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UNIFORMLY ERGODIC THEOREM FOR COMMUTING MULTIOPERATORS

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Abstract. In this paper, we established some uniformly Ergodic theorems by using multioperators satisfying the E-k condition introduce in [3]. One consequence, is that if $I-T$ is quasi-Fredholm and satisfies E-k condition then T is uniformly ergodic. Also we give some conditions for uniform ergodicity of a commuting multioperators satisfies condition E-k. These results are of interest in view of analogous results for unvalued operators (see, for example [2]) also in view of the recent developments in the ergodic theory and its applications.

1. Introduction and main results

Throughout this paper, X is a complex Banach space, and $L(X)$ is the algebra of linear continuous operators acting in X. If there is an integer n for which $T^{n+1}X = T^nX$, then we say that T has finite descent and the smallest integer $d(T)$ for which equality occurs is called the descent of T. If there is exists an integer m for which $kerT^{m+1} = kerT^m$, then T is said to have finite ascent and the smallest integer $a(T)$ for this equality occurs is called ascent of T. If both $a(T)$ and $d(T)$ are finite, then they are equal [1, 38.3]. We say that T is chain-finite and that its chain length is this common minimal value. Moreover [1, 38.4], in this case there is a decomposition of the vector space

$$X = T^{d(T)}X \oplus \ker T^{d(T)}.$$

We now focus on the topological situation: For every $T \in L(X)$ we set

$$M_i(T) = i^{-1}(I + T + T^2 + \dots + T^{i-1}), \quad i = 1, 2, 3, \dots, \tag{1.1}$$

i.e. the averages associated with T, where $I = id_X$ is the identity of X. If $T = (T_1, T_2, \dots, T_n) \in L(X)^n$ is commuting multioperator (briefly, c.m.), we also set

$$M_v(T) = M_{v_1}(T_1)M_{v_2}(T_2)\dots M_{v_n}(T_n), \quad v \in Z^n_+, v \geq e, \tag{1.2}$$

where Z^n_+ is the family of multi-indices of length n (i.e. n-tuples of nonnegative integers) and $e := (1, 1, \dots, 1) \in Z^n_+$. In other words, (1.2) defines the averages associated with T.

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Definition 1.1. A commuting multioperator $T \in L(X)^n$ is said to be uniformly ergodic if the limit
$$\lim_v M_v(T) \tag{1.3}$$
exists in the uniform topology of $L(X)$.

Remark 1.2. (a) If $n = 1$, then (1.3) is automatically fulfilled, and therefore the above definition extends the usual concept of uniformly ergodic operator (see, for example [2]).

(b) If $T = (I, \dots, T_j, I, \dots, I) \in L(X)^n$, then T is uniformly ergodic if and only the $\lim_{v_j} M_{v_j}(T_j)$ exists in the uniform topology of $L(X)$.

Definition 1.3 ([3]). Let $k = (k_1, \dots k_n) \in Z_+^n$ and $T \in L(X)^n$ be a c.m. We say that T satisfies condition E-k if $\lim_v (I - T_j)^{k_j} M_v(T) = 0$ for each $j \in \{1, \dots, n\}$.

It is clear that condition E-k implies condition E-n for any $n \geq k$ Thus we see that the example $T = (T_1, I, \dots, I) \in Z_+^n$ with
$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
This shows that E-2e is strictly weaker than E-e.

Theorem 1.4. *Let $k \in Z_+^n$. Suppose $T \in L(X)^n$ satisfies condition E-k and $\sum_{j=1}^n (I - T_j)^{k_j} X$, $\sum_{j=1}^n (I^* - T_j^*)^{k_j} X^*$ are closed in X and X^* respectively. If $\left[\sum_{j=1}^n (I - T_j)^{k_j} X\right] \cap \left[\cap_{j=1}^n \ker(I - T_j)^{k_j}\right] = \{0\}$. Then T is uniformly ergodic*

Proof. Arguing exactly as in [5, Theorem 1], with δ_T and γ_T given by
$$\oplus_{j=1}^n x_j \to \delta_T(\oplus_{j=1}^n x_j) = \sum_{j=1}^n (I - T_j)^{k_j} x_j \text{ and } x \to \gamma_T(x) = \oplus_{j=1}^n (I - T_j)^{k_j} x.$$
□

Theorem 1.5. *Let $T \in L(X)$ satisfy condition E-r, and one of the following nine conditions:*

(a) *$I - T$ has chain length at most r*
(b) *1 is a pole of the resolvent of order at most r*
(c) *$I - T$ is quasi-Fredholm operator*
(d) *$(I - T)^r X$ is closed and $\ker(I - T)^r$ has a closed T-invariant complement*
(e) *$(I - T)^r X \bigoplus \ker(I - T)^r = (I - T)^r X + \ker(I - T)^r$*
(f) *$(I - T)^m X$ is closed for all $m \geq r$*
(g) *$(I - T)^r X$ is closed*
(h) *$(I - T)^m X$ is closed for some $m \geq r$*
(i) *$I - T$ has finite descent.*

Then T is uniformly ergodic.

Proof. Firstly, from [3, Theorem 6], the above statements (a)–(i) are equivalent. Then, take $G = (T, I, \dots I) \in L(X)^n$ and $k = (r, 1, \dots, 1) \in Z_+^n$; Therefore, we have $\sum_{j=1}^n (I - G_j)^{k_j} X = (I - T)^r X$ is closed, it follows that $\sum_{j=1}^n (I^* - G_j^*)^{k_j} X^* = (I^* - T^*)^r X^*$ is closed, which implies since $(I - T)^r X \cap \ker(I - T)^r = \{0\}$, that G is uniformly ergodic in $L(X)^n$. From Theorem 1.4 this means that T is uniformly ergodic in $L(X)$ □

Theorem 1.6. *Let $k \in Z_+^n$. If $T \in L(X)^n$ A c.m. satisfies condition E-k, such that $\sum_{j=1}^n (I-T_j)$ has chain length at most 1 and $\ker(\sum_{j=1}^n (I-T_j)) = \cap_{j=1}^n \ker(I-T_j)$. Then T is uniformly.*

Proof. There are two cases

Case 1: $d\big(\sum_{j=1}^n (I-T_j)\big) = 0$**.** Then $\sum_{j=1}^n (I-T_j)$ is bijective then $X = \sum_{j=1}^n (I-T_j)X \oplus \ker(\sum_{j=1}^n (I-T_j))$, which implies, since $\cap_{j=1}^n \ker(I-T_j) \subset \ker(\sum_{j=1}^n (I-T_j)) = \{0\}$ that $X = \sum_{j=1}^n (I-T_j)X \bigoplus \cap_{j=1}^n \ker(I-T_j)$, from the [3, Theorem 10] we obtain T is uniformly ergodic.

Case 2: $d(\big(\sum_{j=1}^n (I-T_j)\big)) = 1$**.** Then $\big(\sum_{j=1}^n (I-T_j)\big)X = \big(\sum_{j=1}^n (I-T_j)\big)^2 X$, so $\big(\sum_{j=1}^n (I-T_j)\big)X = \big(\sum_{j=1}^n (I-T_j)\big)^{nr}\big(\sum_{j=1}^n (I-T_j)\big)X$, with $r = \max_{1\le j\le n} k_j$. so $\big(\sum_{j=1}^n (I-T_j)\big)^{nr}$ is a bijection of $\big(\sum_{j=1}^n (I-T_j)\big)X$ onto itself. Which implies, since T satisfies condition E-k, that $M_v(T)|\big(\sum_{j=1}^n (I-T_j)\big)X \to 0$ and since $M_v(T)|\cap_{j=1}^n \ker(I-T_j) = I|\cap_{j=1}^n \ker(I-T_j)$, it follows that T is uniformly ergodic. $\square$

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