equations along the normal to the boundary represents an interesting approach, there are several issues which have not been addressed. First and foremost, there is the question of the well-posedness of the resulting boundary conditions, that is, in the sense that the initial-boundary value problem has a unique solution and this solution depends continuously on the initial and boundary data. In fact, at least in the EC case, the projection of the equations itself does not yield well-posed boundary conditions, because it provides too many conditions, some of which are ill-posed for the system. Therefore, further considerations are necessary in order to single out a subset of well-posed constraint-preserving boundary conditions for the EC formulation (see [25]).

Of course, specifying constraint-preserving boundary conditions for a certain formulation of Einstein's equations does not solve entirely the complicated problem of numerical relativity. There are other aspects that have to be addressed in order to obtain good numerical simulations; for example, the existence of bulk constraint violations, in which existing violations are amplified by the evolution equations (see [18, 19, 36, 45], and references therein). A review of some work done in this direction can be found in the introductory section of [34]. Before we end this very brief review, it should also be mentioned the work done on boundary conditions for Einstein's equations in harmonic coordinates, when Einstein's equations become a system of second order hyperbolic equations for the metric components. The question of the constraints preservation does not appear here, as it is hidden in the gauge choice, i.e., the constraints have to be satisfied only at the initial surface, the harmonic gauge guarantees their preservation in time (see [38, 50, 51], and references therein).

In this paper we address the boundary conditions problem for the classical EC equations derived in [6], linearized with respect to the flat Minkowski spacetime, and with arbitrary lapse density and shift perturbations. This problem has been addressed before in [16] in the case of spherically symmetric black-hole spacetimes in vacuum or with a minimally coupled scalar field, within the EC formulation of Einstein's equations. Here Stewart's idea of imposing the vanishing of the ingoing constraint modes as boundary conditions is employed once again. Then, the radial derivative is eliminated in favour of the time derivative in the expression of the ingoing constraints by using the main evolution system. The emerging set of boundary conditions depends only on the main variables and their time derivative and preserves the constraints. In [17] this technique is refined and employed for the generalized EC formulation [32] when linearized around Minkowski spacetime with vanishing lapse and shift perturbations on a cubic box. Again, the procedure consists in choosing well-posed boundary conditions to the evolution system for the constraint variables and translating them into well-posed boundary conditions for the variables of the main evolution system. The scheme proposed in [17] ends up giving two sets, called "Dirichlet and Neumann-like," of constraint preserving boundary conditions. However, the energy method used in [17] works only for symmetric hyperbolic constraint propagation, which forces the parameter η of the generalized EC system to satisfy the condition $0 < \eta < 2$. Therefore the analysis in [17] does not cover the case $\eta = 4$ required for the standard EC formulation introduced in [6]. In [9] we announced and presented our results on the boundary conditions problem for the standard EC formulation ($\eta = 4$) linearized around the Minkowski spacetime with arbitrary lapse density and shift perturbations in the Penn State Numerical Relativity Seminar. In essence, we introduced the very same sets of boundary conditions that are under scrutiny in this material, i.e., (4.4) and (4.13) (see [9]). Much of this material appeared also in the thesis of the second author [52].

The organization of this paper is as follows: in Section 2 we introduce Einstein's equations and their ADM equations for vacuum spacetime. In Section 3, by densitizing the lapse, linearizing, and defining a set of new variables, we derive the linearized EC first order symmetric hyperbolic formulation around flat spacetime. The equivalence of this formulation with the linearized ADM is proven in the Cauchy problem case. In Section 4 we indicate two distinct sets of well-posed constraint-preserving boundary conditions for the linearized EC. We prove that the linearized EC together with these boundary conditions is equivalent with linearized ADM on polyhedral domains. In Section 5 we indicate boundary conditions for an extended unconstrained system equivalent to the linearized ADM decomposition. In Section 6 we discuss the case of inhomogeneous boundary conditions. We end this work with a summary and a discussion of our results in Section 7. For reader's convenience, in the appendix we review a classical result on the L^2 well-posedness of maximal nonnegative boundary conditions for symmetric hyperbolic systems.

2. Einstein's Equations and the ADM Decomposition

In general relativity, spacetime is a 4-dimensional manifold M of events endowed with a pseudo-Riemannian metric $g_{\alpha\beta}$ that determines the length of the line element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. This metric determines curvature on the manifold, and Einstein's equations relate the curvature at a point of spacetime to the mass-energy there: $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, where $G_{\alpha\beta}$ is the *Einstein tensor*, i.e., the *trace-reversed* Ricci tensor $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$, and $T_{\alpha\beta}$ is the *energy-momentum tensor*. In what follows we will restrict ourselfs to the case of vacuum spacetime, that is $T_{\alpha\beta} = 0$. Einstein's equations can be viewed as equations for geometries, that is, their solutions are equivalent classes under spacetime diffeomorphisms of metric tensors. To break this diffeomorphisms invariance, Einstein's equations must be first transformed into a system having a well-posed Cauchy problem. In other words, the spacetime is foliated and each slice Σ_t is characterized by its intrinsic geometry γ_{ij} and extrinsic curvature K_{ij}, which is essentially the "velocity" of γ_{ij} in the unit normal direction to the slice. Subsequent slices are connected via the lapse function N and shift vector β^i corresponding to the ADM decomposition [10] (also [54]) of the line element

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{2.1}$$

This decomposition allows one to express six of the ten components of Einstein's equations in vacuum as a constrained system of evolution equations for the metric γ_{ij} and the extrinsic curvature K_{ij}:

$$\begin{gathered}\dot\gamma_{ij} = -2NK_{ij} + 2\nabla_{(i}\beta_{j)},\\ \dot K_{ij} = N[R_{ij} + (K^l_l)K_{ij} - 2K_{il}K^l_j] + \beta^l\nabla_l K_{ij} + K_{il}\nabla_j\beta^l + K_{lj}\nabla_i\beta^l - \nabla_i\nabla_j N,\\ R^i_i + (K^i_i)^2 - K_{ij}K^{ij} = 0,\\ \nabla^j K_{ij} - \nabla_i K^j_j = 0.\end{gathered} \tag{2.2}$$

where we use a dot to denote time differentiation and ∇_j for the covariant derivative associated to γ_{ij}. The spatial Ricci tensor R_{ij} has components given by second order spatial differential operators applied to the spatial metric components γ_{ij}. Indices are raised and traces taken with respect to the spatial metric γ_{ij}, and paranthesized indices are used to denote the symmetric part of a tensor.

3. Linearized Einstein–Christoffel

The Einstein-Christoffel or EC formulation [6] is derived from the ADM system with a *densitized lapse*. That is, we replace the lapse N in (2.2) with $\alpha\sqrt{g}$ where α denotes the lapse density. A trivial solution to this system is Minkowski spacetime in Cartesian coordinates, given by $\gamma_{ij} = \delta_{ij}$, $K_{ij} = 0$, $\beta^i = 0$, $\alpha = 1$. In the remainder of the paper we will consider the problem linearized about this solution. To derive the linearization, we write $\gamma_{ij} = \delta_{ij} + \bar g_{ij}$, $K_{ij} = \bar K_{ij}$, $\beta^i = \bar\beta^i$, $\alpha = 1 + \bar\alpha$, where the bars indicate perturbations, assumed to be small. If we substitute these expressions into (2.2) (with $N = \alpha\sqrt{\gamma}$), and ignore terms which are at least quadratic in the perturbations and their derivatives, then we obtain a linear system for the perturbations. Dropping the bars, the system is

$$\dot g_{ij} = -2K_{ij} + 2\partial_{(i}\beta_{j)}, \tag{3.1}$$

$$\dot K_{ij} = \partial^l\partial_{(j}g_{i)l} - \frac{1}{2}\partial^l\partial_l g_{ij} - \partial_i\partial_j g^l_l - \partial_i\partial_j\alpha, \tag{3.2}$$

$$C := \partial^j(\partial^l g_{lj} - \partial_j g^l_l) = 0, \tag{3.3}$$

$$C_j := \partial^l K_{lj} - \partial_j K^l_l = 0, \tag{3.4}$$

where we use a dot to denote time differentiation.

Remark. For the linear system the effect of densitizing the lapse is to change the coefficient of the term $\partial_i\partial_j g^l_l$ in (3.2). Had we not densitized, the coefficient would have been $-1/2$ instead of -1, and the derivation of the linearized EC formulation below would not be possible.

The usual approach to solving the system (3.1)–(3.4) is to begin with initial data $g_{ij}(0)$ and $K_{ij}(0)$ defined on $\mathbb{R}^3$ and satisfying the constraint equations (3.3), (3.4), and to define g_{ij} and K_{ij} for $t > 0$ via the Cauchy problem for the evolution equations (3.1), (3.2). It can be easily shown that the constraints are then satisfied for all times. Indeed, if we apply the Hamiltonian constraint operator defined in (3.3) to the evolution equation (3.1) and apply the momentum constraint operator defined in (3.4) to the evolution equation (3.2), we obtain the first order symmetric hyperbolic system

$$\dot C = -2\partial^j C_j, \quad \dot C_j = -\frac{1}{2}\partial_j C.$$

Thus if C and C_j vanish at $t = 0$, they vanish for all time.

The linearized EC formulation provides an alternate approach to obtaining a solution of (3.1)–(3.4) with the given initial data, based on solving a system with better hyperbolicity properties. If g_{ij}, K_{ij} solve (3.1)–(3.4), define

$$f_{kij} = \frac{1}{2}[\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g^l_l)\delta_{jk} - (\partial^l g_{lj} - \partial_j g^l_l)\delta_{ik}]. \tag{3.5}$$

Then $-\partial^k f_{kij}$ coincides with the first three terms of the right-hand side of (3.2), so

$$\dot K_{ij} = -\partial^k f_{kij} - \partial_i\partial_j\alpha. \tag{3.6}$$

Differentiating (3.5) in time, substituting (3.1), and using the constraint equation (3.4), we obtain

$$\dot f_{kij} = -\partial_k K_{ij} + L_{kij}, \tag{3.7}$$

where

$$L_{kij} = \partial_k\partial_{(i}\beta_{j)} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik} \tag{3.8}$$

The evolution equations (3.6) and (3.7) for K_{ij} and f_{kij}, together with the evolution equation (3.1) for g_{ij}, form the linearized EC system. As initial data for this system we use the given initial values of g_{ij} and K_{ij} and derive the initial values for f_{kij} from those of g_{ij} based on (3.5):

$$f_{kij}(0) = \frac{1}{2}\{\partial_k g_{ij}(0) - [\partial^l g_{li}(0) - \partial_i g^l_l(0)]\delta_{jk} - [\partial^l g_{lj}(0) - \partial_j g^l_l(0)]\delta_{ik}\}. \tag{3.9}$$

In this paper we study the preservation of constraints by the linearized EC system and the closely related question of the equivalence of that system and the linearized ADM system. Our main interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed, but first we consider the result for the pure Cauchy problem in the remainder of this section.

Suppose that K_{ij} and f_{kij} satisfy the evolution equations (3.6) and (3.7) (which decouple from (3.1)). If K_{ij} satisfies the momentum constraint (3.4) for all time, then from (3.6) we obtain a constraint which must be satisfied by f_{kij}:

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l) = 0. \tag{3.10}$$

Note that (3.5) is another constraint that must be satisfied for all time. The following theorem shows that the constraints (3.4), (3.5), and (3.10) are preserved by the linearized EC evolution.

Theorem 1. *Let initial data $g_{ij}(0)$ and $K_{ij}(0)$ be given satisfying the constraints* (3.3) *and* (3.4)*, respectively, and $f_{kij}(0)$ be defined by* (3.9)*. Then the unique solution of the evolution equations* (3.1)*,* (3.6)*, and* (3.7) *satisfies* (3.4)*,* (3.5)*, and* (3.10) *for all time.*

Proof. First we show that the initial data $f_{kij}(0)$ defined in (3.9) satisfies the constraint (3.10). Applying the constraint operator in (3.10) to (3.9) we find

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l)(0) = \frac{1}{2}\partial_j(\partial^l\partial^k g_{kl} - \partial^k\partial_k g_l^l)(0) = \frac{1}{2}\partial_j C(0) = 0. \quad \text{(by (3.3))}$$

It is immediate from the evolution equations that each component K_{ij} satisfies the inhomogeneous wave equation

$$\ddot{K}_{ij} = \partial^k\partial_k K_{ij} - \partial^k L_{kij} - \partial_i\partial_j\dot{\alpha}.$$

Applying the momentum constraint operator defined in (3.4), we see that each component C_j satisfies the homogeneous wave equation

$$\ddot{C}_j = \partial^k\partial_k C_j. \tag{3.11}$$

Now $C_j = 0$ at the initial time by assumption, so if we can show that $\dot{C}_j = 0$ at the initial time, we can conclude that C_j vanishes for all time. But, from (3.6) and the definition of C_j,

$$\dot{C}_j = -\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l), \tag{3.12}$$

which we just proved that vanishes at the initial time. Thus we have shown C_j vanishes for all time, i.e., (3.4) holds. In view of (3.12), (3.10) holds as well. From (3.7) and (3.1) we have

$$\dot{f}_{kij} = \frac{1}{2}\partial_k\dot{g}_{ij} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik}.$$

Applying the momentum constraint operator to (3.1) and using (3.4), it follows that

$$\frac{1}{2}(\partial^l\dot{g}_{li} - \partial_i\dot{g}_l^l) = \partial^l\partial_{[l}\beta_{i]},$$

so $f_{kij} - [\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g_l^l)\delta_{kj} - (\partial^l g_{lj} - \partial_j g_l^l)\delta_{ki}]/2$ does not depend on time. From (3.9), we have (3.5). □

In view of this theorem it is straightforward to establish the key result that for given initial data satisfying the constraints, the unique solution of the linearized EC evolution equations satisfies the linearized ADM system, and so the linearized ADM system and the linearized EC system are equivalent.

Theorem 2. *Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.3) *and momentum constraint* (3.4)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.9)*. Then the unique solution of the linearized EC evolution equations* (3.1)*,* (3.6)*,* (3.7) *satisfies the linearized ADM system* (3.1)*–*(3.4)*.*

Proof. From Theorem 1, we know that $C_j = 0$ for all time, i.e., (3.4) holds. Then from (3.1) and (3.4) we see that $\dot{C} = -2\partial^j C_j = 0$, and, since C vanishes at initial time by assumption, C vanishes for all time, i.e., (3.3) holds as well.

It remains to verify (3.2). From Theorem 1, we also have (3.5). Substituting (3.5) in (3.6) gives (3.2), as desired. □

4. Maximal Nonnegative Constraint Preserving Boundary Conditions

In this section of the paper, we provide maximal nonnegative boundary conditions for the linearized EC system which are constraint-preserving in the sense that the analogue of Theorem 1 is true for the initial–boundary value problem. This will then imply the analogue of Theorem 2. We assume that Ω is a polyhedral domain.

Consider an arbitrary face of $\partial\Omega$ and let n^i denote its exterior unit normal. Denote by m^i and l^i two additional vectors which together n^i form an orthonormal basis. The projection operator orthogonal to n^i is then given by $\tau_i^j := m_i m^j + l_i l^j$ (and does not depend on the particular choice of these tangential vectors). Note that

$$\delta_i^j = n_i n^j + \tau_i^j, \quad \tau_i^j \tau_j^k = \tau_i^k. \tag{4.1}$$

Consequently,

$$v_l w^l = n^j v_j n_i w^i + \tau_l^j v_j \tau_i^l w^i \quad \text{for all } v_l,\ w^l. \tag{4.2}$$

First we consider the following boundary conditions on the face:

$$n^i m^j K_{ij} = n^i l^j K_{ij} = n^k n^i n^j f_{kij} = n^k m^i m^j f_{kij} = n^k l^i l^j f_{kij} = n^k m^i l^j f_{kij} = 0. \tag{4.3}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k \tau^{il} \tau^{jm} f_{kij} = 0, \tag{4.4}$$

and so do not depend on the choice of basis for the tangent space. We begin by showing that these boundary conditions are maximal nonnegative for the hyperbolic system (3.1), (3.6), and (3.7), and so, according to the classical theory of [24] and [35] (also [30, 33, 37, 39, 46, 47], among others), the initial–boundary value problem is well-posed. For convenience, in Appendix A we recall the definition and a classical result due to Rauch [39] on L^2 well-posedness of maximal nonnegative boundary conditions.

Let V denote the vector space of triplets of constant tensors $(g_{ij}, K_{ij}, f_{kij})$ all three symmetric with respect to the indices i and j. Thus $\dim V = 30$. The boundary operator A_n associated to the evolution equations (3.1), (3.6), and (3.7) is the symmetric linear operator $V \to V$ given by

$$\tilde{g}_{ij} = 0, \quad \tilde{K}_{ij} = n^k f_{kij}, \quad \tilde{f}_{kij} = n_k K_{ij}. \tag{4.5}$$

A subspace N of V is called nonnegative for A_n if

$$g_{ij}\tilde{g}^{ij} + K_{ij}\tilde{K}^{ij} + f_{kij}\tilde{f}^{kij} \ge 0 \tag{4.6}$$

whenever $(g_{ij}, K_{ij}, f_{kij}) \in N$ and $(\tilde{g}_{ij}, \tilde{K}_{ij}, \tilde{f}_{kij})$ is defined by (4.5). The subspace is maximal nonnegative if also no larger subspace has this property. Since A_n has six positive, 18 zero, and six negative eigenvalues, a nonnegative subspace is maximal nonnegative if and only if it has dimension 24. Our claim is that the subspace N defined by (4.3) is maximal nonnegative. The dimension is clearly 24. In view of (4.5), the verification of (4.6) reduces to showing that $n^k f_{kij} K^{ij} \geq 0$ whenever (4.3) holds. In fact, $n^k f_{kij} K^{ij} = 0$, that is, $n^k f_{kij}$ and K_{ij} are orthogonal (when (4.3) holds). To see this, we use orthogonal expansions of each based on the normal and tangential components:

$$K_{ij} = n^l n_i n^m n_j K_{lm} + n^l n_i \tau_j^m K_{lm} + \tau_i^l n^m n_j K_{lm} + \tau_i^l \tau_j^m K_{lm}, \tag{4.7}$$

$$n^k f_{kij} = n^l n_i n^m n_j n^k f_{klm} + n^l n_i \tau_j^m n^k f_{klm} + \tau_i^l n^m n_j n^k f_{klm} + \tau_i^l \tau_j^m n^k f_{klm}. \tag{4.8}$$

In view of the boundary conditions (in the form (4.4)), the two inner terms on the right-hand side of (4.7) and the two outer terms on the right-hand side of (4.8) vanish, and so the orthogonality is evident.

Next we show that the boundary conditions are constraint-preserving. This is based on the following lemma.

Lemma 3. *Suppose that α and β^i vanish. Let g_{ij}, K_{ij}, and f_{kij} be a solution to the homogeneous hyperbolic system* (3.1), (3.6), *and* (3.7) *and suppose that the boundary conditions* (4.3) *are satisfied on some face of $\partial\Omega$. Let C_j be defined by* (3.4). *Then*

$$\dot{C}_j n^l \partial_l C^j = 0 \tag{4.9}$$

on the face.

Proof. In fact we shall show that $n^j C_j = 0$ (so also $n^j \dot{C}_j = 0$) and $\tau_j^p n^l \partial_l C^j = 0$, which, by (4.2) implies (4.9). First note that

$$C_j = (\delta_j^m \delta^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im} = (\delta_j^m n^i n^k + \delta_j^m \tau^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im},$$

where we have used the first identity in (4.1). Contracting with n^j gives

$$\begin{aligned} n^j C_j &= (n^m n^i n^k + n^m \tau^{ik} - n^k \delta^{im}) \partial_k K_{im} \\ &= -n^m n^i n^k \dot{f}_{kim} + \tau^{il} \tau_l^k n^m \partial_k K_{im} + n^k \delta^{im} \dot{f}_{kim} \end{aligned}$$

where now we have used the equation (3.7) (with $\beta_i = 0$) for the first and last term and the second identity in (4.1) for the middle term. From the boundary conditions we know that $n^m n^i n^k f_{kim} = 0$, and so the first term on the right-hand side vanishes. Similarly, we know that $\tau^{il} n^m K_{im} = 0$ on the boundary face, and so the second term vanishes as well (since the differential operator $\tau_l^k \partial_k$ is purely tangential). Finally, $n^k \delta^{im} f_{kim} = n^k (n^i n^m + l^i l^m + m^i m^m) f_{kim} = 0$, and so the third term vanishes. We have established that $n^j C_j = 0$ holds on the face.

To show that $\tau_j^p n^l \partial_l C^j = 0$ on the face, we start with the identity

$$\tau_j^p n^l \delta^{mj} \delta^{ik} = \tau^{pm} (n^i n^k + \tau^{ik}) n^l = \tau^{pm} n^i (\delta^{kl} - \tau^{kl}) + \tau^{pm} \tau^{ik} n^l.$$

Similarly

$$\tau_j^p n^l \delta^{kj} \delta^{im} = \tau^{pk} n^l n^i n^m + \tau^{pk} \tau^{im} n^l.$$

Therefore,

$$\begin{aligned} \tau_j^p n^l \partial_l C^j &= \tau_j^p n^l \partial_l (\delta^{mj} \delta^{ik} - \delta^{kj} \delta^{im}) \partial_k K_{im} \\ &= (\tau^{pm} n^i \delta^{kl} - \tau^{pm} n^i \tau^{kl} + \tau^{pm} \tau^{ik} n^l - \tau^{pk} n^l n^i n^m - \tau^{pk} \tau^{im} n^l) \partial_k \partial_l K_{im}. \end{aligned}$$

For the last three terms, we again use (3.7) to replace $\partial_l K_{im}$ with $-\dot{f}_{lim}$ and argue as before to see that these terms vanish. For the first term we notice that $\delta^{kl}\partial_k\partial_l K_{im} = \partial^k\partial_k K_{im} = \ddot{K}_{im}$ (from (3.6) and (3.7) with vanishing α and β^i). Since $\tau^{pm}n^i K_{im}$ vanishes on the boundary, this term vanishes. Finally we recognize that the second term is the tangential Laplacian, $\tau^{kl}\partial_k\partial_l$ applied to the quantity $n^i\tau^{pm}K_{im}$, which vanishes. This concludes the proof of (4.9). □

The next theorem asserts that the boundary conditions are constraint-preserving.

Theorem 4. *Let Ω be a polyhedral domain. Given $g_{ij}(0)$ and $K_{ij}(0)$ on Ω satisfying the constraints* (3.3) *and* (3.4)*, respectively, and $f_{kij}(0)$ defined by* (3.9)*, define g_{ij}, K_{ij}, and f_{kij} for positive time by the evolution equations* (3.1)*,* (3.6)*, and* (3.7) *and the boundary conditions* (4.3)*. Then the constraints* (3.4)*,* (3.5)*, and* (3.10) *are satisfied for all time.*

Proof. Exactly as for Theorem 1 we find that C_j satisfies the wave equation (3.11) and both C_j and $\dot{C}_j$ vanish at the initial time; these facts are unrelated to the boundary conditions. Define the usual energy

$$E(t) = \frac{1}{2}\int_\Omega (\dot{C}_j\dot{C}^j + \partial^l C_j\partial_l C^j)\,dx.$$

Clearly $E(0) = 0$. From (3.11) and integration by parts

$$\dot{E} = \int_{\partial\Omega} \dot{C}_j n^l \partial_l C^j d\sigma. \tag{4.10}$$

Therefore, if $\alpha = 0$ and $\beta^i = 0$, we can invoke Lemma 3, and conclude that E is constant in time. Hence E vanishes identically. Thus C_j is constant, and, since it vanishes at time 0, it vanishes for all time. By (3.12), the constraints (3.10) are also satisfied for all time. This establishes that the constraints (3.4) and (3.10) hold under the additional assumption that α and β^i vanish.

To extend to the case of general α and β^i we use Duhamel's principle. Let $S(t)$ denote the solution operator associated to the homogeneous boundary value problem. That is, given functions $h_{ij}(0)$, $\kappa_{ij}(0)$, $\phi_{kij}(0)$ on Ω, define

$$S(t)(h_{ij}(0), \kappa_{ij}(0), \phi_{kij}(0)) = (h_{ij}(t), \kappa_{ij}(t), \phi_{kij}(t)),$$

where h_{ij}, κ_{ij}, ϕ_{kij} is the solution to the homogeneous evolution equations

$$\dot{h}_{ij} = -2\kappa_{ij}, \quad \dot{\kappa}_{ij} = -\partial^k\phi_{kij}, \quad \dot{\phi}_{kij} = -\partial_k\kappa_{ij},$$

satisfying the boundary conditions and assuming the given initial values. Then Duhamel's principle represents the solution g_{ij}, K_{ij}, f_{kij} of the inhomogeneous initial-boundary value problem (3.1), (3.6), (3.7), (4.3) as

$$\begin{aligned}&(g_{ij}(t), K_{ij}(t), f_{kij}(t))\\ &= S(t)(g_{ij}(0), K_{ij}(0), f_{kij}(0)) + \int_0^t S(t-s)(2\partial_{(i}\beta_{j)}, -\partial_i\partial_j\alpha(s), L_{kij}(s))\,ds.\end{aligned} \tag{4.11}$$

Now it is easy to check that the Hamiltonian constraint (3.3) is satisfied when g_{ij} is replaced by $2\partial_{(i}\beta_{j)}$ (for any smooth vector function β^i), the momentum constraint (3.4) is satisfied when K_{ij} is replaced by $-\partial_i\partial_j\alpha(s)$ (for any smooth function α), and the constraint (3.10) is satisfied when f_{kij} is replaced by $L_{kij}(s)$ defined by (3.8) (for any smooth vector function β^i). Hence the integrand in (4.11) satisfies the constraints by the result for the homogeneous case, as does the first term on the right-hand side, and thus the constraints (3.4) and (3.10) are indeed satisfied by K_{ij} and f_{kij}, respectively.

The proof of the fact that the constraints (3.5) are satisfied for all time follows exactly as in Theorem 1.

Note that the boundary conditions (4.3) play a crucial role in proving that the momentum constraints (3.4) are preserved for all time; the preservation of the constraints (3.5) and (3.10) being a consequence of this fact. □

The analogue of Theorem 2 for the initial–boundary value problem follows from the preceding theorem exactly as before.

Theorem 5. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.3) *and momentum constraint* (3.4)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.9)*. Then the unique solution of the linearized EC initial–boundary value problem* (3.1)*,* (3.6)*,* (3.7)*, together with the boundary conditions* (4.3) *satisfies the linearized ADM system* (3.1)*–*(3.4) *in Ω.*

We close this section by noting a second set of boundary conditions which are maximal nonnegative and constraint-preserving. These are

$$n^i n^j K_{ij} = m^i m^j K_{ij} = l^i l^j K_{ij} = m^i l^j K_{ij} = n^k n^i m^j f_{kij} = n^k n^i l^j f_{kij} = 0, \quad (4.12)$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il}\tau^{jm} K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} = 0. \quad (4.13)$$

Now when we make an orthogonal expansion as in (4.7), (4.8), the outer terms on the right-hand side of the first equation and the inner terms on the right-hand side of the second equation vanish (it was the reverse before), so we again have the necessary orthogonality to demonstrate that the boundary conditions are maximal nonnegative. Similarly, to prove the analogue of Lemma 3, for these boundary conditions we show that the tangential component of $\dot{C}_j$ vanishes and the normal component of $n^l \partial_l C^j$ vanishes (it was the reverse before). Otherwise the analysis is essentially the same as for the boundary conditions (4.3).

5. Extended EC System

In this section we indicate an extended initial boundary value problem whose solution solves the linearized ADM system (3.1)–(3.4) in Ω. This approach could present advantages from the numerical point of view since the momentum constraint is "built-in," and so controlled for all time. The new system consists of (3.1), (3.7), and two new sets of equations corresponding to (3.6)

$$\dot{K}_{ij} = -\partial^k f_{kij} + \frac{1}{2}(\partial_i p_j + \partial_j p_i) - \partial^k p_k \delta_{ij} - \partial_i \partial_j \alpha, \quad (5.1)$$

and to a new three dimensional vector field p_i defined by

$$\dot{p}_i = \partial^l K_{li} - \partial_i K^l_l. \quad (5.2)$$

Observe that the additional terms that appear on the right-hand side of (5.1) compared with (3.6) are nothing but the negative components of the formal adjoint of the momentum constraint operator applied to p_i.

Let $\tilde{V}$ be the vector space of quadruples of constant tensors $(g_{ij}, K_{ij}, f_{kij}, p_k)$ symmetric with respect to the indices i and j. Thus $\dim \tilde{V} = 33$. The boundary operator $\tilde{A}_n : \tilde{V} \to \tilde{V}$ in this case is given by

$$\tilde{g}_{ij} = 0,\ \tilde{K}_{ij} = n^k f_{kij} - \frac{1}{2}(n_i p_j + n_j p_i) + n^k p_k \delta_{ij},\ \tilde{f}_{kij} = n_k K_{ij},\ \tilde{p}_i = -n^l K_{il} + n_i K^l_l. \tag{5.3}$$

The boundary operator $\tilde{A}_n$ associated to the evolution equations (3.1), (5.1), (3.7), and (5.2) has six positive, 21 zero, and six negative eigenvalues. Therefore, a nonnegative subspace is maximal nonnegative if and only if it has dimension 27. We claim that the following boundary conditions are maximal nonnegative for (3.1), (5.1), (3.7), and (5.2),

$$\begin{aligned} n^i m^j K_{ij} = n^i l^j K_{ij} &= n^k n^i n^j f_{kij} \\ &= n^k (m^i m^j f_{kij} + p_k) \\ &= n^k (l^i l^j f_{kij} + p_k) \\ &= n^k m^i l^j f_{kij} = 0. \end{aligned} \tag{5.4}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k (\tau^{il} \tau^{jm} f_{kij} + \tau^{lm} p_k) = 0, \tag{5.5}$$

and so do not depend on the choice of basis for the tangent space.

Let us prove the claim that the subspace $\tilde{N}$ defined by (5.4) is maximal nonnegative. Obviously, $\dim \tilde{N} = 27$. Hence, it remains to be proven that $\tilde{N}$ is also nonnegative. In view of (5.3), the verification of non-negativity of $\tilde{N}$ reduces to showing that

$$n^k f_{kij} K^{ij} - n^i p^j K_{ij} + n^k p_k K^l_l \geq 0 \tag{5.6}$$

whenever (5.4) holds. In fact, we can prove that the left-hand side of (5.6) vanishes pending (5.4) holds. From the boundary conditions (in the form (5.5)) and the orthogonal expansions (4.7) and (4.8) of K_{ij} and f_{kij}, respectively, the first term on the right-hand side of (5.6) reduces to $n^k \tau^{il} \tau^{jm} f_{kij} K_{lm} = -n^k p_k \tau^{lm} K_{lm}$. Then, combining the first and third terms of the left-hand side of (5.6) gives $-n^k p_k \tau^{ij} K_{ij} + n^k p_k \delta^{ij} K_{ij} = n^k p_k n^i n^j K_{ij}$. Finally, by using the orthogonal decomposition $p^j = n^k p_k n^j + \tau^{kj} p_k$ and the first part of the boundary conditions (5.5) the second term of the left-hand side of (5.6) is $-n^k p_k n^i n^j K_{ij} - p_k n^i \tau^{kj} K_{ij} = -n^k p_k n^i n^j K_{ij}$, which is precisely the negative sum of the first and third terms of the left-hand side of (5.6). This concludes the proof of (5.6).

Theorem 6. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (3.3) *and momentum constraints* (3.4)*, respectively, $f_{kij}(0)$ is defined by* (3.5)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3.1)*,* (5.1)*,* (3.7)*, and* (5.2)*, together with the boundary conditions* (5.4)*, satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3.1)*–*(3.4) *in Ω.*

Proof. Observe that the solution of the initial boundary value problem (3.1), (3.6), (3.7), and (4.3) (boundary conditions), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3.1), (5.1), (3.7), and (5.2), together with the boundary conditions (5.4). The conclusion follows from Theorem 5. □

We close by indicating a second set of maximal nonnegative boundary conditions (corresponding to (4.12)) for (3.1), (5.1), (3.7), and (5.2) for which Theorem 6 holds as well. These are

$$\begin{aligned} n^i n^j K_{ij} &= m^i m^j K_{ij} = m^i l^j K_{ij} \\ &= l^i l^j K_{ij} = n^k n^i m^j f_{kij} - m^k p_k \\ &= n^k n^i l^j f_{kij} - l^k p_k = 0, \end{aligned} \tag{5.7}$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il}\tau^{jm} K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} - \tau^{kl} p_k = 0. \tag{5.8}$$

6. Inhomogeneous Boundary Conditions

In this section we provide a formal method of constructing well-posed constraint-preserving *inhomogeneous* boundary conditions for (3.1), (3.6), and (3.7) corresponding to the two sets of boundary conditions (4.3) and (4.12), respectively. The first set of inhomogeneous boundary conditions corresponds to (4.3) and can be written in the following form

$$n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} = n^k n^i n^j \tilde{f}_{kij} = n^k m^i m^j \tilde{f}_{kij} = n^k l^i l^j \tilde{f}_{kij} = n^k m^i l^j \tilde{f}_{kij} = 0, \tag{6.1}$$

where $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given in $\overline{\Omega}$ for all time and satisfying the constraints (3.4) and (3.10), respectively.

The analogue of Theorem 4 for the inhomogeneous boundary conditions (6.1) is true.

Theorem 7. *Let Ω be a polyhedral domain. Given $g_{ij}(0)$ and $K_{ij}(0)$ on Ω satisfying the constraints* (3.3) *and* (3.4)*, respectively, and $f_{kij}(0)$ defined by* (3.9)*, define g_{ij}, K_{ij}, and f_{kij} for positive time by the evolution equations* (3.1)*,* (3.6)*, and* (3.7) *and the boundary conditions* (6.1)*. Then the constraints* (3.4)*,* (3.5)*, and* (3.10) *are satisfied for all time.*

Proof. Observe that g_{ij}, $\tilde{K}_{ij}$, and $\tilde{f}_{kij}$ satisfy (3.1), (3.6), and (3.7) with the forcing terms replaced by $2\partial_{(i}\beta_{j)}$, $-\partial_i\partial_j\alpha - \partial^k F_{kij} - \dot{\kappa}_{ij}$, and $L_{kij} - \partial_k\kappa_{ij} - \dot{F}_{kij}$, respectively. Exactly as in Theorem 4, it follows that $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ satisfy (3.4) and (3.10), respectively, for all time. Thus, K_{ij} and f_{kij} satisfy (3.4) and (3.10), respectively, for all time. Finally, same arguments as in Theorem 1 show that the constraints (3.5) are also preserved through evolution for all time. □

The analogue of Theorem 5 for the case of the inhomogeneous boundary conditions (6.1) follows from the preceding theorem by using the same arguments as in the proof of Theorem 2.

Theorem 8. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.3) *and momentum constraint* (3.4)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.9)*. Then the unique solution of the linearized EC initial–boundary value problem* (3.1)*,* (3.6)*,* (3.7)*, together with the* inhomogeneous *boundary conditions* (6.1) *satisfies the linearized ADM system* (3.1)*–*(3.4) *in Ω.*

Note that there is a second set of inhomogeneous boundary conditions corresponding to (4.12) for which Theorem 7 and Theorem 8 remain valid. These are

$$n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} = l^i l^j \tilde{K}_{ij} = m^i l^j \tilde{K}_{ij} = n^k n^i m^j \tilde{f}_{kij} = n^k n^i l^j \tilde{f}_{kij} = 0, \quad (6.2)$$

where again $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given and satisfying the constraints (3.4) and (3.10), respectively.

Similar considerations can be made for the extended system introduced in the previous section. There are two sets of *inhomogeneous* boundary conditions for which the extended system produces solutions of the linearized ADM system (3.1)–(3.4) on a polyhedral domain Ω. These are

$$\begin{aligned} n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} &= n^k n^i n^j \tilde{f}_{kij} \\ &= n^k (m^i m^j \tilde{f}_{kij} + p_k) \\ &= n^k (l^i l^j \tilde{f}_{kij} + p_k) \\ &= n^k m^i l^j \tilde{f}_{kij} = 0 \end{aligned} \quad (6.3)$$

and

$$\begin{aligned} n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} &= m^i l^j \tilde{K}_{ij} \\ &= l^i l^j \tilde{K}_{ij} \\ &= n^k n^i m^j \tilde{f}_{kij} - m^k p_k \\ &= n^k n^i l^j \tilde{f}_{kij} - l^k p_k = 0, \end{aligned} \quad (6.4)$$

where $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ are defined as before.

The next theorem is an extension of Theorem 6 to the case of inhomogeneous boundary conditions.

Theorem 9. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (3.3) *and momentum constraints* (3.4)*, respectively, $f_{kij}(0)$ is defined by* (3.5)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3.1)*,* (5.1)*,* (3.7)*, and* (5.2)*, together with the* inhomogeneous *boundary conditions* (6.3) *(or* (6.4)*), satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3.1)–(3.4) *in Ω.*

Proof. Note that the solution of the initial boundary value problem (3.1), (3.6), (3.7), and (6.1) (or (6.2), respectively), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3.1), (5.1), (3.7), and (5.2), together with the boundary conditions (6.3) (or (6.4), respectively). The conclusion follows from Theorem 8. □

7. Concluding Remarks

We have studied the boundary conditions problem for the standard EC formulation of Einstein's equations linearized about the Minkowski spacetime. In Section 4, we indicate two sets of maximal nonnegative boundary conditions (4.3) and (4.12), respectively, which are consistent with the constraints. These boundary conditions were announced in [9] and overlap with the boundary conditions found in [17] for the generalized EC formulation for $0 < \eta < 2$ with vanishing shift and lapse density perturbations. However, the energy method of [17] works only for the generalized EC formulation with $0 < \eta < 2$; the standard EC formulation corresponds to $\eta = 4$. Moreover, we prove that our boundary conditions are well-posed and consistent with the constraints in the more general case of arbitrary shift and lapse density perturbations by using a new argument involving the Duhamel's principle. Also, our approach emphasizes the relation between the ADM formulation and the constrained evolution of the EC system in the linearized context. Besides, our method is simpler, yet effective, and seems to be easily transferable to other formulations and/or other background spacetimes. In fact, other Einstein's hyperbolic formulations, e.g., Alekseenko–Arnold [4], are analyzed in [52] by using the same method. A subclass of the boundary conditions presented in this paper and introduced previously in [9] has been pointed out by Frittelli and Gomez in [25] (in the case of vanishing shift vector) as an example to their Einstein boundary conditions, that is, the vanishing of the projection of Einstein's tensor along the normal to the boundary.

One of the main results in this paper is the construction of an extended symmetric hyperbolic system which incorporates the momentum constrains as main variables. For this extended system, we construct two sets of maximal nonnegative boundary conditions and establish its relationship with the linearized ADM formulation. Such a construction could serve as a model of how to control the bulk constraint violations by making the constraints part of the main evolution system, and so keeping them under control for all time. To the best of our knowledge, this is a new approach regarding the bulk constraint violations control.

We also make some considerations about how inhomogeneous boundary conditions consistent with the constraints could be constructed.

In some places, our methods of proof interfere with the techniques used in [16] and [17], e.g., using the trading of normal derivatives for tangential and temporal ones and the use of the energy method to prove that the constraints are preserved. We apply these techniques to the slightly more general case of polyhedral domains (as opposed to cubic boxes) and in a more systematic way. This could be of potential interest to the case of curved boundary domains, for which the derivative components trading techniques introduce new terms related to the geometry of the boundaries (see [52], Section 4.2, for the analysis of a model problem similar to the linearized EC formulation on curved domains). It is also expected that these or similar techniques will be useful in the nonlinear case. For the interested reader, we point out the work done in [34], where the authors construct new boundary conditions for the nonlinear KST form [32] of the Einstein equations (which includes the EC formulation). Their boundary conditions are designed to prevent the influx of constraint violations and physical gravitational waves into the computational domain. However, as specified in [34], there is no rigorous mathematical well-posedness theoretical ground yet for these kind of boundary conditions, as opposed to the simpler case of maximal nonnegative boundary conditions.

8. Appendix: Maximal nonnegative boundary conditions for symmetric hyperbolic systems

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $T > 0$. We introduce the notations $\mathbb{O} = (0,T)\times\Omega$ and $\Gamma = (0,T)\times\partial\Omega$. Consider the first order differential operator $L := \partial_t + \sum_{i=1}^n A_i(t,x)\partial_i + B(t,x)$, where $A_i \in \mathrm{Lip}(\overline{\mathbb{O}})$, $B \in$

$L^\infty(\overline{\mathbb{O}})$, and $(B+B^*)/2 - \sum_{i=1}^n \partial_i A_i \in L^\infty(\mathbb{O})$. We suppose that L is symmetric, that is, $A_i = A_i^*$ on $\overline{\mathbb{O}}$. Our interest is in solving the initial–boundary value problem

$$Lu = f(t,x) \text{ in } \mathbb{O}, \quad u(0,\cdot) = g \text{ in } \Omega, \quad u(t,x) \in N(t,x) \text{ for } (t,x) \in [0,T] \times \partial\Omega, \tag{8.1}$$

where $N(t,x)$ is a Lipshitz continuous map from $[0,T] \times \partial\Omega$ to the subspaces of $\mathbb{C}^n$. Set n^i be the outer unit normal to Γ, and denote by $A_n(t,x)$ the *boundary matrix/operator* $A_n(t,x) := \sum_{i=1}^n n^i(x) A_i(t,x)$. We assume that Γ is characteristic of constant multiplicity in the sense that $\dim\ker A_n$ is constant on each component of Γ. We next suppose that N is *maximal nonnegative*, that is, the following two conditions hold on $\overline{\Gamma}$:

$$\langle A_n(t,x)v, v\rangle \geq 0, \ \forall (t,x) \in \overline{\Gamma}, \ \forall v \in N(t,x) \tag{8.2}$$

and

$$\dim N(t,x) = \#\text{ nonnegative eigenvalues of } A_n(t,x) \text{ counting multiplicity.} \tag{8.3}$$

The maximality condition (8.3) implies that the boundary subspace N cannot be enlarged while preserving (8.2).

Let $\mathbb{H}_{\mathbb{O}} := \{u \in L^2(\mathbb{O}) : Lu \in L^2(\mathbb{O})\}$. It is easy to prove that $\mathbb{H}_{\mathbb{O}}$ is a Hilbert space with respect to the inner product $\langle u,u\rangle_{\mathbb{H}_{\mathbb{O}}} := \langle u,u\rangle_{L^2(\mathbb{O})} + \langle Lu, Lu\rangle_{L^2(\mathbb{O})}$.

Theorem 10 (L^2 well-posedness, [39, Theorem 9])**.** *For any $f \in L^1((0,T) : L^2(\Omega))$ and $g \in L^2(\Omega)$ there is a unique $u \in \mathbb{H}_{\mathbb{O}}$ satisfying* (8.1)*. In addition, $u \in C((0,T) : L^2(\Omega))$,*

$$\sup_{0\leq t\leq T} \|u(t)\|_{L^2(\Omega)} \leq C\|f\|_{L^1((0,T):L^2(\Omega))} + \|g\|_{L^2(\Omega)},$$

and

$$\|u(t_2)\|_{L^2(\Omega)} - \|u(t_1)\|_{L^2(\Omega)} \leq \int_{t_1}^{t_2} \|f(\sigma)\|_{L^2(\Omega)} + C\|u(\sigma)\|_{L^2(\Omega)}\, d\sigma.$$

References

[1] Abrahams, A., Anderson, A., Choquet–Bruhat, Y., York, J. (1997), *Geometrical hyperbolic systems for general relativity and gauge theories*, Class. Quantum Grav. **14**, A9–A22.

[2] Alekseenko, A. M. (2005), *Constrained evolution and differential boundary conditions*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).

[3] Alekseenko, A. M. (2005), *Constraint preserving boundary conditions for the linearized BSSN formulation*, gr-qc/0405080.

[4] Alekseenko, A. M., Arnold, D. N. (2003), *A new symmetric hyperbolic formulation for the Einstein equations*, Phys. Rev. D **68** (also gr-qc/0210071).

[5] Anderson, M., Matzner, R. A. (2004), *Extended Lifetime in Computational Evolution of Isolated Black Holes*, gr-qc/0307055.

[6] Anderson, A., York, J. W. (1999), *Fixing Einstein's equations*, Phys. Rev. Lett. **82**, pp. 4384-4387.

[7] Arnold, D. N. (2002), *Numerical Relativity*, ICM 2002 Satellite Conference on Nonlinear PDE: Theory and Approximation, Hong Kong, (http://www.ima.umn.edu/ arnold/talks.html).

[8] Arnold, D. N. (2005), *Introduction to the Einstein Field Equations*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).

[9] Arnold, D. N., Tarfulea, N. (2003), *Constraint-preserving boundary conditions for linearized Einstein–Christoffel formulation*, Numerical Relativity Seminar, Penn State University (http://gravity.psu.edu/numrel/nrlunch/2003/fall.html#2003_09_25_Arnold).

[10] Arnowitt, A., Deser, S., Misner, C. (1962) in *Gravitation: An introduction to current research*, edited by L. Witten, Wiley, New York.

[11] Ashtekar, A. (1987), *New Hamiltonian formulation of general relativity.*, Phys. Rev. D (3) **36**, no. 6, pp. 1587–1602.

[12] Bardeen J. M., Buchman, L. T.(2002), *Numerical Tests of Evolution Systems, Gauge Conditions, and Boundary Conditions for 1D Colliding Gravitational Plane Waves*, Phys. Rev. D **65**, 064037.

[13] Bona, C., Masso, J. (1992), *Hyperbolic evolution system for numerical relativity*, Phys. Rev. Lett. **68**, pp. 1097–1099.

[14] Bona, C., Masso, J., Seidel, E., Stela, J. (1997), *First order hyperbolic formalism for numerical relativity*, Phys. Rev. D (3) **56**, pp. 3405–3415.

[15] Baumgarte, T. W., Shapiro, S. L. (1999), *Numerical integration of Einstein's field equations*, Phys. Rev. D (3) **59**, 024007.

[16] Calabrese, G., Lehner, L., Tiglio, M. (2002), *Constraint-preserving boundary conditions in numerical relativity*, Phys. Rev. D **65** (also gr-qc/0111003).

[17] Calabrese, G., Pullin, J., Reula, O., Sarbach,O., Tiglio, M. (2003), *Well posed constraint-preserving boundary conditions for the linearized Einstein equations*, Commun. Math. Phys. **240**, pp. 377-395.

[18] Choptuik, M. W., Hirschmann, E. W., Liebling, L.S., Pretorius, F. (2003), *Critical Collapse of the Massless Scalar Field in Axisymmetry*, Phys. Rev. D **68** 044007.

[19] Choptuik, M. W., Hirschmann, E. W., Liebling, L.S., Pretorius, F. (2003), *An Axisymmetric Gravitational Collapse Code*, Class. Quant. Grav. **20** pp. 1857–1878.

[20] Choquet–Bruhat, Y. (1973), *The Cauchy Problem*, in *Gravitation: An Introduction to Current Research*, ed. L. Witten (New York: Wiley).

[21] Choquet–Bruhat, Y., Ruggeri, T. (1983), *Hyperbolicity of the 3+1 System of Einstein Equations*, Commun. Math. Phys. **89**, pp. 269–275.

[22] Friedrich, H., Nagy, G. (1999), *The Initial Boundary Value Problem for Einstein's Vacuum Field Equations*, Comm. Math. Phys. **201**, pp. 619–655.

[23] Friedrich, H., Rendall, A. (2000), *The Cauchy Problem for the Einstein Equations*, Lecture Notes in Phys. **540**, pp. 127–223.

[24] Friedrichs, K.O. (1958), *Symmetric positive linear differential equations*, Comm. Pure Appl. Math. **11**, pp. 333–418.

[25] Frittelli, S., Gomez, R. (2004), *Einstein boundary conditions in relation to constraint propagation for the initial-boundary value problem of the Einstein equations*, Phys. Rev. D **69**, no. 12, 124020, 10 pp. .

[26] Frittelli, S., Gomez, R. (2003), *Boundary conditions for hyperbolic formulations of the Einstein equations*, Class. Quant. Grav. **20**, pp. 2379-2392.

[27] Frittelli, S., Gomez, R. (2003), *First-order quasilinear canonical representation of the characteristic formulation of the Einstein equations*, Phys. Rev. D **68**, no. 8, 084013, 6 pp. .

[28] Frittelli, S., Gomez, R. (2003), *Einstein boundary conditions for the* $3+1$ *Einstein equations*, Phys. Rev. D **68**, no. 4, 044014, 6 pp. .

[29] Frittelli, S., Reula, O. S. (1996), *First-order symmetric-hyperbolic Einstein equations with arbitrary fixed gauge*, Phys. Rev. Lett. **76**, pp. 4667-4670.

[30] Gustafsson, B., Kreiss, H. O., Oliger, J. (1995), *Time dependent problems and difference methods*, Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York.

[31] Holst, M., Lindblom, L., Owen, R., Pfeiffer, H. P., Scheel, M. A., Kidder L. E. (2004), *Optimal Constraint Projection for Hyperbolic Evolution Systems*, Phys.Rev. D70 (2004) 084017.

[32] Kidder, L., Scheel, M., Teukolsky, S. (2001), *Extending the lifetime of 3D black hole computations with a new hyperbolic system of evolution equations*, Phys. Rev. D (3), **64**, pp. 064017-064030.

[33] Kreiss, H. O., Lorenz, J. (1989), *Initial Boundary Value Problems and Navier-Stokes Equations*, Pure and Applied Mathematics **136**, Accademic Press.

[34] Lawrence E. Kidder, Lee Lindblom, Mark A. Scheel, Luisa T. Buchman, Harald P. Pfeiffer (2005), *Boundary Conditions for the Einstein Evolution System*, Phys. Rev. D **71**, 064020.

[35] Lax, P.D., Phillips, R.S. (1960), *Local Boundary Conditions for Dissipative Symmetric Linear Differential Operators*, Comm. Pure Appl. Math. **13**, pp. 427–455.

[36] Lindblom, L., Scheel, M. A., Kidder, L. E., Pfeiffer, H. P., Shoemaker, D., Teukolsky, S. A. (2004), *Controlling the Growth of Constraints in Hyperbolic Evolution Systems*, Phys. Rev. D **69**, 124025.

[37] Majda, A., Osher, S. (1975), *Initial-boundary value problem for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. **28**, pp. 607–675.
[38] Pretorius, F. (2005), *Toward binary black hole simulations in numerical relativity*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).
[39] Rauch, J. (1985), *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc. **291**, pp. 167–187.
[40] Reula, O. (1998), *Hyperbolic Methods for Einstein's Equations*, Living Rev. Relativ. **1**, 37 pp. (electronic).
[41] Reula, O. (2005), *On strong hyperbolicity of Einstein's equations*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).
[42] Sarbach, O. (2005), *Boundary conditions for Einstein's equations*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).
[43] Sarbach, O., Tiglio, T. (2005), *Boundary conditions for Einstein's field equations: mathematical and numerical analysis*, Journal of Hyperbolic Differential Equations, 2 (2005), pp. 839–883.
[44] Scheel, M. (2005), *Constraint-preserving boundary conditions for the Einstein evolution system*, PCA Workshop III: Relativistic Astrophysics, IPAM–UCLA (http://www.ipam.ucla.edu/abstract.aspx?tid=5419).
[45] Schnetter, E. (2003), *Gauge fixing for the simulation of black hole spacetimes*, Ph.D. Thesis, University of Tuebingen.
[46] Secchi, P. (1996), *The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity*, Differential Integral Equations **9**, pp. 671–700.
[47] Secchi, P. (1996), *Well-posedness of characteristic symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **134**, no. 2, pp. 155–197.
[48] Shibata, M., Nakamura, T. (1995), *Evolution of three-dimensional gravitational waves: harmonic slicing case*, Phys. Rev. D (3) **52**, no. 10, 5428–5444.
[49] Stewart, J. M. (1998), *The Cauchy problem and the initial-boundary value problem in numerical relativity*, Classical Quantum Gravity 15, no. 9, 2865–2889.
[50] Szilagyi, B., Schmidt, B., Winicour, J. (2002), *Boundary conditions in linearized harmonic gravity*, Phys. Rev. D **65**, no. 6, 064015 , 15 pp.
[51] Szilagyi, B., Winicour, J. (2003), *Well-Posed Initial-Boundary Evolution in General Relativity*, Phys. Rev. D **68**, no. 4, 041501, 5 pp.
[52] Tarfulea, N. (2004), *Constraint Preserving Boundary Conditions for Some Hyperbolic Formulations of Einstein's equations*, Ph.D. Thesis, University of Minnesota, gr-qc/0508014.
[53] Tarfulea, N. (2005), *Constraint preserving boundary conditions in numerical relativity*, BIRS Workshop on Numerical Relativity (http://bh0.physics.ubc.ca/BIRS05/).
[54] York, J. W., Jr. (1979), in *Sources of Gravitational Radiation*, edited by L. L. Smarr, Cambridge University Press, Cambridge, England, pp. 83–126.

Douglas N. Arnold
Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota 55455, USA
E-mail address: `arnold@ima.umn.edu`
URL: `http://www.ima.umn.edu/~arnold`

Nicolae Tarfulea
Department of Mathematics, Purdue University Calumet, Hammond, Indiana 46323, USA
E-mail address: `tarfulea@calumet.purdue.edu`
URL: `http://ems.calumet.purdue.edu/tarfulea`