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COMMON FIXED POINTS FOR LIPSCHITZIAN SEMIGROUPS

SAMIR LAHRECH, ABDERRAHIM MBARKI, ABDELMALEK OUAHAB

Abstract. Lim and Xu [4] established a fixed point theorem for uniformly Lipschitzian mappings in metric spaces with uniform normal structure. Recently, Huang and Hong [1] extended hyperconvex metric space version of this theorem, by showing a common fixed point theorem for left reversible uniformly k-Lipschitzian semigroups. In this paper, we extend Huang and Hong's theorem to metric spaces with uniform normal structure.

1. Introduction and main results

Throughout this paper, (X, d) stands for a metric space, a nonempty family $\mathcal{F}$ of subsets of X is said to define a convexity structure on X if it is stable by intersection. Recall that a subset of X is said admissible if it is an intersection of closed balls. We denote, by $\mathcal{A}(X)$ the family of all admissible subsets of X. Obviously, $\mathcal{A}(X)$ defines a convexity structure on X. In this paper any convexity structure $\mathcal{F}$ on X is always assumed to contain $\mathcal{A}(X)$. For $r \geq 0$ and x in X and a bounded subset M of X, we adopt the following notation:

$$
\begin{gathered}
B(x,r) \text{ is the closed ball centered at } x \text{ with radius } r,\\
r(x,M) = \sup\{d(x,y) : y \in M\},\\
\delta(M) = \sup\{r(x,M) : x \in M\},\\
R(M) = \inf\{r(x,M) : x \in M\}.
\end{gathered}
$$

Definition 1.1 ([2])**.** A metric space (X, d) is said to have normal (resp. uniform normal) structure if there exists a convexity structure $\mathcal{F}$ on X such that $R(A) < \delta(A)$ (resp. $R(A) \leq c\delta(A)$ for some constant $c \in (0,1)$) for all A in $\mathcal{F}$ which is bounded and $\delta(A) > 0$. It is also said that $\mathcal{F}$ is normal and (resp. uniformly normal).

The uniform normal structure coefficient $N(X)$ of X relative to $\mathcal{F}$ is the number

$$\sup\{\frac{R(A)}{\delta(A)} : A \in \mathcal{F} \text{ is bounded and } \delta(A) > 0\}.$$

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Definition 1.2 ([3]). Let (X, d) a metric space and $\mathcal{T}$ is the family of subsets of X consisting of X and sets which are complements of closed balls of X. The weak topology (also called ball topology) on X is the topology whose open sets are generated by $\mathcal{T}$.

It is clear that X is compact in the weak topology if and only if every subfamily of $\mathcal{A}(X)$ with the finite intersection property has nonempty intersection.

Kulesza and Lim proved the following result.

Lemma 1.3 ([3]). *Every complete metric space with uniform normal structure is compact in the weak topology.*

For a bounded subset A of X, the admissible hull of A, denoted by $ad(A)$, is the set

$$\cap\{B : A \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.$$

The following definition is a net version of [4, def. 5].

Definition 1.4 ([1]). A metric space (X, d) is said to have the property (P) if given any two bounded nets $\{x_i\}_{i\in I}$ and $\{z_i\}_{i\in I}$ in X, one can find some $z \in \cap_{i\in I} \operatorname{ad}\{z_j : j \geq i\}$ such that

$$\overline{\lim}_{i\in I} d(z, x_i) \leq \overline{\lim}_{j\in I} \overline{\lim}_{i\in I} d(z_j, x_i),$$

where $\overline{\lim}_{i\in I} d(z, x_i) = \inf_{\beta\in I} \sup_{i\geq\beta} d(z, x_i)$.

Remark 1.5. If X has uniform normal structure, then $\cap_{i\in I} \operatorname{ad}\{z_j : j \geq i\} \neq \emptyset$ (by Lemma 1.3). Also, if X is a weakly compact convex subset of a normed linear space, then admissible hulls are closed convex and $\cap_{i\in I} \operatorname{ad}\{z_j : j \geq i\} \neq \emptyset$ by weak compactness of X and that X possesses property (P) follows directly from the weak lower semicontinuity of the function $x \longmapsto \overline{\lim}_{i\in I} \|x_i - x\|$.

The following Lemma is a net version of [4, lemma. 5].

Lemma 1.6. *Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure. Then for any net $\{x_i\}_{i\in I}$ in X and any $\overline{c} > N(X)$, the normal structure coefficient with respect to the given convexity structure $\mathcal{F}$, there exists a point $z \in X$ satisfying the properties:*

(i) $\overline{\lim}_{i\in I} d(z, x_i) \leq \overline{c}\delta(\{x_i\}_{i\in I})$;

(ii) $d(z, y) \leq \overline{\lim}_{i\in I} d(x_i, y)$ *for all* $y \in X$.

Proof. Using the Lemma 1.3 to conclude that $\cap_{i\in I} A_i \neq \emptyset$ for any deceasing net $\{A_i\}_{i\in I}$ of admissible subsets of X, the rest of the proof of lemma is the same as that in Lim et al. [4]. □

Let S be a semigroup of selfmaps on a metric space (X, d). For any $x \in X$ (resp. $b \in S$), we denote by Sx (resp. bS) the subset $\{gx : g \in S\}$ (resp $\{bg : g \in S\}$) of X (resp. of S). Recall that a semigroup S is said to be left reversible if, for any f, g in S, there are a, b in S such that $fa = gb$. Examples of left reversible semigroups include all commutative semigroups and all groups.

Let S be a left semigroup. For a, b in S we say that $a \geq b$ if $a \in bS \cup \{b\}$. Then $(S, \geq)$ is a directed set. In what follows in this paper, we deal only with "$\geq$".

Definition 1.7 ([1]). A semigroup S acting on a metric space (X, d) is said to be a uniformly k-Lipschitzian semigroup if

$$d(tx, ty) \leq kd(x, y)$$

for all t in S and all x, y in X.

If S is a left reversible semigroup, then $(S, \geq)$ is a linearly directed set if any a, b in S satisfy either $a \geq b$ or $b \geq a$. For example, if $\Delta = \{T_s : s \in [0, \infty)\}$ is a family of selfmaps on $\mathbb{R}$ such that $T_{h+t}(x) = T_h T_t(x)$ for all h, t in $[0, \infty)$ and $x \in \mathbb{R}$, then $(\Delta, \geq)$ is a linearly directed left reversible semigroup.

Our main result is as follows.

Theorem 1.8. *Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure and let S be a left reversible uniformly k-Lipschitzian semigroup of selfmaps on X such that $k < N(X)^{-1/2}$ and $(S, \geq)$ is a linearly directed set. Then S has a common fixed point z in X.*

Proof. Choose a constant $\overline{c}$, $1 > \overline{c} > N(X)$, such that $k < \overline{c}^{-1/2}$. Fix an $x_0 \in X$. For $t \in S$, denote tx_0 by $x_{0,t}$. Then $\{x_{o,t}\}$ is a net in X. By Lemma 1.6, we can inductively construct a sequence $\{x_j\} \subset X$ such that for each integer $j \leq 0$,

(a) $\overline{\lim}_{t \in S} d(x_{j+1}, x_{j,t}) \leq \overline{c}\delta(Sx_j)$;
(b) $d(x_{j+1}, y) \leq \overline{\lim}_{t \in S} d(x_{j,t}, y)$ for all y in X.

Write
$$D_j = \overline{\lim}_{t \in S} d(x_{j+1}, x_{j,t}) \text{ and } h = \overline{c}k^2 < 1.$$
The rest of the proof of Theorem is the same as that in Huang and Hong [1]. □

Remark 1.9. It can be seen from the above that the conclusion of main theorem is still valid if we only assume that $\mathcal{A}(X)$, the family of all admissible subsets of X, is uniformly normal.

The following corollary follows immediately from the main Theorem.

Corollary 1.10 (Huang and Hong [1]). *Let (X, d) be a bounded hyperconvex metric space with both property (P) and let S be a left reversible uniformly k-Lipschitzian semigroup of selfmaps on X such that $k < \sqrt{2}$ and $(S, \geq)$ is a linearly directed set. Then S has a common fixed point z in X.*

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Samir Lahrech
Département de Mathématiques, Université Oujda, 60000 Oujda, Morocco
E-mail address: lahrech@sciences.univ-oujda.ac.ma

Abderrahim Mbarki
Current address: National school of Applied Sciences, P.O. Box 669, Oujda University, Morocco
E-mail address: ambarki@ensa.univ-oujda.ac.ma

Abdelmalek Ouahab
Département de Mathématiques, Université Oujda, 60000 Oujda, Morocco
E-mail address: ouahab@sciences.univ-oujda.ac.ma