2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 163–172.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF TWO NONTRIVIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS

ABDEL R. EL AMROUSS, FOUZIA MORADI, MIMOUN MOUSSAOUI

Abstract. This paper concerns the existence of multiple nontrivial solutions for some nonlinear problems. The first nontrivial solution is found using a minimax method, and the second by computing the Leray-Schauder index and the critical group near 0.

1. Introduction

We consider the Dirichlet problem

$$\begin{gathered} -\Delta u = \lambda_k u + f(u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function satisfying the Carathéodory conditions, and $0 < \lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots$ is the sequence of eigenvalues of the problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Let us denote by $E(\lambda_j)$ the λ_j-eigenspace and by $F(s)$ the primitive $\int_0^s f(t)\, dt$.

There are several works studying the problem

$$\begin{gathered} -\Delta u = \lambda_k u + f(x,u) + h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.2}$$

where $h \in L^2(\Omega)$; see for example [4, 5, 6, 8, 9]. We write

$$\begin{gathered} l_\pm(x) = \liminf_{s\to\pm\infty} \frac{f(x,s)}{s}, \quad k_\pm(x) = \limsup_{s\to\pm\infty} \frac{f(x,s)}{s}, \\ L_\pm(x) = \liminf_{s\to\pm\infty} \frac{2F(x,s)}{s^2}, \quad K_\pm(x) = \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2}\,. \end{gathered}$$

In [6], the solvability of (1.2) for every $h \in L^2(\Omega)$, is ensured when

$$0 < \upsilon_k \leq l_\pm(x) \leq k_\pm(x) \leq \upsilon_{k+1} < \lambda_{k+1} - \lambda_k,$$

2000 *Mathematics Subject Classification.* 58E05, 35J65, 56J20.
Key words and phrases. Variational elliptic problem; resonance; critical group; Morse theory; minimax method.

Published September 20, 2006.

where υ_k and υ_{k+1} are constants.

However, in the autonomous case $f(x,s) = f(s)$, De Figuerido and Gossez [5] introduced a density condition that requires $\frac{f(s)}{s}$ to be between 0 and $\alpha = \lambda_{k+1} - \lambda_k$ as $s \to \pm\infty$, and showed the existence of solution for any h. Next in [4], Costa and Oliveira proved an existence result for (1.2) under the following conditions:

$$0 \le l_\pm(x) \le k_\pm(x) \le \lambda_{k+1} - \lambda_k \quad \text{uniformly for a.e } x \in \Omega, \tag{1.3}$$

$$0 \preceq L_\pm(x) \le K_\pm(x) \preceq \lambda_{k+1} - \lambda_k \quad \text{uniformly for a.e } x \in \Omega. \tag{1.4}$$

Here the relation $a(x) \preceq b(x)$ indicates that $a(x) \le b(x)$ on Ω, with strict inequality holding on subset of positive measure.

Later in [9], the authors proved an existence result in situation $L_\pm(x) = 0$ for a.e $x \in \Omega$ and $K_\pm(x) = \lambda_{k+1} - \lambda_k$ for a.e $x \in \Omega$. They replaced (1.4) by classical resonance conditions of Ahmad-Lazer-Paul on two sides of (1.4) and showed that (1.2) is solvable. More recently, in [8], the author interested to study the existence of two nontrivial solutions in the case $k = 1$ and under other weaker conditions cited above.

The aim of this paper is to generalize the above result for $k \ge 1$. We assume the following assumptions:

- (F0) $|f'(s)| \le c(|s|^p + 1)$, $s \in \mathbb{R}$, $p < \frac{4}{n-2}$ if $n \ge 3$ and no restriction if $n = 1, 2$.
- (F1) $sf(s) \ge 0$ for $|s| \ge r > 0$ and
$$\limsup_{s \to \pm\infty} \frac{f(s)}{s} \le \lambda_{k+1} - \lambda_k = \alpha.$$
- (F2) $\lim_{\|v\| \to \infty, v \in E(\lambda_k)} \int F(v(x))dx = +\infty$.
- (F3) There exists $\eta \in \mathbb{R}$, $0 < \eta < \alpha$, such that
$$\liminf_{n \to +\infty} \frac{\mu(G_n)}{n} > 0$$
where $G_n = \{s \in] - n, n[, s \ne 0, \text{ and } \frac{f(s)}{s} \le \alpha - \eta\}$ and μ denotes the Lebesgue measure on $\mathbb{R}$.
- (F4) $f'(0) + \lambda_k < \lambda_1$

Theorem 1.1. *Let f be C^1 function, with $f(0) = 0$, that satisfies the conditions (F0)-(F4). Then* (1.1) *has at least two nontrivial solutions.*

This paper is organized as follows: In section 2, we give some technical lemmas and some results of critical groups. The proof of our result is carried out in section 3.

2. Preliminaries Lemmas

Let us consider the functional defined on $H_0^1(\Omega)$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\lambda_k \int u^2 dx - \int F(u)dx.$$

where $H_0^1(\Omega)$ is the usual Sobolev space obtained through the completion of $C_c^\infty(\Omega)$ with respect to the norm induced by the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v dx, \quad u, v \in H_0^1(\Omega).$$

It is well known that under a linear growth condition on f, the functional Φ is well defined on $H_0^1(\Omega)$, weakly lower semi-continuous and $\Phi \in C^1(H_0^1, \mathbb{R})$, with

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \nabla v dx - \lambda_k \int uv dx - \int f(u) v dx, \quad \text{for } u, v \in H_0^1(\Omega).$$

Consequently, the weak solutions of the problem (1.1) are the critical points of the functional Φ. Moreover, under the condition(F0), Φ is a C^2 functional with the second derivative given by

$$\Phi''(u)v.w = \int \nabla v \nabla w dx - \lambda_k \int vw dx - \int f'(u) vw dx,$$

for $u, v, w \in H_0^1(\Omega)$.

Since we are going to apply the variational characterization of the eigenvalues, we shall decompose the space $H_0^1(\Omega)$ as $E = E_- \oplus E_k \oplus E_{k+1} \oplus E_+$, where E_- is the subspace spanned by the λ_j- eigenfunctions with $j < k$ and E_j is the eigenspace generated by the λ_j-eigenfunctions and E_+ is the orthogonal complement of $E_- \oplus E_k \oplus E_{k+1}$ in $H_0^1(\Omega)$ and we shall decompose for any $u \in H_0^1(\Omega)$ as following $u = u^- + u^k + u^+$ where $u^- \in E_-$, $u^k \in E_k$, $u^{k+1} \in E_{k+1}$ and $u^+ \in E_+$. We can verify easily that

$$\int |\nabla u|^2 \, dx - \lambda_i \int |u|^2 \, dx \geq \delta_i \|u\|^2 \quad \forall u \in \oplus_{j \geq i+1} E_j \tag{2.1}$$

$$\int |\nabla u|^2 \, dx - \lambda_i \int |u|^2 \, dx \leq -\delta_i \|u\|^2 \quad \forall u \in \oplus_{j \leq i} E_j, \tag{2.2}$$

where $\delta_i = \min\{1 - \frac{\lambda_i}{\lambda_{i+1}}, \frac{\lambda_i}{\lambda_{i-1}} - 1\}$.

2.1. **A compactness condition.**

To apply minimax methods for finding critical points of Φ, we need to verify that Φ satisfies a compacteness condition of the Palais-Smail type which was introduced by Cerami [2], and recently was generalized by the first author in [7].

Definition. Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$.
(i) A sequence (u_n) is said to be a $(C)_c$ sequence, at the level $c \in \mathbb{R}$, if there is a sequence $\epsilon_n \to 0$, such that

$$\Phi(u_n) \to c \tag{2.3}$$

$$\|u_n\| \langle \Phi'(u_n), v\rangle_{H_0^1, H^{-1}} \leq \epsilon_n \|v\| \quad \forall v \in H_0^1. \tag{2.4}$$

(ii) A functional $\Phi \in C^1(E, \mathbb{R})$, is said to satisfy a condition $(C)_c$, at the level $c \in \mathbb{R}$, if every $(C)_c$ sequence (u_n), possesses a convergent subsequence.

Now, we present some technical lemmas.

Lemma 2.1. *Let $(u_n) \subset H_0^1(\Omega)$ and $(p_n) \subset L^\infty(\Omega)$ be sequences, and let A a nonnegative constant such that*

$$0 \leq p_n(x) \leq A \quad \text{a.e. in } \Omega \text{ and for all } n \in \mathbb{N}$$

and $p_n \rightharpoonup 0$ in the weak topology of L^∞, as $n \to \infty$. Then, there are subsequences $(u_n), (p_n)$ satisfying the above conditions, and there is a positive integer n_0 such that for all $n \geq n_0$,*

$$\int p_n u_n ((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)) \, dx \geq \frac{-\delta_k}{2} \|u_n^+ + u_n^{k+1}\|^2. \tag{2.5}$$

Proof. Since $p_n \geq 0$ a.e. in Ω, we see that

$$\begin{aligned} &\int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)) \\ &\geq -\int p_n (u_n^+ + u_n^{k+1})^2\,dx \\ &\geq -\Big[\int p_n \big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\big)^2\,dx\Big]\|u_n^+ + u_n^{k+1}\|^2. \end{aligned} \tag{2.6}$$

Moreover, by the compact imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ and $p_n \rightharpoonup 0$ in the weak* topology of L^∞, when $n \to \infty$, then there are subsequences $(u_n), (p_n)$ such that

$$\int p_n \Big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\Big)^2\,dx \to 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\int p_n \Big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\Big)^2\,dx \leq \frac{\delta_k}{2}. \tag{2.7}$$

Combining inequalities (2.6) and (2.7), we get inequality (2.5). $\square$

Lemma 2.2. *Let $(u_n) \subset H_0^1(\Omega)$ be a (C) sequence. If*

$$f_n(x) = \frac{f(u_n(x))}{u_n(x)}\chi_{[|u_n(x)| \geq r_\epsilon]} \rightharpoonup 0$$

in the weak topology of L^∞, as $n \to \infty$. Then, there is subsequence (u_n) such that $(\|u_n^- + (u_n^+ + u_n^{k+1})\|)_n$ is uniformly bounded in n.*

Proof. Since $(u_n)_n \subset H_0^1$ be a (C) sequence, (2.3) and (2.4) are satisfied. Now, we prove that the sequence $(\|u_n^- + u_n^+ + u_n^{k+1}\|)_n$ is uniformly bounded in n. Take $v = (u_n^- + u_n^k) - (u_n^+ + u_n^{k+1})$ in (2.4), $p_n(x) = f_n(x)$, and

$$\begin{aligned} \Lambda = \Big\{ &-\int |\nabla u_n^-|^2 + \lambda_k \int |u_n^-|^2\,dx + \int |\nabla(u_n^+ + u_n^{k+1})|^2 \\ &- \lambda_k \int |u_n^+ + u_n^{k+1}|^2\,dx + \int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+))\,dx\Big\} \\ \Gamma = \Big\{&\epsilon_n + \int_{|u_n(x)| \leq r_\epsilon} |f(u_n(x)||(u_n^+ + u_n^{k+1}) - (u_n^- + u_n^k)|\,dx\Big\}. \end{aligned}$$

Then $\Lambda \leq \Gamma$. By the Poincaré inequality, from (2.1), (2.2), (2.5), and $\Lambda \leq \Gamma$, it follows that there exists constants A_ϵ and B_ϵ such that

$$\frac{\delta_k}{2}\|u_n^- + (u_n^+ + u_n^{k+1})\|^2 \leq \epsilon_n + A_\epsilon\|u_n^- + (u_n^+ + u_n^{k+1})\| + B_\epsilon.$$

This gives that $(\|u_n^- + (u_n^+ + u_n^{k+1})\|)_n$ is uniformly bounded in n. $\square$

Lemma 2.3. *Let $(u_n) \subset H_0^1(\Omega)$ such that $\|u_n^- + (u_n^+ + u_n^{k+1})\|$ is uniformly bounded in n and there exists A such that if $A \leq \Phi(u_n)$, then*

$$\int F(\frac{u_n^k}{2})\,dx \leq M.$$

Proof. From $A \leq \Phi(u_n)$, and Poincaré inequality, we have

$$\int F(\frac{u_n^k}{2})dx \leq -A + \int [F(\frac{u_n^k}{2}) - F(u_n)]dx + \frac{1}{2}\|u_n^- + u_n^+ + u_n^{k+1}\|^2. \tag{2.8}$$

Since $f \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ satisfy (F1), there exists two functions $\gamma, h : \Omega \to \mathbb{R}$ such that

$$f(t) = t\gamma\left(t\right) + h\left(t\right)$$

with $0 \leq \gamma\left(t\right) = \frac{f(t)}{t}\chi[|t| \geq r] \leq \lambda_{k+1} - \lambda_k$ and $h(t) = f(t)\chi[|t| < r]$. However, by the mean value theorem, we get

$$\begin{aligned}
\int [F(\frac{u_n^k}{2}) - F(u_n)]dx &= \int_\Omega \int_0^1 f(t\frac{u_n^k}{2} + (1-t)u_n)dt(\frac{u_n^k}{2} - u_n)dx \\
&= \int_\Omega \int_0^1 h(t\frac{u_n^k}{2} + (1-t)u_n)dt(\frac{u_n^k}{2} - u_n)dx \\
&\quad + \int_\Omega \int_0^1 \gamma(t\frac{u_n^k}{2} + (1-t)u_n)[t(\frac{u_n^k}{2} - u_n)^2 + (\frac{u_n^k}{2} - u_n)u_n]
\end{aligned} \tag{2.9}$$

Set $t_1 = \min\{t \in [0,1] : \int_0^1 h(t\frac{u_n^k}{2} + (1-t)u_n) \neq 0\}$ and $t_2 = \max\{t \in [0,1] : \int_0^1 h(t\frac{u_n^k}{2} + (1-t)u_n) \neq 0\}$. It is clear that

$$(t_2 - t_1)|\frac{u_n^k}{2} - u_n| \leq 2r. \tag{2.10}$$

So that using (2.9),(2.10) and the Poincaré inequality, and an elementary inequality

$$(\frac{a}{2} - b)^2 + (\frac{a}{2} - b)b \leq (a-b)^2.$$

We have

$$\begin{aligned}
&\int [F(\frac{u_n^k}{2}) - F(u_n)]dx \\
&\leq \int_\Omega \int_{t_1}^{t_2} h(t\frac{u_n^k}{2} + (1-t)u_n)dt\big(\frac{u_n^k}{2} - u_n\big)dx + \frac{\lambda_{k+1} - \lambda_k}{4\lambda_1}\|u_n^- + u_n^+ + u_n^{k+1}\|^2 \\
&\leq 2r \sup_{|s| \leq r} |f(s)| \operatorname{meas}(\Omega) + \frac{\lambda_{k+1} - \lambda_k}{4\lambda_1}\|u_n^- + u_n^+ + u_n^{k+1}\|^2.
\end{aligned} \tag{2.11}$$

From (2.8) and (2.11), there exists $M > 0$ such that

$$\int F(\frac{u_n^k}{2})\,dx \leq M.$$

□

2.2. **Critical groups.** Let H be a Hilbert space and $\Phi \in C^1(H, \mathbb{R})$ satisfying the Palais-Smaile condition or the Cerami condition. Set $\Phi^c = \{u \in H \mid \Phi(u) \leq c\}$ and denote by $H_q(X, Y)$ the q-th relative singular homology group with integer coefficient. The critical groups of Φ at an isolated critical point u with $\Phi(u) = c$ are defined by

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}); \quad q \in Z.$$

where U is a closed neighborhood of u.

Let $K = \{u \in H \mid \Phi'(u) = 0\}$ be the set of critical points of Φ and $a < \inf_K \Phi$. The critical groups of Φ at infinity are defined by

$$C_q(\Phi, \infty) = H_q(H, \Phi^a); \quad q \in Z$$

We will use the notation $\deg(\Phi', U, 0)$ for the Leray-Schauder degree of Φ with respect to the set U and the value 0. Denote also by $\operatorname{index}(\Phi', u)$ the Leray-Schauder index of Φ' at critical point u. Recall that this quantity is defined as $\lim_{r\to 0} \deg(\Phi', B_r(u), 0)$, if this limit exists, where $B_r(u)$ is the ball of radius r centered at u.

Proposition 2.4 ([3]). *If u is a mountain pass point of Φ, then*

$$C_q(\Phi, u) = \delta_{q,1} Z.$$

Proposition 2.5 ([1]). *Assume that $H = H^+ \oplus H^-$, Φ is bounded from below on H^+ and $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in H^-$. Then*

$$C_\mu(\Phi, \infty) \neq 0, \quad \text{with } \mu = \dim H^- < \infty.$$

3. Proof of Theorem 1.1

First, we prove that Φ satisfies the Cerami condition.

Lemma 3.1. *Under the assumptions (F0)–(F3), Φ satisfies the $(C)_c$ condition on $H_0^1(\Omega)$, for all $c \in \mathbb{R}$.*

Proof. Let $(u_n)_n \subset H_0^1$ be a $(C)_c$ sequence, i.e

$$\Phi(u_n) \to c \tag{3.1}$$

$$\|u_n\| \langle \Phi'(u_n), v\rangle_{H_0^1, H^{-1}} \le \epsilon_n \|v\| \quad \forall v \in H_0^1, \tag{3.2}$$

where $\epsilon_n \to 0$. It clearly suffices to show that $(u_n)_n$ remains bounded in H_0^1. Assume by contradiction. Defining $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that $z_n \rightharpoonup z$ weakly in H_0^1, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω. By the linear growth of f , the sequence $\big(\frac{f(u_n(x))}{\|u_n\|}\big)_n$ remains bounded in L^2, then for a subsequence, we have

$$\frac{f(u_n(x))}{\|u_n\|} \rightharpoonup \zeta \quad \text{in } L^2.$$

and by standard arguments based on assumptions F0),F1), ζ can be written as $\zeta(x) = m(x)z(x)$, where m satisfies (see [4]).

$$0 \le m(x) \le \lambda_{k+1} - \lambda_k \quad \text{a.e. in } \Omega.$$

However, divide (3.2) by $\|u_n\|^2$ and goes to the limit we obtain

$$\frac{\langle \Phi'(u_n), v\rangle}{\|u_n\|} = \int \nabla z_n \nabla v - \lambda_k \int z_n v - \int \frac{f(u_n)}{\|u_n\|} v dx \to 0$$

for every $v \in H_0^1$. On the other hand, since z_n converges to z weakly in H_0^1, strongly in L^2 and $\frac{f(u_n(x))}{\|u_n\|}$ converges weakly in L^2 to ζ, we deduce

$$\frac{\langle \Phi'(u_n), v\rangle}{\|u_n\|} \to \int \nabla z \nabla v - \lambda_k \int z v - \int \zeta v dx = 0 \quad \forall v \in H_0^1(\Omega). \tag{3.3}$$

Claim: We will prove that $z_n \to z$ strongly in H_0^1. Indeed, taking $v = z$ in (3.3) we have

$$\|z\|^2 = \lambda_k \int z^2 + \int m(x)z^2. \tag{3.4}$$

On the other hand, by (3.2) it results

$$\frac{\langle \Phi'(u_n), u_n\rangle}{\|u_n\|^2} \to 1 - \lambda_k \int z^2 - \int m(x)z^2 = 0. \tag{3.5}$$

From (3.4) and (3.5), it follows $\|z\| = 1$. Since $z_n \rightharpoonup z$, $\|z_n\| \to \|z\|$ and $H_0^1(\Omega)$ is convex uniformly space the claim follows. So that, z is a nontrivial solution of problem

$$\begin{gathered} -\Delta z = (\lambda_k + m(x))z \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.6}$$

We now distinguish three cases: i) $\lambda_k < m(x)+\lambda_k$ and $m(x)+\lambda_k < \lambda_{k+1}$ on subset of positive measure; (ii) $m(x) + \lambda_k \equiv \lambda_k$; (iii) $m(x) + \lambda_k \equiv \lambda_{k+1}$.

Case i: We have z is a nontrivial solution of problem (3.6), then 1 is an eigenvalue of this problem. On the other hand, by strict monotonicity $\lambda_k\left(\lambda_k + m(x)\right) < 1$ and $\lambda_{k+1}\left(\lambda_k + m(x)\right) > 1$, which gives a contradiction.

Case ii: By (F1), for $\varepsilon > 0$, there exists a constant $r_\varepsilon > r$ such that

$$0 \le \frac{f(s)}{s} \le \lambda_{k+1} - \lambda_k + \varepsilon \quad \forall |s| \ge r_\varepsilon \tag{3.7}$$

Put $f_n(x) = \frac{f\left(u_n(x)\right)}{u_n(x)}\chi\{|u_n(x)| \ge r_\varepsilon\}$, which remains bounded in L^∞, passing if necessary to a subsequence, $f_n \to l$ in the weak* topology of L^∞. By (3.7), the L^∞-function l satisfies

$$0 \le l(x) \le \lambda_{k+1} - \lambda_k + \varepsilon \quad \text{a.e.in } \Omega$$

Multiply f_n by z_n^2, integrate on Ω and going to the limit, to have

$$\int f_n z_n^2 dx = \int \frac{f\left(u_n(x)\right)}{\|u_n\|} z_n \to \int m(x)z^2 dx = \int l(x)z^2 dx = 0.$$

By the unique continuation Property of Δ and $l \ge 0$, we deduce that $l \equiv 0$ a.e.in Ω. Then, by lemma 2.2 and lemma 2.3 there exists $M > 0$ such that

$$\int F(\frac{u_n^k}{2})dx \le M.$$

This is a contradiction with assumption (F2) and $\|u_n^k\| \to +\infty$.

Case iii: If $m(x) \equiv \lambda_{k+1} - \lambda_k$. Dividing (3.1) by $\|u_n\|^2$, we obtain

$$\frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\|z_n\|^2 - \frac{\lambda_k}{2}\int z_n^2 - \int \frac{F(u_n(x))}{\|u_n\|^2}dx \to 0, \quad \text{as } n \to \infty.$$

However, it results that

$$\lim_{n\to+\infty} \int \frac{F(u_n(x))}{\|u_n\|^2}dx = \frac{1}{2}\alpha \int z^2 dx.$$

Applying Fatou's lemma, we have

$$\int_{z>0} (\alpha - K_+)z^2 dx + \int_{z<0} \left(\alpha - K_-\right) z^2 dx \le 0.$$

This is a contradiction with assumption (F3), since (F3) is equivalent to $K_{\pm} = \limsup_{s\to\pm\infty} \frac{2F(s)}{s^2} < \alpha$. (see [10]). The proof of lemma is complete. $\square$

Lemma 3.2. *Under the hypothesis of Theorem 1.1, the functional Φ has the following properties:*

(i) $\Phi(w) \to +\infty$, *as* $\|w\| \to +\infty$, $w \in W^+ = E_{k+1} \oplus E_+$.
(ii) $\Phi(v) \to -\infty$, *as* $\|v\| \to +\infty$, $w \in W^- = E_k \oplus E_-$.

Proof. (i) Φ is coercive on W^+. Indeed, the assumption (F3) is equivalent to $K_{\pm} = \limsup_{s\to\pm\infty} \frac{2F(s)}{s^2} < \alpha$. Thus, there exists an $B_\varepsilon \geq 0$ such that

$$F(s) \leq \frac{\alpha}{2}s^2 - \varepsilon s^2 + B_\varepsilon \quad \forall s \in \mathbb{R}.$$

Hence, for every $w \in W^+$, we obtain

$$\begin{aligned}
\Phi(w) &= \frac{1}{2}\|w\|^2 - \frac{\lambda_k}{2}\int w^2 - \int F(w)\,dx \\
&\geq \frac{\lambda_{k+1} - \lambda_k}{2\lambda_{k+1}}\|w\|^2 - \frac{\alpha - 2\varepsilon}{2}\int w^2 - B_\varepsilon|\Omega| \\
&\geq \frac{\varepsilon}{\lambda_{k+1}}\|w\|^2 - B_\varepsilon|\Omega|.
\end{aligned}$$

However, $\Phi(w) \to +\infty$, as $\|w\| \to +\infty$.
(ii) Assume by contradiction that there exists a constant $B > 0$ and a sequence $(v_n) \subset V$ with $\|v_n\| \to \infty$ such that

$$B \leq \Phi(v_n) \leq -\delta\|v_n^-\|^2.$$

Therefore, by lemma 2.3, since $\|v_n^-\|$ is bounded, there exists $M > 0$ such that

$$\int F(\frac{v_n^k}{2})\,dx \leq M$$

which contradicts (F2). $\square$

Lemma 3.3. *Under the condition (F4), the functional Φ has the following properties:*

(i) *There is an $R > 0$ and $\beta > 0$ such that $\Phi \geq \beta$ on $\partial B_R(0)$.*
(ii) $C_q(\Phi, 0) = \delta_{q,0}Z$

Proof. (i) We start by proving the first assertion. On one hand, it is easy to see that if $\lambda_k + f'(0) \leq 0$ we have

$$\Phi''(0)u.u \geq \|u\|^2.$$

On the other hand, where $\lambda_k + f'(0) > 0$, the Poincaré's inequality gives that

$$\Phi''(0)u.u = \|u\|^2 - \lambda_k\int u^2 - \int f'(0)u^2dx \geq \big(1 - \frac{\lambda_k + f'(0)}{\lambda_1}\big)\|u\|^2$$

Put $\gamma = 1 - \frac{\lambda_k + f'(0)}{\lambda_1}$ and by (F4), we have $\gamma > 0$ and

$$\Phi''(0)u.u \geq \gamma\|u\|^2.$$

Taylor's formula implies

$$\Phi(u) = \frac{1}{2}\Phi''(0)u.u + o(\|u\|^2) \geq \big(\frac{\gamma}{2} + \frac{o(\|u\|^2)}{\|u\|^2}\big)\|u\|^2$$

with $\frac{o(\|u\|^2)}{\|u\|^2} \to 0$, as $\|u\| \to 0$. Consequently, the assertion (i) follows.
(ii) Since $u = 0$ is a local mininum of Φ, we have

$$C_q(\Phi, 0) = \delta_{q,0} Z.$$

□

Lemma 3.4. *The functional Φ has at least one critical point u_0, such that*

$$C_q(\Phi, u_0) = \delta_{q,1} Z.$$

Proof. According to (ii) of Lemma 3.2, Φ is anti-coercive on W^- we can find an $e \in H_0^1$ such that $\|e\| \geq M > R$ and$\Phi(e) \leq 0$. So by mountain pass theorem, there exists a critical point u_0 of mountain pass type, such that

$$C_1(\Phi, u_0) \neq 0.$$

By proposition 2.4, it results that $C_q(\Phi, u_0) = \delta_{q,1} Z$. The proof of lemma is complete. □

Proof of Theorem 1.1. For this proof we distinguish two cases.

Case 1: If $k = 1$, we assume that $\{0, u_0\}$ is the critical set of Φ and let $R > 0$, such that $\{0, u_0\} \subset B_R(0)$. By the Riesz representation theorem we can write

$$\langle \Phi'(u), v \rangle = \langle u, v \rangle - \langle Nu, v \rangle, \quad \text{for all } u, v \in H_0^1(\Omega)$$

where $\langle u, v \rangle = \int_\Omega \nabla u \nabla v$ and $\langle Nu, v \rangle = \int [\lambda_1 u + f(u)] v \, dx$. So that, $\Phi' = I - N$ and By the Sobolev embedding theorem, N is compact. We see that Φ' has the form Identity-compact, so that Leary-Shauder techniques are applicable

$$\begin{aligned} \deg(\Phi', B_R(0), 0) &= \operatorname{index}(\Phi', 0) + index(\Phi', u_0) \\ &= \sum_{q=0}^{\infty} (-1)^q \dim C_q\,(\Phi, 0) + \sum_{q=0}^{\infty} (-1)^q \dim C_q\,(\Phi, u_0) \\ &= 1 - 1 = 0 \end{aligned} \tag{3.8}$$

In a similar way we can define a compact map $T : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$\langle Tu, v \rangle = \int (\lambda_1 + \mu) uv \, dx$$

where $0 < \mu < \lambda_2 - \lambda_1$. Now we claim that there is a priori bound in $H_0^1(\Omega)$ for all possible solutions of the family of equations (see [12])

$$\begin{gathered} -\Delta u - \lambda_1 u = (1 - t)\mu u + tf(u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The homotopy invariance of Leray-Schauder degree implies

$$\deg(\Phi', B_R(0), 0) = \deg(I - T, B_R(0), 0) = -1.$$

This contradicts (3.8).

Case 2: If $k \geq 2$, by Lemma 3.1, the functional Φ satisfies the condition (C). Since Φ is weakly lower semi continuous and coercive on W^+, Φ is bounded from below on W^+. Moreover, by (ii) of Lemma 3.2, Φ is anti-coercive on W^-, thus we can apply the proposition 2.5 and we conclude that

$$C_\mu(\Phi, \infty) \neq 0$$

where $\mu = \dim W^- \geq k \geq 2$. It follows from the Morse inequality that Φ has a critical point u_1 with

$$C_\mu(\Phi, u_1) \neq 0.$$

Since $\mu \neq 1$ and $\mu \neq 0$, then the problem (1.1) has at least two nontrivial solutions. The proof of theorem is complete. □

References

[1] T. Bartsh, S. J. Li, *Critical point theory for asymptotically quadratic functionals and applications with resonance*, Nonlinear Analysis, T. M. A. 28, 419-441, 1997.

[2] G. Cerami, *Un criterio de esistenza per i punti critici su varietá ilimitate*, Rc. Ist. Lomb. Sci. Lett. 121, 332-336, 1978.

[3] K. C. Chang, *Infinite dimensional Morse theory and Multiple solutions problems*, Birkhäuser, Boston, 1993.

[4] D. G. Costa & A. S. Oliveira, *Existence of solution for a class of semilinear elliptic problems at double resonance*, Bol. Soc. Bras. Mat. 19. 21-37, 1988.

[5] D. G. DeFigueiredo & J. P. Gossez, *Conditions de non résonance pour certains problémes elliptiques semi-linéaires*, C. R. Acad. Sc. Paris, 302, pp. 543-545, (1986).

[6] C. L. Dolph, *Nonlinear integral equations of the Hammertein type*, Trans. Amer. Math. SOC. 66, pp. 289-307, (1949).

[7] A. R. El Amrouss, *Critical point theorems and applications to differential equations*, Acta Mathematica Sinica, English Series, 21 no 1, 129-142, 2005.

[8] A. R. El Amrouss, *Multiple solutions of a resonant semilinear elliptic problems*, to appear.

[9] A. R. El Amrouss & M. Moussaoui, *Resonance at two consecutive eigenvalues for semilinear elliptic problem: A variational approach*, Ann. Sci. Math. Qubec, 23 no 2, 157-171, 1999.

[10] J.P. Gossez & P. Omari, *Periodic solutions of a second order ordinary differential equation; a necessary and sufficient condition for nonresonance*, J. Diff. Equations 94 No. 1, 67-82, 1991.

[11] J. Mawhin, & M. Willem, *Critical point theory and Hamiltonien systems*, Springer-Verlag, New York, 1989.

[12] M. N. Nkashama, *Density condition at infinity and resonance in nonlinear elliptic partial differential equations*, Nonlinear Analysis, Theory, Methods & Applications, Vol, 22, No 3, pp. 251-265, 1994.

Abdel R. El Amrouss
University Mohamed 1er, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: amrouss@sciences.univ-oujda.ac.ma

Fouzia Moradi
University Mohamed 1er, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: foumoradi@yahoo.fr

Mimoun Moussaoui
University Mohamed 1er, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: moussaoui@sciences.univ-oujda.ac.ma