2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 249–254.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF NON-NEGATIVE SOLUTIONS FOR NONLINEAR EQUATIONS IN THE SEMI-POSITONE CASE

NAJI YEBARI, ABDERRAHIM ZERTITI

Abstract. Using the fibring method we prove the existence of non-negative solution of the p-Laplacian boundary value problem $-\Delta_p u = \lambda f(u)$, for any $\lambda > 0$ on any regular bounded domain of $\mathbb{R}^N$, in the special case $f(t) = t^q - 1$.

1. Introduction and main results

In this paper we are interested in finding nonnegative solutions to the equation

$$\begin{gathered} -\Delta_p u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

for some specific f in the non positone case ($f(0) < 0$), under assumptions stated below.

Here Ω is a connected and bounded subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ in $C^{1,\alpha}$. We set

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

When $p = 2$, this type of problem in the nonpositone case can be studied via the shooting method. Existence of a radially symmetric nonnegative solution for $\lambda > 0$ sufficiently small have been obtained in [1, 2] and nonexistence of such a solution for $\lambda > 0$ large have been established in [1, 3], in the framework of the semi positone case and f is superlinear. Observe that, since $f(0) < 0$, the constant 0 is an upper solution of (1.1) and as a consequence it is not possible, in general, to apply the usual techniques (for example: the method of upper and lower solutions, etc.) and we shall work in the framework of the so-called fibration method introduced by Pohozaev in [5], and then developed in [6, 7, 8]. We shall assume that f has the form

$$f(t) = t^q - 1, \quad \text{with } q > p - 1 \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35J65, 35J25.
Key words and phrases. Superlinear semi-positone problems; variational methods; fibring method.

Published September 20, 2006.

To avoid the noncompactness problem we shall always assume that the problem is subcritical, in the sense of the critical exponent for Ω,

$$p^\star = \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N, \\ +\infty & \text{if } p \geq N. \end{cases} \tag{1.3}$$

Let

$$u = \theta v, \quad \theta = \lambda^{-(\frac{1}{q-p+1})}, \quad \mu = \lambda^{\frac{q}{q-p+1}} > 0\,. \tag{1.4}$$

and

$$\begin{gathered} -\Delta_p u = u^q - \mu \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

It can be seen that (1.1), (1.2), (1.4) and (1.5) are equivalent. Also let p and q satisfy

$$0 < p - 1 < q < p^* - 1 \tag{1.6}$$

where p^* is given by (1.3). Concerning μ, we shall assume its positivity.

By a solution of (1.5), we mean a $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ function which is a critical point of the functional

$$E(v) = -\frac{1}{p}\int_\Omega |\nabla v|^p dx + \frac{1}{q+1}\int_\Omega |v|^{q+1} dx - \mu \int_\Omega |v(x)| dx$$

and therefore satisfies

$$\int_\Omega (|\nabla u|^{p-2}\nabla u.\nabla\varphi - (u^q - \mu)\varphi) dx = 0$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Our main result is as follows.

Theorem 1.1. *Let assumptions* (1.3) *and* (1.6) *be satisfied. Then there exists a nontrivial nonnegative solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of problem* (1.1) *for any $\lambda > 0$. Moreover, $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$.*

2. Proof of the main theorem

The proof is based on the fibering method and is divided into five stages.

Step 1: We introduce the Euler functional associated with (1.5) as follows

$$E(u) = -\frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q+1}\int_\Omega |u|^{q+1} dx - \mu \int_\Omega |u(x)| dx$$

According to the fibering method, we set

$$u(x) = rv(x), \tag{2.1}$$

where $r \in \mathbb{R}^+$ and $v \in W_0^{1,p}(\Omega)$. Then we obtain

$$\widetilde{E}(r,v) = E(r,v) = -\frac{|r|^p}{p}\int_\Omega |\nabla v|^p dx + \frac{|r|^{q+1}}{q+1}\int_\Omega |v|^{q+1} dx - \mu r \int_\Omega |v(x)| dx \tag{2.2}$$

We introduce the fibering functional

$$\int_\Omega |\nabla v|^p dx = 1 \tag{2.3}$$

Under condition (2.3) the functional $\widetilde{E}$ takes the form

$$\widetilde{E}(r,v) = -\frac{r^p}{p} + \frac{r^{q+1}}{q+1}\int_\Omega |v|^{q+1}dx - \mu r\int_\Omega |v(x)|dx \tag{2.4}$$

The bifurcation equation is

$$0 = \frac{\partial \widetilde{E}}{\partial r} = -r^{p-1} + r^q\int_\Omega |v|^{q+1}dx - \mu\int_\Omega |v(x)|dx \tag{2.5}$$

which gives

$$-r^p + r^{q+1}\int_\Omega |v|^{q+1}dx - \mu r\int_\Omega |v(x)|dx = 0. \tag{2.6}$$

Let set

$$\widetilde{E}(v) = E(r(v)v) \tag{2.7}$$

Step 2: Let us consider the variational problem

$$M_0 = \sup\big\{\widetilde{E}(v);\ v \in W_0^{1,p}(\Omega)/\int_\Omega |\nabla v|^p dx = 1\big\}. \tag{2.8}$$

It follows that

$$\widetilde{E}(v) = \min_{r\geq 0}\widetilde{E}(r,v) = \min_{r\geq 0}\{-\frac{r^p}{p} + \frac{r^{q+1}}{q+1}\int_\Omega |v|^{q+1}dx - \mu r\int_\Omega |v(x)|dx\} < 0, \tag{2.9}$$

as a matter of fact, (2.6) gives

$$-\frac{r^p(v)}{p} = -\frac{r^{q+1}(v)}{p}\int_\Omega |v|^{q+1}dx + \mu\frac{r(v)}{p}\int_\Omega |v(x)|dx,$$

On the other hand,

$$\begin{aligned}\widetilde{E}(v) &= E(r(v)v)\\ &= -\frac{r^{q+1}(v)}{p}\int_\Omega |v|^{q+1}dx + \mu\frac{r(v)}{p}\int_\Omega |v(x)|dx\\ &\quad+\frac{r^{q+1}(v)}{p}\int_\Omega |v|^{q+1}dx - \mu r(v)\int_\Omega |v(x)|dx,\end{aligned}$$

which gives

$$\widetilde{E}(v) = \frac{(p-q-1)}{(q+1)p}r^{q+1}(v)\int_\Omega |v|^{q+1}dx - \mu r(v)(1-\frac{1}{p})\int_\Omega |v(x)|dx \tag{2.10}$$

By (1.6), $\widetilde{E}(v) < 0$.

Let us prove the following Lemma.

Lemma 2.1. *The sequence maximizing problem* (2.8) *is bounded in* $W_0^{1,p}(\Omega)$.

Proof. Let (v_n) be a maximizing sequence for (2.8). We set

$$v_n(x) = c_n + \overline{v}_n(x) \tag{2.11}$$

with

$$\int_\Omega \overline{v}_n(x)dx = 0\,. \tag{2.12}$$

Since

$$\int_\Omega |\nabla v_n|^p dx = \int_\Omega |\nabla \overline{v}_n|^p dx = 1 \tag{2.13}$$

and by the Sobolev embedding theorems (the Poincare-Wirtinger inequality), the sequence $(\overline{v}_n)$ is bounded in $W^{1,p}(\Omega)$. From the bifurcation equation (2.5), we obtain

$$r_n^p = r_n^{q+1}\int_\Omega |c_n+\overline{v}_n|^{q+1}dx - \mu r_n\int_\Omega |c_n+\overline{v}_n|dx. \tag{2.14}$$

Let us assume that

$$c_n \to +\infty, \quad \text{as } n\to+\infty\,. \tag{2.15}$$

Then

$$\int_\Omega |1+\frac{\overline{v}_n}{c_n}|^{q+1}dx = \frac{1}{c_n^{q+1}r_n^{q-p+1}} + \frac{\mu}{c_n^q r_n^q}\int_\Omega |1+\frac{\overline{v}_n}{c_n}|dx\,. \tag{2.16}$$

By embedding results, there exists $C>0$ such that

$$\|\overline{v}_n\|_{W^{1,p}(\Omega)} \le C, \quad \forall n\in\mathbb{N}$$

Using (2.15) and since by assumption (1.6) the space $W^{1,p}(\Omega)$ is compactly embedded in $L^{q+1}(\Omega)$. We may assume that $(\overline{v}_n)$ converges strongly in latter space. Then from (2.16) we have

$$\int_\Omega |1+\frac{\overline{v}_n}{c_n}|^{q+1}dx \to |\Omega|>0, \quad \text{as } n\to+\infty. \tag{2.17}$$

The proof is complete. □

Hence, we can assume that the sequence (v_n) converges weakly in $W_0^{1,p}(\Omega)$. By assumption (1.6), it follows that $v_n\to\overline{v}$ in $L^{q+1}(\Omega)$. This implies that

$$\|\nabla v_0\|_p \le \liminf_{n\to+\infty}\|\nabla v_n\|_p\,.$$

Since

$$\|\nabla v_n\|_p^p = \int_\Omega |\nabla v_n|^p dx = 1,$$

we obtain

$$0\le \|\nabla v_0\|_p^p = \int_\Omega |\nabla v_0|^p dx \le 1. \tag{2.18}$$

Now we shall prove the equality

$$\int_\Omega |\nabla v_0|^p dx = 1. \tag{2.19}$$

We assume the contrary; i.e, that

$$\int_\Omega |\nabla v_0|^p dx < 1. \tag{2.20}$$

Note that

$$0 < \int_\Omega |\nabla v_0|^p dx\,. \tag{2.21}$$

Otherwise, if $\int_\Omega |\nabla v_0|^p dx = 0$, $v_0=c_0$ is a constant, and from (2.8), we have for all $\epsilon>0$ there exist $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ we have

$$M_0-\epsilon < \widetilde{E}(v_n) < M_0.$$

Let $\theta\in]0,1[$. Then

$$\widetilde{E}(\theta v_n)-\epsilon \le M_0-\epsilon < \widetilde{E}(v_n) < M_0 \tag{2.22}$$

by (2.10). We recall that

$$\widetilde{E}(v_n) = \frac{(p-q-1)}{(q+1)p} r_n^{q+1} \int_\Omega |v_n|^{q+1} dx - \mu r_n (\frac{p-1}{p}) \int_\Omega |v_n(x)| dx \,.$$

Using (2.22), we see that

$$(1-\theta)\mu r_n (\frac{p-1}{p}) \int_\Omega |v_n(x)| dx < \epsilon \text{ for all } n \geq n_0.$$

Then we have

$$r_n \int_\Omega |v_n(x)| dx \to r_0 \int_\Omega |v_0(x)| dx = 0$$

as $n \to +\infty$ and $v_0 = c_0 = 0$ which gives $\widetilde{E}(v_0) = 0$. This contradicts $M_0 < 0$.

Due to (2.20) and (2.21), there exists $\theta > 1$ (i.e., $\theta^p = 1/\int_\Omega |\nabla v_0(x)|^p dx > 1$) such that $v_* = \theta v_0$ satisfies

$$\int_\Omega |\nabla v_*(x)|^p dx = 1$$

and

$$\begin{aligned}
\widetilde{E}(v_*) = \widetilde{E}(\theta v_0) &= \min_{r\geq 0} \big\{ -\frac{r^p}{p} + \frac{r^{q+1}}{q+1} \theta^{q+1} \int_\Omega |v_0|^{q+1} dx - \mu r \theta \int_\Omega |v_0(x)| dx \big\} \\
&= \min_{\rho \geq 0} \big\{ -\frac{\rho^p}{p\theta^p} + \frac{\rho^{q+1}}{q+1} \int_\Omega |v_0|^{q+1} dx - \mu \rho \int_\Omega |v_0(x)| dx \big\} \\
&> \min_{\rho \geq 0} \big\{ -\frac{\rho^p}{p} + \frac{\rho^{q+1}}{q+1} \int_\Omega |v_0|^{q+1} dx - \mu \rho \int_\Omega |v_0(x)| dx \big\}.
\end{aligned}$$

Thus,

$$\widetilde{E}(v_*) > \widetilde{E}(v_0).$$

This inequality contradicts the definition of (2.8). Thus, we have obtained a solution to the variational problem.

Step 3:

$$\widetilde{E}(v_0) = \sup \big\{ \widetilde{E}(v);\ v \in W_0^{1,p}(\Omega) / \int_\Omega |\nabla v|^p dx = 1 \big\}$$

The fibering method implies $r = r_0 = r(v_0)$ where $r_0 > 0$ and

$$\begin{aligned}
&- \frac{r_0^p}{p} + \frac{r_0^{q+1}}{q+1} \int_\Omega |v_0|^{q+1} dx - \mu r_0 \int_\Omega |v_0(x)|\, dx \\
&= \min_{r \geq 0} \big\{ -\frac{r^p}{p} + \frac{r^{q+1}}{q+1} \int_\Omega |v|^{q+1} dx - \mu r \int_\Omega |v(x)| dx \big\}
\end{aligned}$$

To complete the proof, we must show that the equation (1.5) is verified. We can assume that v_0 is nonnegative by replacing v_n by $|v_n|$. Moreover, there exists a Lagrange multiplier σ such that

$$\widetilde{E}'(v_0).h = \sigma \Big(\int_\Omega |\nabla(.)|^p dx \Big)'(v_0).h \quad \forall h \in W_0^{1,p}(\Omega). \tag{2.23}$$

From the above equation, and by taking v_0 as test function, we have

$$r_0 \big\{ \int_\Omega ((r_0 v_0)^q - \mu) v_0 dx \big\} = p\sigma \int_\Omega |\nabla(v_0)|^p dx = p\sigma \,.$$

By (2.6) we obtain $\sigma = \frac{r_0^p}{p} > 0$. Then we can write

$$\widetilde{E}'(v_0) = p\sigma(-\Delta_p v_0)$$

which is equivalent to

$$-\Delta_p(r_0 v_0) = (r_0 v_0)^q - \mu\,.$$

Then if we set $u = r_0 v_0 \geq 0$, we can see that u is a solution of problem (1.5).

Step 4: For $u \geq 0$, we have $\widetilde{E}(v_0) < 0$, thus the solution $u \geq 0$ is non trivial.

Step 5: We have obtained the nonnegative nontrivial solution u to problem (1.5). A standard bootstrap argument (see Drabek [4]) shows that $u \in L^\infty(\Omega)$. Then the asserted regularity of $u \in C^{1,\alpha}_{\mathrm{Loc}}(\Omega)$ follows by Tolksdorf [9]. Thus the theorem is proved.

References

[1] D. Arcoya and A. Zertiti, *Existence and non-existence of radially symmetric non-negative solutions for a class of semi-positone problems in a annulus*, Rendiconti di Matematica, Serie VII, Vol. 14, Roma (1994), 625-646.

[2] K. J. Brown, A. Castro and R. Shivaji, *Nonexistence of radially symmetric solutions for a class of semipositone problems*, Differential and Integral Eqns., 2(4) (1989) pp. 541-545.

[3] A. Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems*, Proc. Roy. Soc. Edin., 108(A)(1988), pp. 291-302.

[4] P. Drabek, *Strongly nonlinear degenerate and singular elliptic problems,* in Nonlinear Partial Differential Equations (Fes, 1994), Pitman Res. Notes Maths. Ser.343, Longman, Harlow, UK, 1996, pp. 112-146.

[5] S. I. Pohozaev, *On eigenfunctions of quasilinear elliptic problems,* Mat. Sb. 82, 192-212 (1970).

[6] S. I. Pohozaev, *On one approach to nonlinear equations,* Dokl. Akad. Nauk. 247, 1327-1331(in Russian)(1979).

[7] S. I. Pohozaev, *On a constructive method in the calculus of variations,* Dokl. AKad. Nauk. 288, 1330-1333 (in Russian)(1988).

[8] S. I. Pohozaev, *On fibering method for the solution of nonlinear boundary value problems,* Trudy Mat.Inst. Steklov 192, 140-163 (1997).

[9] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations,* J. Differential Equations, 51 (1984), pp. 126-150.

Naji Yebari

Département de Mathématiques, Université Abdelmalek Essaadi, B. P. 2121, Tetouan, Maroc

E-mail address: nyebari@hotmail.com, nyebari@fst.ac.ma

Abderrahim Zertiti

Département de Mathématiques, Université Abdelmalek Essaadi, B. P. 2121, Tetouan, Maroc

E-mail address: zertiti@fst.ac.ma