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EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION FOR A NON HOMOGENEOUS PROBLEM OF FOURTH ORDER WITH WEIGHTS

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Abstract. In this work we study the existence of a positive solutions to the non homogeneous equation

$$\Delta(|\Delta u|^{p-2}\Delta u) = m|u|^{q-2}u$$

with Navier boundary conditions, where $1 < p, q < p_2^*$ and $m \in L^\infty(\Omega) \setminus \{0\}$, $m \geq 0$. In the case $p > q$ and $m \in \mathcal{C}(\overline{\Omega})$, we prove the uniqueness of this solution.

1. Introduction

We consider the following problem with Navier boundary conditions

$$\begin{gathered}\Delta_p^2 u = m|u|^{q-2}u \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega,\\ u = \Delta u = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{1.1}$$

Here Ω is a smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, Δ_p^2 is the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $m \in L^\infty(\Omega) \setminus \{0\}, m \geq 0$ and $p, q \in]1, p_2^*[$, $p \neq q$ where

$$p_2^* = \begin{cases} \frac{Np}{N-2p} & \text{if } p < N/2,\\ +\infty & \text{if } p \geq N/2.\end{cases}$$

In [9], we proved that the problem (1.1), without the second condition, has an infinity of solutions in the case $p > q$ by using the fundamental multiplicity theorem, but for $p < q$ we have applied the mountain-pass lemma to prove the existence of nontrivial solution. Finally we have studied the regularity of these solutions. In this work we are interested by the existence of a positive solution then in the case $p > q$ we prove the uniqueness of this solution. Notice that our approach does not use the fundamental multiplicity theorem and the mountain-pass lemma. We can refer the reader to [6] for the existence of a positive solution and to [8] for the uniqueness.

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Similar results as ours, but with p-Laplacian operator, were studied by authors [8, 2].

2. Preliminaries

In this paper, we consider the transformation of Poisson problem used by Drábek and Ôtani [3]. We recall some properties of the Dirichlet problem for the Poisson equation

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

It is well known that (2.1) is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $f \in L^p(\Omega)$ and for any $p \in]1, +\infty[$.

We denote by: $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,
$\|u\|_p = (\int_\Omega |u|^p dx)^{1/p}$ the norm in $L^p(\Omega)$,
$\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{1/p}$ the norm in X,
$\|u\|_\infty$ the norm in $L^\infty(\Omega)$,
and $\langle \cdot, \cdot \rangle$ is the duality bracket between $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $p' = p/(p-1)$. Denote by Λ the inverse operator of $-\Delta : X \to L^p(\Omega)$. The following lemma gives us some properties of the operator Λ (c.f. [3, 7]).

Lemma 2.1. (i) (Continuity): *There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{2,p} \le c_p \|f\|_p$$

holds for all $p \in]1, +\infty[$ and $f \in L^p(\Omega)$.

(ii) (Continuity) *Given $k \in \mathbb{N}^*$, there exists a constant $c_{p,k} > 0$ such that*

$$\|\Lambda f\|_{W^{k+2,p}} \le c_{p,k} \|f\|_{W^{k,p}}$$

holds for all $p \in]1, +\infty[$ and $f \in W^{k,p}(\Omega)$.

(iii) (Symmetry) *The equality*

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u \cdot \Lambda v dx$$

holds for all $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p \in]1, +\infty[$.

(iv) (Regularity) *Given $f \in L^\infty(\Omega)$, we have $\Lambda f \in C^{1,\alpha}(\bar{\Omega})$ for all $\alpha \in]0, 1[$; moreover, there exists $c_\alpha > 0$ such that*

$$\|\Lambda f\|_{C^{1,\alpha}} \le c_\alpha \|f\|_\infty.$$

(v) (Regularity and Hopf-type maximum principle) *Let $f \in C(\bar{\Omega})$ and $f \ge 0$ then $w = \Lambda f \in C^{1,\alpha}(\bar{\Omega})$, for all $\alpha \in]0, 1[$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.*

(vi) (Order preserving property) *Given $f, g \in L^p(\Omega)$ if $f \le g$ in Ω, then $\Lambda f < \Lambda g$ in Ω.*

Note that for all $u \in X$ and all $v \in L^p(\Omega)$, we have $v = -\Delta u$ if and only if $u = \Lambda v$.

Let us denote N_p the Nemytskii operator defined by

$$N_p(v)(x) = \begin{cases} |v(x)|^{p-2} v(x) & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0. \end{cases}$$

Then for all $v \in L^p(\Omega)$ and all $w \in L^{p'}(\Omega)$, we have $N_p(v) = w$ if and only if $v = N_{p'}(w)$.

For $v = -\Delta u$ which means that $u = \Lambda v$. As $X \hookrightarrow L^q(\Omega)$, then $\Lambda v \in L^q(\Omega) \forall v \in L^p(\Omega)$. We define the functionals $F, G : L^p(\Omega) \to \mathbb{R}$ as follows:

$$F(v) = \frac{1}{p}\|v\|_p^p \quad \text{and} \quad G(v) = \frac{1}{q}\int_\Omega m|\Lambda v|^q dx.$$

Then it is clear that F and G are well defined on $L^p(\Omega)$, and are of class $\mathcal{C}^1$ on $L^p(\Omega)$ and for all $v \in L^p(\Omega)$ we have $F'(v) = N_p(v)$ and $G'(v) = \Lambda(mN_q(\Lambda v))$ in $L^{p'}(\Omega)$.

The operator Λ enables us to transform problem (1.1) to another problem which we shall study in the space $L^p(\Omega)$.

Definition 1. We say that $u \in X \setminus \{0\}$ is a solution of problem (1.1), if $v = -\Delta u$ is a solution of the problem: Find $v \in L^p(\Omega) \setminus \{0\}$, $v > 0$, such that

$$N_p(v) = \Lambda(mN_q(\Lambda v)) \quad \text{in } L^{p'}(\Omega). \tag{2.2}$$

3. Existence of a positive solution

For solutions of (2.2) we understand critical points of the associated Euler-Lagrange functional $E \in \mathcal{C}^1(L^p(\Omega))$, which are given by

$$E(v) = F(v) - G(v).$$

As in [4, 10], we introduce the modified Euler-Lagrange functional defined on $\mathbb{R} \times L^p(\Omega)$ by

$$A(t, v) = E(tv).$$

If v is an arbitrary element of $L^p(\Omega)$, $\partial_t A(., v)$ (resp. $\partial_{tt} A(., v)$)are the first (resp. second) derivative of the real valued function: $t \mapsto A(t, v)$. Since the functional A is even in t and that we are interested by the positive solutions, we limit our study for $t > 0$.

Theorem 3.1. *Problem* (1.1) *has a positive solution.*

To prove theorem 3.1, we need the following preliminary results.

Case $p > q$: Let v be an arbitrary element of $L^p(\Omega) \setminus \{0\}$. It is clair that the real valued function $t \mapsto A(t, v)$ is decreasing on $]0, t(v)[$, increasing on $]t(v), +\infty[$ and attains its unique minimum for $t = t(v)$, where

$$t(v) = \Big(\frac{qG(v)}{pF(v)}\Big)^{\frac{1}{p-q}}. \tag{3.1}$$

On the other hand, a direct computation gives

$$A(t(v), v) = \Big(\frac{1}{p} - \frac{1}{q}\Big)\frac{(qG(v))^{\frac{p}{p-q}}}{(pF(v))^{\frac{q}{p-q}}} < 0.$$

Furthermore we have proved in [9] that E is bounded bellow and coercive. We deduce that A is also bounded bellow and if

$$\alpha = \inf_{v \in L^p(\Omega) \setminus \{0\}} A(t(v), v), \tag{3.2}$$

we get $-\infty < \alpha < 0$. Let $(v_n) \subset L^p(\Omega) \setminus \{0\}$ be a minimizing sequence of (3.2). Put $V_n = t(v_n)v_n$. Since E is coercive the sequence (V_n) is bounded.

Lemma 3.2. *The sequence (V_n) satisfies*
$$\liminf_{n\to+\infty} \|V_n\|_p > 0.$$

Proof. Suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim_{n\to+\infty} \|V_n\| = 0$. It follows that $\lim_{n\to+\infty} E(V_n) = 0$; i.e. $\alpha = 0$, which is impossible since $A(t(v_n), v_n) < 0$. $\square$

Lemma 3.3. *If $\mathbb{S}$ is the unit sphere of $L^p(\Omega)$, we have*
$$\alpha = \inf_{v\in\mathbb{S}, v\geq 0} A(t(v), v).$$

Proof. For every $v \in L^p(\Omega)$, we have $|\Lambda v| \leq \Lambda|v|$ and since $p > q$, we get
$$A(t(v), v) \geq \big(\frac{1}{p} - \frac{1}{q}\big)\frac{(qG(|v|))^{\frac{p}{p-q}}}{(pF(|v|))^{\frac{q}{p-q}}} = A(t(|v|), |v|).$$
On the other hand the relation (3.1) implies that $\forall r > 0$ *and* $\forall v \in L^p(\Omega) \setminus \{0\}$, $t(v) = \frac{1}{r}t(\frac{v}{r})$. We deduce that
$$\alpha = \inf_{v\in\mathbb{S}, v\geq 0} A(t(v), v), \tag{3.3}$$
where $\mathbb{S}$ is the unit sphere of $L^p(\Omega)$. $\square$

Note that the minimizing sequences considered up to here are in $\mathbb{S}$ and are nonnegative.

Lemma 3.4. *Let $(v_n) \subset \mathbb{S}$ be a minimizing sequence of (3.3), then $(V_n) := (t(v_n)v_n)$ is Palais-Smale sequence for the functional E.*

Proof. We have $E(V_n) \to \alpha$. We show that
$$E'(V_n) \to 0 \quad \text{in } L^{p'}(\Omega).$$
Note that for every $v \in L^p(\Omega) \setminus \{0\}$, we have $\partial_t A(t(v), v) = 0$ and $\partial_{tt} A(t(v), v) \neq 0$. The implicit function theorem implies that $v \to t(v)$ is $\mathcal{C}^1$ since A is. Let us introduce the $\mathcal{C}^1$ functional B defined on $\mathbb{S}$ by
$$B(v) = A(t(v), v) = E(t(v)v).$$
Then
$$\alpha = \inf_{v\in\mathbb{S}, v\geq 0} B(v) \quad \text{and} \quad \lim_{n\to+\infty} B(v_n) = \alpha$$
Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, \|\cdot\|_p)$ to the functional B, we conclude that
$$|B'(v)(\varphi)| \leq \frac{1}{n}\|\varphi\|_p, \quad \text{for every } \varphi \in T_{u_n}\mathbb{S},$$
where $T_{v_n}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point v_n. Moreover, for every $\varphi \in T_{v_n}\mathbb{S}$, one has
$$\begin{aligned} B'(v_n)(\varphi) &= \partial_t A(t(v_n), v_n)t'(v_n)(\varphi) + \partial_v A(t(v), v)(\varphi) \\ &= \partial_v A(t(v), v)(\varphi), \end{aligned}$$
since $\partial_t A(t(v), v) = 0$, where $t'(v)$ denotes the derivative of $v \mapsto t(v)$ at the point v. Furthermore, let $P : L^p(\Omega) \setminus \{0\} \to \mathbb{R} \times \mathbb{S}$,
$$v \mapsto (P_1(v), P_2(v)) = (\|v\|_p, \frac{v}{\|v\|_p}).$$

Applying Hölder's inequality, for every $(v,\varphi) \in L^p(\Omega) \setminus \{0\} \times L^p(\Omega)$ we have

$$\|P_2'(v)(\varphi)\|_p \le 2\frac{\|\varphi\|_p}{\|v\|_p}.$$

From lemma 3.2 and by the fact that $\|V_n\|_p = t(v_n)$, there is a positive constant C such that

$$t(v_n) \ge C, \quad \forall n \in \mathbb{N}.$$

Then for every $\varphi \in L^p(\Omega)$ we get

$$\begin{aligned}
|E'(V_n)(\varphi)| &= |\partial_t A(P_1(V_n), P_2(V_n))P_1'(V_n)(\varphi) + \partial_v A(P_1(V_n), P_2(V_n))P_2'(V_n)(\varphi)| \\
&= |\partial_v A(t(v_n), v_n)P_2'(V_n)(\varphi)| \\
&= |B'(v_n)P_2'(V_n)(\varphi)| \\
&\le \frac{1}{n}\|P_2'(V_n)(\varphi)\|_p \\
&\le \frac{2}{n}\frac{\|\varphi\|_p}{C}.
\end{aligned}$$

We easily conclude that $\lim_{n\to+\infty} E'(V_n) = 0$ in $L^{p'}(\Omega)$. $\square$

Case $p < q$**:** If v is an arbitrary element of $L^p(\Omega) \setminus \{0\}$, the real valued function $t \mapsto A(t,v)$ is increasing on $]0, t(v)[$, decreasing on $]t(v), +\infty[$ and attains its unique maximum for $t = t(v)$, where

$$t(v) = \big(\frac{pF(v)}{qG(v)}\big)^{\frac{1}{q-p}}. \tag{3.4}$$

Lemma 3.5. *If $p < q$, there exists a positive constant $c(p,q,\Omega,m)$ which depends uniquely of p, q, Ω and m such that $A(t(v), v) \ge c(p,q,\Omega,m)$.*

Proof. A direct computation gives

$$A(t(v), v) = \big(\frac{1}{p} - \frac{1}{q}\big)\frac{(pF(v))^{\frac{q}{q-p}}}{(qG(v))^{\frac{p}{q-p}}}.$$

Hence

$$A(t(v), v) \ge \big(\frac{1}{p} - \frac{1}{q}\big)\frac{1}{\|m\|_\infty^{\frac{p}{q-p}}}\big(\frac{\|v\|_p}{\|\Lambda v\|_q}\big)^{\frac{pq}{q-p}}.$$

The assertion (i) of Lemma 2.1 and the fact that $X \hookrightarrow L^q(\Omega)$ imply that there exists positive constants c_q and c such that

$$A(t(v), v) \ge \big(\frac{1}{p} - \frac{1}{q}\big)\frac{1}{(c_q c)^{\frac{pq}{q-p}}\|m\|_\infty^{\frac{p}{q-p}}}\big(\frac{\|v\|_p}{\|v\|_p + \|\Lambda v\|_p}\big)^{\frac{pq}{q-p}}.$$

Finally the assertion (i) of lemma2.1 implies that there exits a positive constant c_p such that

$$A(t(v), v) \ge \big(\frac{1}{p} - \frac{1}{q}\big)\frac{1}{(c_q c_p c)^{\frac{pq}{q-p}}\|m\|_\infty^{\frac{p}{q-p}}}.$$

We take $c_(p,q,\Omega,m) = \big(\frac{1}{p} - \frac{1}{q}\big)\frac{1}{(c_q c_p c)^{\frac{pq}{q-p}}\|m\|_\infty^{\frac{p}{q-p}}}$. $\square$

Put

$$\alpha = \inf_{v \in L^p(\Omega)\setminus\{0\}} A(t(v), v).$$

Then Lemma 3.5 implies $\alpha > 0$.

Lemma 3.6. *If $\mathbb{S}$ is the unit sphere of $L^p(\Omega)$, we have*
$$\alpha = \inf_{v\in\mathbb{S}, v\geq 0} A(t(v), v).$$

Proof. For every $v \in L^p(\Omega) \setminus \{0\}$, we have
$$A(t(v), v) = \big(\frac{1}{p} - \frac{1}{q}\big)\frac{(pF(v))^{\frac{q}{q-p}}}{(qG(v))^{\frac{p}{q-p}}}.$$
Since $|\Lambda v| \leq \Lambda|v|$, we get
$$A(t(v), v) \geq \big(\frac{1}{p} - \frac{1}{q}\big)\frac{pF(|v|)^{\frac{q}{q-p}}}{qG(|v|)^{\frac{p}{q-p}}} = A(t(|v|), |v|).$$
On the other hand, the relation (3.4) implies that for every $r > 0$ and for every $v \in L^p(\Omega) \setminus \{0\}$, $t(v) = \frac{1}{r}t(\frac{v}{r})$. Hence
$$\alpha = \inf_{v\in\mathbb{S}, v\geq 0} A(t(v), v). \tag{3.5}$$
□

Let (v_n) be a minimizing sequence of (3.5), as in the case $p > q$, we put
$$V_n = t(v_n)v_n.$$
The proof of the following lemmas can be done like in the previous case.

Lemma 3.7. $\liminf_{n\to+\infty} \|V_n\|_p > 0$.

Lemma 3.8. *Let $(v_n) \subset \mathbb{S}$ be a minimizing sequence of (3.3). Then $(V_n) := (t(v_n)v_n)$ is Palais-Smale sequence for the functional E.*

Proof of theorem 3.1. In our paper [9] we showed that E verifies the Palais-Smale condition. Then by lemma 3.4 and lemma 3.8, we deduce that there is a subsequence of (V_n), still noted by (V_n) such that $V_n \to V$, $V \in L^p(\Omega)\backslash\{0\}$ and $V \geq 0$. Moreover, since $E'(V_n) \to 0$, then $E'(V) = 0$. i.e. V is a nonnegative solution of problem (2.2). Hence
$$N_p(V) = \Lambda(mN_q(\Lambda V)). \tag{3.6}$$
The assertion (vi) of lemma 2.1, the relation (3.6) and the fact that $m \in L^p(\Omega)\backslash\{0\}$, $m \geq 0$ enable us to claim that $N_p(V) > 0$ and $V > 0$. Furthermore $U = \Lambda V$ is a positive solution of problem (1.1). □

4. Uniqueness of the positive solution

Theorem 4.1. *If $m \in \mathcal{C}(\overline{\Omega})$, $m \geq 0$ and $p > q$, then* (1.1) *has a unique nonnegative solution.*

Problem (2.2) is equivalent to the problem: Find $v \in L^p(\Omega) \setminus \{0\}$, $v > 0$ such that
$$N_p(v) = \|m^{1/q}\Lambda v\|_q^{q-p}\|m^{1/q}\Lambda v\|_q^{p-q}\Lambda(mN_q(\Lambda v)) \quad \text{in } L^{p'}(\Omega). \tag{4.1}$$
To prove that problem (2.2) has a unique nonnegative solution, we will study the principal positive eigenvalue of the eigenvalue problem: Find $v \in L^p(\Omega) \setminus \{0\} \times \mathbb{R}_+^*$ such that
$$N_p(v) = \lambda\|m^{1/q}\Lambda v\|_q^{p-q}\Lambda(mN_q(\Lambda v)) \quad \text{in} \quad L^{p'}(\Omega). \tag{4.2}$$

Consider the functionals f and g defined on $L^p(\Omega)$ by

$$f(v) = \frac{1}{p}\|v\|_p \quad \text{and} \quad g(v) = \frac{1}{p}\Big(\int_\Omega m|\Lambda v|^q dx\Big)^{\frac{p}{q}}.$$

Hence problem (4.2) is equivalent to the problem: Find $(v, \lambda) \in L^p(\Omega) \setminus \{0\} \times \mathbb{R}_+^*$ such that

$$f'(v) = \lambda g'(v) \quad \text{in } L^{p'}(\Omega). \tag{4.3}$$

Define

$$\lambda_1 = \inf_{v \in M} f(v),$$

where $M = \{v \in L^p(\Omega)/g(v) = 1\}$. We need the preliminary results.

Lemma 4.2. (i) *λ_1 is the first positive eigenvalue of problem (4.2). Moreover v_1 is an eigenfunction associated with λ_1 if and only if*

$$f(v_1) - \lambda_1 g(v_1) = 0 = \inf_{v \in L^p(\Omega)\setminus\{0\}} f(v) - \lambda_1 g(v).$$

(ii) *Every eigenfunction associated with λ_1 is positive or negative.*

Proof. (i) The functional f is weakly semi-continuous below and coercive on M. Since g is weakly continuous, then M is weakly closed. Hence there is $v_1 \in M$ such that $f(v_1) = \lambda_1 = \lambda_1 g(v_1)$.

The p-homogeneity of f and g implies that λ_1 is an eigenvalue of problem (4.2) if and only if

$$\forall v \in L^p(\Omega) \setminus \{0\}, \quad \lambda_1 \leq \frac{f(v)}{|g(v)|}$$

if and only if for all $v \in L^p(\Omega) \setminus \{0\}$,

$$f(v) - \lambda_1 g(v) \geq f(v) - \lambda_1 |g(v)| \geq 0 = f(v_1) - \lambda_1 g(v_1).$$

Now we show that λ_1 is the first positive eigenvalue: Suppose on the contrary that there exits $\lambda \in]0, \lambda_1[$ and $v \in L^p(\Omega) \setminus \{0\}$ such that $f(v) - \lambda g(v) = 0$. Then we get

$$0 = f(v_1) - \lambda_1 g(v_1) \leq f(v) - \lambda_1 g(v) < f(v) - \lambda g(v) = 0,$$

which is a contradiction.
(ii) Let v be an eigenfunction associated with λ_1. From the assertion (i) and by the fact that $|\Lambda v| \leq \Lambda |v|$, we get

$$0 = f(v) - \lambda_1 g(v) \leq f(|v|) - \lambda_1 g(|v|) \leq f(v) - \lambda_1 g(v) = 0.$$

Therefore, $|v|$ an is eigenfunction associated with λ_1. From the assertion in lemma 2.1(vi) and by the fact that

$$N_p(|v|) = \lambda_1 \Lambda(m N_q(|v|),$$

we deduce that $|v| > 0$ in Ω. Hence v is positive or negative in Ω. □

Lemma 4.3. *If v and w are positive eigenfunctions of* (2.2) *associated with λ_1, then the functions* max *and* min *defined in Ω by* $\max(x) = \max(v(x), w(x))$ *and* $\min(x) = \min(u(x), w(x))$ *are also solutions of* (2.2) *associated with λ_1.*

To prove lemma 4.3 we need the following results.

Lemma 4.4. *Let a, b, c and p be reals such that $a \geq 0$, $b \geq 0$ and $p > 1$. If $c \geq \max\{b - a, 0\}$, then*

$$|a + c|^p + |b - c|^p \geq a^p + b^p.$$

For the proof of the above lemma see for example [3].

Lemma 4.5. *Let a, b, c and d be in $\mathbb{R}_+$ such that $a \geq \max(c, d)$. If $a + b \geq c + d$, then for every $p \in [1, +\infty[$, $a^p + b^p \geq c^p + d^p$.*

Proof. If $b \geq \min(c, d)$ or $a \geq c + d$ it is evident. Else, set $\alpha = a - d$ and $\beta = c - b$. We can suppose that $d \leq c$. Since $a < c + d$ and $a + b \geq c + d$ we deduce that $\alpha < c$ and $\beta \leq \alpha$. Then

$$a^p + b^p = |d + \alpha|^p + |c - \beta|^p \geq |d + \alpha|^p + |c - \alpha|^p.$$

As $\alpha \geq c - d$, then from lemma 4.4 we conclude that $a^p + b^p \geq c^p + d^p$. $\square$

Proof of lemma 4.3. If u and v are two positive eigenfunctions associated with λ_1, we claim that

$$\begin{aligned} &\Big(\int_\Omega m|\Lambda \max(u, v)|^q dx\Big)^{\frac{p}{q}} + \Big(\int_\Omega m|\Lambda \min(u, v)|^q dx\Big)^{\frac{p}{q}} \\ &\geq \Big(\int_\Omega m|\Lambda u|^q dx\Big)^{\frac{p}{q}} + \Big(\int_\Omega m|\Lambda v|^q dx\Big)^{\frac{p}{q}}. \end{aligned} \tag{4.4}$$

Indeed, we have

$$\max(u, v) = u + \frac{v - u + |v - u|}{2}.$$

Then the fact that for every $w \in L^p(\Omega)$, $\Lambda|w| \geq |\Lambda w|$ enables us to deduce that

$$\Lambda \max(u, v) \geq \Lambda u + \frac{\Lambda v - \Lambda u + |\Lambda v - \Lambda u|}{2} = \max(\Lambda u, \Lambda v).$$

Hence

$$\begin{aligned} \int_\Omega m|\Lambda \max(u, v)|^q dx) &\geq \int_\Omega m|\max(\Lambda u, \Lambda v)|^q dx \\ &\geq \max(\int_\Omega m|\Lambda u|^q dx, \int_\Omega m|\Lambda v|^q dx). \end{aligned}$$

Therefore, from lemma 4.5 we conclude inequality (4.4). If we put

$$\phi(w) = f(w) - \lambda_1 g(w) \quad \forall w \in L^p(\Omega),$$

from (4.4) and from lemma 4.2, we deduce that

$$0 \leq \phi(\max(u, v)) + \phi(\min(u, v) \leq \phi(u) + \phi(v) = 0$$

and $\phi(\max(u, v)) = \phi(\min(u, v)) = 0$. Thus, $\min(u, v)$ and $\max(u, v)$ are eigenfunctions associated with λ_1. $\square$

Lemma 4.6. *Every eigenfunction of problem (2.2) is in $\mathcal{C}(\overline{\Omega})$.*

Proof. If v is an eigenfunction of problem (2.2) associated with a positive eigenvalue λ, then

$$v = \lambda^{1/(p-1)} N_{p'}(\|m^{1/q}\Lambda w\|_q^{p-q} \Lambda(m N_q(\Lambda v))). \tag{4.5}$$

Since $|\Lambda v| \leq \Lambda|v|$, we get

$$|v| \leq \lambda^{1/(p-1)} \|m\|_\infty^{\frac{1}{p-1}} \|m^{1/q}\Lambda w\|_q^{\frac{p-q}{p-1}} N_{p'}(\Lambda N_q(|\Lambda v|)). \tag{4.6}$$

We showed in our paper [9] that $N_{p'}(\Lambda N_q(|\Lambda v|)) \in \mathcal{C}(\overline{\Omega})$. Hence from (4.6) we deduce that $v \in L^\infty(\Omega)$ and from (4.5) and the assertion in lemma 2.1(iv) it follows that $v \in \mathcal{C}(\overline{\Omega})$. $\square$

Proposition 4.7. *The eigenvalue λ_1 is simple and every positive eigenfunction is associated with λ_1.*

Proof. Let v and w be two positive eigenfunctions associated with λ_1. For $x_0 \in \Omega$ set $k = v(x_0)/w(x_0)$ and $\max_k(x) = \max(v(x), kw(x))$. Lemma 4.3 enables us to claim that $\max_k$ is a solution of problem (2.2) associated with λ_1. Since

$$\begin{gathered} N_p(v) = \lambda_1 \Lambda(mN_p(\Lambda v)), \\ N_p(w) = \lambda_1 \Lambda(mN_p(\Lambda w)), \\ N_p(\max{}_k) = \lambda_1 \Lambda(mN_p(\Lambda \max{}_k)), \end{gathered}$$

Lemma 4.6 and lemma 2.1 imply that $N_p(v), N_p(w), N_p(\max_k) \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ and $N_p(v), N_p(w)$ are positive in Ω. Then

$$N_p(v)/N_p(w) \in \mathcal{C}^1(\Omega).$$

For any unit vector e, we have

$$N_p(v)(x_0 + te) - N_p(v)(x_0) \le N_p(\max{}_k)(x_0 + te) - N_p(\max{}_k)(x_0)$$

and

$$N_p(kw)(x_0 + te) - N_p(kw)(x_0) \le N_p(\max{}_k)(x_0 + te) - N_p(\max{}_k)(x_0).$$

Dividing these inequalities by $t > 0$ and $t < 0$ and letting t tend to $0^{\pm}$, we get

$$\nabla N_p(v)(x_0) = \nabla N_p(\max{}_k)(x_0) = k^{p-1} \nabla N_p(w)(x_0).$$

Thus

$$\begin{aligned} \nabla\big(\frac{N_p(v)}{N_p(w)}\big)(x_0) &= \nabla\big(\frac{N_p(v)}{N_p(w)}\big)(x_0) \\ &= \frac{(\nabla(N_p(v))(x_0) N_p(w)(x_0) - N_p(v)(x_0)\nabla(N_p(w))(x_0))}{(N_p(w)(x_0))^2} = 0. \end{aligned}$$

Hence

$$N_p\big(\frac{v}{w}\big) = \frac{N_p(v)}{N_p(w)} = \text{const} = k^{p-1} \quad \text{in } \Omega$$

and

$$\frac{v}{w} = k \quad \text{in } \Omega.$$

Now we show that every positive eigenfunction is associated with λ_1: Let $\lambda > \lambda_1$, suppose that problem (2.2) has a positive eigenfunction w associated with λ and let v be a positive solution of problem (2.2) associated with λ_1, we have

$$N_p(v) = \lambda_1 \Lambda(mN_p(\Lambda v)) \quad \text{and} \quad N_p(w) = \lambda \Lambda(mN_p(\Lambda w)).$$

Then from the assertion in lemma 2.1(v) we deduce that $N_p(v)$ and $N_p(w)$ are in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$, and

$$\partial(N_p(v))/\partial n < 0, \quad \partial(N_p(w))/\partial n < 0 \quad \text{on } \partial\Omega.$$

It follows that $N_p(v)/N_p(w)$ is in $\mathcal{C}(\overline{\Omega})$. Set

$$a = \max_{x \in \overline{\Omega}} N_p(v)(x)/N_p(w)(x).$$

We deduce that $N_p(v) \le aN_p(w)$. The monotonicity of $N_{p'}$ implies

$$v \le a^{\frac{1}{p-1}} w.$$

Since problem (2.2) is homogeneous, $a^{\frac{1}{p-1}}w$ is also a solution of problem (2.2), we may assume without loss of generality that $v \le w$. Then, from the assertion of lemma 2.1(vi) and by the monotonicity of N_q, we get

$$\begin{aligned} N_p(v) &= \lambda_1 \|m^{1/q}\Lambda v\|_q^{p-q}\Lambda(mN_q(\Lambda v)) \\ &\le \|m^{1/q}\Lambda w\|_q^{p-q}\lambda_1\Lambda(mN_q(\Lambda w)) \\ &= \lambda\|m^{1/q}\Lambda cw\|_q^{p-q}\Lambda(mN_q(\Lambda cw)) \\ &= N_p(cw), \end{aligned}$$

where

$$c = (\lambda_1/\lambda)^{1/(p-1)} < 1.$$

Hence it follows by the monotonicity of $N_{p'}$ that $v < cw$. Repeating this argument n times, we obtain $0 \le v \le c^n w$. Therefore by letting n tend to infinity, we deduce that $v \equiv 0$. This is a contradiction. $\square$

Proof of theorem 4.1. Let v and w be two positive solutions of problem (4.1). Then v and w are eigenfunctions associated with the eigenvalues $\|m^{1/q}\Lambda v\|_q^{q-p}$ and $\|m^{1/q}\Lambda w\|_q^{q-p}$ respectively. From proposition 4.7 we deduce that

$$\|m^{1/q}\Lambda v\|_q^{q-p} = \|m^{1/q}\Lambda w\|_q^{q-p} = \lambda_1$$

and there is $k > 0$ such that $w = kv$. It follows that $v = w$. $\square$

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