2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 155–162.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NOTE ON THE NODAL LINE OF THE P-LAPLACIAN

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Abstract. In this paper, we prove that the length of the nodal line of the eigenfunctions associated to the second eigenvalue of the problem

$$-\Delta_p u = \lambda\rho(x)|u|^{p-2}u \quad \text{in } \Omega$$

with the Dirichlet conditions is not bounded uniformly with respect to the weight.

1. Introduction

In this paper we consider the nonlinear elliptic boundary-value problem

$$\begin{gathered} -\Delta_p u = \lambda\rho(x)|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, Ω is a bounded and smooth domain in $\mathbb{R}^N$ $(1 < p < +\infty)$ and $\rho \in L^\infty(\Omega)$ is an indefinite weight such that

$$\operatorname{meas}(\Omega_\rho^+) \neq 0 \quad \text{with} \quad \Omega_\rho^+ = \{x \in \Omega \mid \rho(x) > 0\}.$$

Several authors have been studied the spectrum $\sigma(-\Delta_p)$ of p-Laplacian, precisely around of the first and the second eigenvalue. In particular Anane [1] proved that the spectrum $\sigma(-\Delta_p)$ contains a positive non-decreasing sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ such that $\lambda_n \to +\infty$ by using the Ljusternik-Schnirelmann, where

$$\lambda_n^{-1} = \lambda_n(\Omega,\rho)^{-1} = \sup_{K\in\mathcal{A}_n}\inf_{v\in K}\int_\Omega \rho(x)|v|^p dx$$

and

$$\mathcal{A}_n = \{K \subset W_0^{1,p}(\Omega) : K \text{ is symmetrical compact and } \gamma(K) \geq n\}.$$

Moreover, he showed that the first eigenvalue is simple and isolated, and that the first eigenfunction corresponding to λ_1 does not change the sign in Ω. In [2] they have showed that the second eigenvalue of the spectrum $\sigma(-\Delta_p)$ is exactly λ_2. The complete determination of this spectrum remains unanswered question. It is useful to announce that in the linear case $(p = 2)$, the spectrum is perfectly given [4, 6].

2000 *Mathematics Subject Classification.* 58E05, 35J65, 56J20.
Key words and phrases. Nonlinear eigenvalue problem; p-Laplacian; nodal line.

Published September 20, 2006.

Let us consider a solution (u, λ) of problem $P(\Omega, \rho)$. We denote by

$$\mathcal{Z}(u) = \{x \in \Omega : u(x) = 0\}$$

the nodal set of u, $\mathcal{N}(u)$ is the number of connected components of $\Omega\backslash\mathcal{Z}(u)$, $\mathcal{C}(u)$ is the set of connected components of $\Omega\backslash\mathcal{Z}(u)$,

$$\mathcal{N}(\lambda) = \max\{\mathcal{N}(u) \mid (u, \lambda) \text{ solution of } P(\Omega, \rho)\} \quad \text{and} \quad \mathcal{N}(\lambda_n) = \mathcal{N}(n).$$

Recently Cuesta, de Figueiredo and Gossez proved that $\mathcal{N}(2) = 2$ [3].

The main result in this paper is the generalization of the work of Kappeler and Ruf [5], in which they affirmed that the length of the nodal lines is not bounded uniformly with respect to the weights in dimension $N = 2$ and $p = 2$. In this work, we locate in the case $p > N$ and we prove that for all real number $L > 0$, there exists a weights $\rho \in L^\infty(\Omega)$ and an eigenfunction associated to the second eigenvalue of (1.1) such that the length of $\mathcal{Z}(u)$ is largest than L. The proof is relatively simpler than that given by Kappeler and Ruf; in which they use the uniform convergence of the gradient.

2. On the measure of nodal sets

In this section, we will extend the result of Kappeler and Ruf [5] in the case $p \neq 2$.

2.1. **Main result.** We consider the case $p > N$. Let Γ be a surface of class C^1 which subdivides Ω in two nodal components Ω_1 and Ω_2 such that

$$\mu_1 \leq \nu_1 \tag{2.1}$$

where μ_1 (respectively ν_1) is the first eigenvalue of $P(\Omega_1, 1)$ (respectively $P(\Omega_2, 1)$. Let v_1 (respectively w_1) the associated eigenfunction. For $n \in \mathbb{N}^*$, let

$$\Omega'_n = \{x \in \Omega_1 : \operatorname{dist}(x, \Gamma) < \frac{1}{n+1}\},$$
$$\Omega_n = \Omega_1\backslash\Omega'_n$$

where $\Gamma \subset \partial\Omega'_n$ of class C^1. Then, we denote by v_1^n the eigenfunction associated to μ_1^n the first eigenvalue of (1.1) with $\rho = 1$.

Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of decreasing positive real numbers such that

$$a_n = \frac{\mu_1^n}{\nu_1} \tag{2.2}$$

which tends to the limit $a \in \mathbb{R}_+^*$ $(0 < a = \frac{\mu_1}{\nu_1} \leq 1)$. Let $(\rho_n)_{n\in\mathbb{N}^*}$ be a sequence of weight functions defined by

$$\rho_n(x) = -r_n 1_{\Omega'_n}(x) + a_n 1_{\Omega_n}(x) + 1_{\Omega_2}(x), \tag{2.3}$$

for all $x \in \Omega$, where $r_n > 0$ such that $\lim_{n\to+\infty} \frac{c_n d_n}{r_n^{(p-1)/p^2}} = 0$ with c_n and d_n are strictly positive constants of immersion and interpolation.

Let us denote u_2^n the eigenfunction associated to the second eigenvalue λ_2^n of (1.1).

Theorem 2.1. *There exists a subsequence of $(u_2^n)_{n\in\mathbb{N}^*}$ still denoted by $(u_2^n)_{n\in\mathbb{N}^*}$ such that*

(i) *The sequence* $(u_2^n)_{n\in\mathbb{N}^*}$ *converges weakly to* $(\alpha\overline{v_1}+\beta\overline{w_1})$ *in* $W_0^{1,p}(\Omega)$, *for some scalars* α , β *not all null, where* $\overline{v_1}$ *(respectively* $\overline{w_1}$ *) is the extension of* v_1 *(respectively* w_1*) by zero in* $\overline{\Omega}$.

(ii) *If* U *and* V *are two opens of* $\mathbb{R}^N$ *such that* $\overline{U}\subset\Omega_1$ *and* $\overline{V}\subset\Omega_2$. *Then for* n *enough large, we have*

$$\overline{U}\cap\mathcal{Z}(u_2^n)=\overline{V}\cap\mathcal{Z}(u_2^n)=\emptyset$$

and u_2^n *change the sign on* $U\cup V$

To prove this result, we need the following preliminary lemmas.

Lemma 2.2. *The following inequalities are true independently of* $(r_n)_{n\in\mathbb{N}^*}$*:*

(i) $0<b^{-1}\lambda_2(\Omega,1)\le\lambda_2^n\le\nu_1$, *where* $b=\begin{cases}1 & \text{if } \mu_1<\nu_1\\ a_1 & \text{if } \mu_1=\nu_1,\end{cases}$

(ii) $\|\Delta_p u_2^n\|_{L^{p'}(\Omega_2)}^{p'}\le M\nu_1^{p'}$,

(iii) $\|\Delta_p u_2^n\|_{L^{p'}(\Omega_n)}^{p'}\le M(a_1\nu_1)^{p'}$, *where* M *is the Sobolev-Poincaré constant.*

Proof. (i) Let F_2 be the vector subspace of $W_0^{1,p}(\Omega)$ spanned by $\{\overline{v_1^n},\overline{w_1}\}$, where $\overline{v_1^n}$ (resp. $\overline{w_1}$) is the extension by zero of v_1^n (resp. w_1) in $\Omega\setminus\Omega_n$ (resp. $\Omega\setminus\Omega_1$). Let S_2 denote the unit sphere of F_2. For all $v\in S_2$ such that $v=\alpha\overline{v_1^n}+\beta\overline{w_1}$, we have

$$|\alpha|^p+|\beta|^p=1\quad\text{and}\quad\int_\Omega\rho_n(x)|v(x)|^p\,dx=|\alpha|^p a_n\frac{1}{\mu_1^n}+|\beta|^p\nu_1^{-1}.$$

Using (2.3), we get

$$\int_\Omega\rho_n(x)|v(x)|^p\,dx=\frac{1}{\nu_1}$$

In particular

$$\frac{1}{\nu_1}\le\inf_{v\in S_2}\{\int_\Omega\rho_n(x)v(x)dx\}\le\frac{1}{\lambda_2^n}.$$

Since $\rho_n(x)\le b$,

$$\frac{\lambda_2(\Omega,b)}{b}=\lambda_2(\Omega,b)\le\lambda_2^n(\Omega,\rho_n)=\lambda_2^n$$

(ii) It is sufficient to notice that

$$-\Delta_p u_2^n=\lambda_2^n|u_2^n|^{p-2}u_2^n\quad\text{a.e. on }\Omega_2$$

and by using the Sobolev-Poincaré inequality, we have

$$\int_{\Omega_2}|-\Delta_p u_2^n|^{p'}\le M(\lambda_2^n)^{p'}\|\nabla u_2^n\|_{L^p(\Omega)}^p\le M\nu_1^{p'}.$$

(iii) Using $-\Delta_p u_2^n=\lambda_2^n a_n|u_2^n|^{p-2}u_2^n$ a.e. on Ω_n and with the same argument as above, one gets

$$\int_{\Omega_n}|-\Delta_p u_2^n|^{p'}\le M(a_1\nu_1)^{p'}.$$

□

Remark 2.1. (1) *By the lemma 2.2 we can choose the sequence* $(\rho)_{n\in\mathbb{N}^*}$ *such that* $(\lambda_2^n)_{n\in\mathbb{N}^*}$ *converges to the positive limit* λ_2. *For all* $p>N$, u_2^n *converges weakly in* $W_0^{1,p}(\Omega)$ *and strongly in* $C(\overline{\Omega})$ *to* $u_2\in W_0^{1,p}(\Omega)$.

(2) $\rho_n(x)\le b$ *for all* $x\in\Omega$.

(3) $u_2 \neq 0$ *in* $L^p(\Omega)$.

Lemma 2.3. *With the above notation,* $\lambda_2 = \nu_1$

Proof. To prove this lemma, we proceed in three steps:
First step: We show that

$$-\Delta_p u_2 = \lambda_2 a|u_2|^{p-2}u_2 \quad \text{a.e. on } \Omega_1 \tag{2.4}$$

Indeed; for $m \geq 1$ and by the lemma 2.2 , we have

$$\|\Delta_p u_2^n\|_{L^{p'}(\Omega_m)}^{p'} \leq \|\Delta_p u_2^n\|_{L^{p'}(\Omega_n)}^{p'} \leq M(a_1\nu_1)^{p'} \quad \forall n \geq m$$

hence

$$\|\Delta_p u_2^n\|_{(W^{1,p}(\Omega_m))'}^{p'} \leq M(a_1\nu_1)^{p'} \quad \forall n \geq m.$$

It follows that there exists a subsequence, still denoted by $(-\Delta_p u_2^n)$, such that $-\Delta_p u_2^n \rightharpoonup T_m$ weakly in the sapce $(W^{1,p}(\Omega_m))'$. By remark 2.1, $u_2^n \rightharpoonup u_2$ weakly in $W^{1,p}(\Omega_m)$. Since

$$-\Delta_p u_2^n = \lambda_2^n a_n |u_2^n|^{p-2}u_2^n \quad \text{a.e. on } \Omega_m \text{ for all } n \geq m,$$

we have

$$\lim_{n\to+\infty}\langle -\Delta_p u_2^n, u_2^n\rangle_m = \langle T_m, u_2\rangle_m$$

where $\langle\cdot,\cdot\rangle_m$ is the duality bracket between $W^{1,p}(\Omega_m)$ and its dual $(W^{1,p}(\Omega_m))'$. However, $-\Delta_p$ is an operator of type (M), consequently $T_m = -\Delta_p u_2$ is in the space $(W^{1,p}(\Omega_m))'$. We deduce that

$$-\Delta_p u_2 = \lambda_2 a|u_2|^{p-2}u_2 \quad \text{a.e. on } \Omega_m \ \forall m \geq 1$$

hence (2.4). Similarly, we prove that

$$-\Delta_p u_2 = \lambda_2 |u_2|^{p-2}u_2 \quad \text{a.e. on } \Omega_2 \tag{2.5}$$

Second step: We show that

$$u_{2_{/\partial\Omega_i}} = 0 \quad \text{in } L^p(\partial\Omega_i) \text{ for } i = 1, 2.$$

Indeed, it follows from (2.2) and since $\partial\Omega_n'$ is of C^1 and $(\Omega \cap \partial\Omega_2) \subset \partial\Omega_n'$, we have

$$\|u_{2_{/\Omega\cap\partial\Omega_2}}^n\|_{L^p(\Omega\cap\partial\Omega_2)} \leq c_n d_n \|u_2^n\|_{W^{1,p}(\Omega_n')}^{\sigma} \|u_2^n\|_{L^p(\Omega_n')}^{1-\sigma}, \tag{2.6}$$

where c_n is the constant of the immersion $W^{\sigma,p}(\Omega_n') \hookrightarrow L^p(\partial\Omega_n')$; $\sigma = \frac{1}{p}$ and d_n is the constant of the interpolation of the inequality

$$\|u\|_{W^{\sigma,p}(\Omega_n')} \leq d_n \|u\|_{W^{1,p}(\Omega_n')}^{\sigma} \|u\|_{L^p(\Omega_n')}^{1-\sigma} \quad \text{for all } \ u \in W^{\sigma,p}(\Omega_n').$$

The two norms of the second member in (2.6) can be estimated as follows:

$$\|u_2^n\|_{W^{1,p}(\Omega_n')}^p \leq \|u_2^n\|_{L^p(\Omega_n')}^p + \|\nabla u_2^n\|_{L^p(\Omega_n')}^p \leq M + 1, \tag{2.7}$$

where M is the constant of the Sobolev-Poincaré of lemma 2.2. Moreover, since (u_2^n, λ_2^n) is a solution of $P(\Omega, \rho_n)$, by (2.3) and lemma 2.2, we get

$$r_n \int_{\Omega_n'} |u_2^n|^p dx \leq \int_{\Omega_2} |u_2^n|^p dx + a_n \int_{\Omega_n} |u_2^n|^p dx\,.$$

Since $b \geq 1$, we deduce that

$$\int_{\Omega_n'} |u_2^n|^p dx \leq \frac{b}{r_n} \int_{\Omega} |u_2^n|^p dx \leq \frac{b}{r_n} M. \tag{2.8}$$

Thus, by (2.6), (2.7) and (2.8), we have

$$\|u^n_{2/\Omega\cap\partial\Omega_2}\|_{L^p(\Omega\cap\partial\Omega_2)} \le c_n d_n (M+1)^{\frac{\sigma}{p}} (\frac{bM}{r_n})^{\frac{1-\sigma}{p}}.$$

However, $\lim_{n\to+\infty} \frac{c_n d_n}{r_n^{(1-\sigma)/p}} = 0$ and $u_2 \in W^{1,p}_0(\Omega)$, consequently $u_{2/\partial\Omega_2} = 0$ in $L^p(\partial\Omega_2)$. Similarly, we have $u_{2/\partial\Omega_1} = 0$ in $L^p(\partial\Omega_1)$ because $(\Omega\cap\partial\Omega_1) \subset \partial\Omega'_n$.

Third step: We establish that $\lambda_2 = \nu_1 = \eta_1$. Indeed, since $u_2 = 0$ in $L^p(\partial\Omega_1)$ and $L^p(\partial\Omega_2)$ with $u_2 \in W^{1,p}_0(\Omega)$, we have $u_2 \in W^{1,p}_0(\Omega_1)$ and $u_2 \in W^{1,p}_0(\Omega_2)$. Moreover, if we use (2.4), (2.5) and the remark 2.1, then $(u_{2/\Omega_1}, \lambda_2)$ and $(u_{2/\Omega_2}, \lambda_2)$ are respectively solutions of problems (1.1) with $\Omega = \Omega_1$ and with $\Omega = \Omega_2$. We have by lemma 2.2,

$$\lambda_2 = \lim_{n\to+\infty} \lambda^n_2 \le \nu_1,$$

where ν_1 is the first eigenvalue of (1.1) with $\Omega = \Omega_2$. We conclude that $\lambda_2 = \nu_1 = \eta_1$.

Lemma 2.4. *The sequence $(f_n)_{n\in\mathbb{N}^*}$ admits a subsequence which converges weakly in $W^{1,p}_0(\Omega)$ to $\overline{v_1}$, where*

$$f_n = \frac{(u^n_2)^+}{(\frac{a}{p})^{1/p}\|(u^n_2)^+\|_{L^p(\Omega)}}.$$

with $(u^n_2)^+ = max\{0, u^n_2\}$.

Proof. It is known that

$$\int_\Omega |\nabla(u^n_2)^+|^p dx = \lambda^n_2 \int_\Omega \rho_n |(u^n_2)^+|^p dx\,.$$

If we multiply by $\big((\frac{a}{p})^{1/p}\|(u^n_2)^+\|^p_{L^p(\Omega)}\big)^{-p}$, then

$$\int_\Omega |\nabla f_n|^p dx = \lambda^n_2 \int_\Omega \rho_n |f_n|^p dx \le \lambda^n_2 b \int_\Omega |f_n|^p dx = b\lambda^n_2 \frac{p}{a}. \tag{2.9}$$

Thus, by lemmas 2.2 and 2.3, we have

$$\int_\Omega |\nabla f_n|^p dx \le \frac{p}{a} b\lambda_2 < +\infty \quad \forall n \in \mathbb{N}^*.$$

So, for a subsequence of the sequence $(f_n)_{n\in\mathbb{N}^*}$, still denoted $(f_n)_{n\in\mathbb{N}^*}$, we have $f_n \rightharpoonup f$ weakly in $W^{1,p}_0(\Omega)$, then $f_n \to f$ strongly in $C(\overline{\Omega})$ with $p > N$. Since $\|f_n\|_{L^p(\Omega)} = (\frac{p}{a})^{1/p}$ then $\|f\|_{L^p(\Omega)} \ne 0$. Hence $f \ne 0$ on Ω, since $u_{2/\Omega_2} = \beta w_1 < 0$,

$$f = 0 \quad \text{on } \Omega_2 \tag{2.10}$$

a fortiori $f = 0$ on $\partial\Omega_2$ and $f = 0$ on $\partial\Omega_1$. It results that $f \in W^{1,p}_0(\Omega_1)$. According to (2.9), we have

$$\begin{aligned}\int_\Omega |\nabla f_n|^p dx &= \lambda^n_2 \Big(a_n \int_\Omega |f_n|^p dx - r_n \int_{\Omega'_n} |f_n|^p dx + \int_{\Omega_2} |f_n|^p dx\Big)\\ &\le \lambda^n_2 \Big(a_n \int_{\Omega_n} |f_n|^p dx + \int_{\Omega_2} |f_n|^p dx\Big).\end{aligned}$$

Hence, $\liminf_{n\to+\infty}\int_\Omega |\nabla f_n|^p dx \le \lambda_2\big(a\int_{\Omega_1}|f|^p dx+\int_{\Omega_2}|f|^p dx\big)$. From (2.10), we deduce that

$$\lim_{n\to+\infty}\inf\int_\Omega |\nabla f_n|^p dx \le \lambda_2 a\int_\Omega |f|^p\, dx.$$

Thus

$$\int_{\Omega_1}|\nabla f|^p dx \le \lambda_2 a\int_\Omega |f|^p dx = \lambda_2 p.$$

We have $\lambda_2=\eta_1$ being the first eigenvalue of (1.1) with $\Omega=\Omega_1$ and $\rho=a$; consequently

$$\lambda_2=\frac{1}{p}\int_{\Omega_1}|\nabla f|^p dx \quad \text{and} \quad f=\overline{v_1}.$$

□

2.2. **Proof of the main result.** (i) From lemma 2.3, u_2 is an eigenfunction associated to ν_1 (resp. η_1). So, there exists $\alpha,\beta\in\mathbb{R}^n$ such that

$$u_2=\alpha v_1+\beta w_1 \quad \text{with } |\alpha|^p+|\beta|^p>0.$$

(ii) We distinguish two possible cases:
First case: If $\alpha\neq 0$ and $\beta\neq 0$, we can assume that $\alpha>0$ and $\beta<0$ (the other cases will be treated in the same way). As $u_2=\alpha v_1$ on Ω_1 and v_1 is a positive eigenfunction of class C^1 on Ω_1, then $\exists x_0\in\overline{U}$ such that $\min\{u_2(x):x\in\overline{U}\}=u_2(x_0)>0$. By lemma 2.2, u_2^n converges uniformly ($p>N$) to u_2 in $\overline{\Omega}$, consequently for $\epsilon=u_2(x_0)>0$ there exists $n_0(\overline{U})\in\mathbb{N}$ such that for all $n\ge n_0(\overline{U})$, we have

$$u_2^n(x)>\frac{\epsilon}{2} \quad \forall x\in\overline{U}$$

i.e. $(\overline{U}\cap\mathcal{Z}(u_2^n))=\emptyset$ for all $n\ge n_0(\overline{U})$. It is the same for $(\overline{V}\cap\mathcal{Z}(u_2^n))=\emptyset$ for all $n\ge n_0(\overline{V})$. We announce here that according to the lemma 2.2 the case where $\alpha\beta>0$ does not intervene.
Second case: If $\alpha=0$ or $\beta=0$. We consider now the case where $\alpha=0$ and $\beta<0$. The other cases will be treated in the same way. By lemma 2.4, there exists a subsequence, still denoted $(f_n)_{n\in\mathbb{N}}$, which converges uniformly to $f=\overline{v_1}$ in $\overline{\Omega}$. Moreover $\overline{v_1}>0$ in $\overline{U}$, and there exists $x_0\in\overline{U}$ such that

$$f(x_0)=\min\{f(x)=\overline{v_1}(x):x\in\overline{U}\}>0.$$

Thus, for $\epsilon=f(x_0)>0$, there exists $n_0(\overline{U})\in\mathbb{N}^*$ such that for all $n\ge n_0(\overline{U})$ we have

$$f_n(x)>\frac{\epsilon}{2} \quad \forall x\in\overline{U}.$$

i.e for all $n\ge n_0(\overline{U})$, $\overline{U}\cap\mathcal{Z}(u_2^n)=\emptyset$. Therefore, since $\beta<0$, it is the same for $\overline{V}\cap\mathcal{Z}(u_2^n)=\emptyset$ for all $n\ge n_0(\overline{V})$.

We remark here that by (i) of the lemma 2.2 the case where $\alpha=\beta=0$ does not intervene. □

2.3. **Consequences of the main result.**

Corollary 2.5. *If Γ is a surface of class C^1 in Ω which subdivides Ω in two connected components, then for all neighborhood $[\Gamma]_\epsilon$ of Γ, there exists a weight $\rho_\epsilon \in L^\infty(\Omega)$ and an eigenfunction u associated to the second eigenvalue of $P(\Omega, \rho_\epsilon)$ such that $\mathcal{Z}(u) \subset [\Gamma]_\epsilon$; where $[\Gamma]_\epsilon = \{x \in \Omega : d(x, \Gamma) \le \epsilon\}$.*

Proof. We distinguish two cases
First case: $\overline{\Gamma} \cap \partial\Omega = \emptyset$. Let $\varepsilon > 0$, we consider $U = \Omega_1 \setminus ([\partial\Omega]_\epsilon \cup [\Gamma]_\epsilon)$ and $V = \Omega_2 \setminus ([\partial\Omega]_\epsilon \cup [\Gamma]_\epsilon)$, where $[\partial\Omega]_\epsilon = \{x \in \Omega : d(x, \Omega) \le \epsilon\}$. Since $\overline{\Gamma} \cap \partial\Omega = \emptyset$, we can choose ϵ enough small, so that $[\partial\Omega]_\epsilon \cap [\Gamma]_\epsilon = \emptyset$. By theorem 2.1, there exists $n \in \mathbb{N}^*$ such that $\mathcal{Z}(u_2^n) \subset [\partial\Omega]_\epsilon \cup [\Gamma]_\epsilon$. We assume that $\mathcal{Z}(u_2^n) \cap [\partial\Omega]_\epsilon \neq \emptyset$ then there exists a nodal component D_ϵ of u_2^n included in $[\partial\Omega]_\epsilon$. Thus $(u^n_{2/D_\epsilon}, \lambda_2^n)$ is a solution of the problem $P(D_\epsilon, \rho_{n/D_\epsilon})$ with λ_2^n its first eigenvalue [1, 7]. By the remark 2.1, we have

$$\lambda_2^n = \lambda_1(D_\epsilon, \rho_{n/D_\epsilon}) \ge \lambda_1(D_\epsilon, b)$$

we have $\text{meas}(D_\epsilon) \to 0$ when $\epsilon \to 0$, consequently $\lambda_2^n = \lambda_1(D_\epsilon, \rho_n) \to +\infty$ when $\epsilon \to 0$ which is absurd with lemma 2.2. So $\mathcal{Z}(u_2^n) \subset [\Gamma]_\epsilon$.
Second case: $\overline{\Gamma} \cap \partial\Omega \neq \emptyset$. Let $\epsilon > 0$, there exists a surface Γ'_ϵ of C^1 which subdivide Ω in two connected components such that

$$\Gamma'_\epsilon \subset [\Gamma]_\epsilon \quad \text{and} \quad \overline{\Gamma'_\epsilon} \cap \partial\Omega = \emptyset$$

Let $\eta > 0$ (enough small) so that $[\Gamma'_\epsilon]_\eta \subset [\Gamma]_\varepsilon$ and $[\partial\Omega]_\eta \cap [\Gamma'_\epsilon]_\eta = \emptyset$, finally we conclude the result by applying the proof of the first case with Γ'_ϵ. ☐

Remark 2.2. The result of the corollary 2.5 remains true even if Γ is not of class C^1, only it is enough to approach Γ by a surface Γ' of class C^1 which located in $[\Gamma]_\epsilon$.

Corollary 2.6. *For all $L > 0$, there exists $\rho \in L^\infty(\Omega)$ and an eigenfunction u associated to the second eigenvalue of $P(\Omega, \rho)$ such that the length of $\mathcal{Z}(u)$ is larger than L.*

Proof. Let $L > 0$, there exists a surface Γ is of class C^1 in Ω which subdivide Ω in two connected components such that

$$\overline{\Gamma} \cap \partial\Omega = \emptyset \quad \text{and} \quad \text{meas}(\Gamma) > L + 1.$$

For $\epsilon > 0$ (enough small) we consider $[\Gamma]_\epsilon$ and $[\partial\Omega]_\epsilon$ two neighborhood of Γ and $\partial\Omega$ respectively such that

$$[\partial\Omega]_\epsilon \cap [\Gamma]_\epsilon = \emptyset$$

Denote by $U = \Omega_1 \setminus ([\partial\Omega]_\epsilon \cup [\Gamma]_\epsilon)$ and $V = \Omega_2 \setminus ([\partial\Omega]_\epsilon \cup [\Gamma]_\epsilon)$ two open. In virtue of the Theorem 2.1 and of the Corollary 2.5, $\exists n \in \mathbb{N}^*$ such that

$$U \cap \mathcal{Z}(u_2^n) = V \cap \mathcal{Z}(u_2^n) = \emptyset \quad \text{and} \quad \mathcal{Z}(u_2^n) \subset [\Gamma]_\epsilon.$$

Let us suppose that for an infinity of $\epsilon > 0$, u_2^n admits a nodal component D_ϵ included in $[\Gamma]_\epsilon$. So

$$\lambda_2^n = \lambda_1(D_\epsilon, \rho_{n/D_\epsilon}) \ge \lambda_1(D_\epsilon, b).$$

Since $\lim_{\epsilon\to 0} \text{meas}(D_\epsilon) = 0$, it follows that $\lambda_2^n \ge \lim_{\epsilon\to 0} \lambda_1(D_\epsilon, b) = +\infty$ which is absurd with the lemma 2.2 . Thus, for ϵ enough small, there exists $n \in \mathbb{N}$ such that $\mathcal{Z}(u_2^n)$ is a closed surface in $[\Gamma]_\epsilon$ with $\mathcal{Z}(u_2^n) = \partial W$ where W is an open containing Ω_i^ϵ which is an open included in Ω_i such that $\partial\Omega_i^\epsilon \subset \partial[\Gamma]_\epsilon$. So if $\text{meas}(\Gamma) > L + 1$, then $\exists\epsilon > 0$ (enough small) and $\exists n \in \mathbb{N}^*$ such that $\text{meas}(\mathcal{Z}(u_2^n) > L$ for $i = 1, 2$. ☐

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