2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 109–117.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NONRESONANCE CONDITIONS FOR A SEMILINEAR WAVE EQUATION IN ONE SPACE DIMENSION

AOMAR ANANE, OMAR CHAKRONE, ABDELLAH ZEROUALI

Abstract. In this paper we study the existence of periodic weak solutions for semilinear wave equations in one space dimension in the case of nonresonance.

1. Introduction

In this paper we consider the existence of periodic solutions for the wave equation

$$\begin{gathered}\Box u = \alpha u + \beta u_x - \gamma u_t + g(x,t,u) + h(x,t) \quad \text{in } Q,\\ u(x,t+2\pi) = u(x,t) \quad \text{in }]0,\pi[\times\mathbb{R},\\ u(0,t) = u(\pi,t) = 0 \quad \forall t\in\mathbb{R},\end{gathered} \tag{1.1}$$

where $Q =]0,\pi[\times]0,2\pi[$, $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is the D'Alembertian, $(\alpha,\beta,\gamma)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, h is a given function in $L^2(Q)$, and $g:]0,\pi[\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is 2π-periodic in t and a Carathéodory function (i.e. measurable in (x,t) for each $s\in\mathbb{R}$ and continuous in s for almost all $(x,t)\in Q$).

We are interested in the nonresonance for the problem (1.1) (i.e. in the condition for the function g such that there exist a solution $u\in L^2(Q)$ for any given $h\in L^2(Q)$). We will assume that g satisfies the following conditions:

(C1) $g(x,t,s)$ is nondecreasing in s;
(C2) for $s\neq r$, $(x,t)\in Q$, we have

$$e^{\beta\frac{x}{2}}\Big(\frac{g(x,t,r)-g(x,t,s)}{r-s}\Big) \geq \frac{\beta^2}{4} - \alpha;$$

(C3) for all $R>0$, there exists $\phi_R\in L^2(Q)$ such that a.e. $(x,t)\in Q$,

$$\max_{|s|\leq R}|g(x,t,s)| \leq \phi_R(x,t);$$

2000 *Mathematics Subject Classification.* 35J65, 35J25.
Key words and phrases. D'Alembertian; homotopy argument; eigenvalues.

Published September 20, 2006.

(C4) a.e. $(x,t) \in Q$, we have
$$\begin{aligned} \lambda_k - \frac{\gamma^2}{4} + \frac{\beta^2}{4} - \alpha < l(x,t) &:= \liminf_{|s|\to+\infty} \frac{g(x,t,s)}{s} \\ &\le \limsup_{|s|\to+\infty} \frac{g(x,t,s)}{s} := k(x,t) \\ &< \lambda_{k+1} - \frac{\gamma^2}{4} + \frac{\beta^2}{4} - \alpha, \end{aligned}$$
where λ_k and λ_{k+1} are two consecutive eigenvalues of the D'Alembertian, and $\sigma(\Box)$ denotes the spectrum of the D'Alembertian.

Problem (1.1) has been studied with conditions of resonances by several authors mention in particular: In the case $\alpha = \beta = \gamma = 0$ and $h = 0$ Benaoum in [3, 4], Mustonen and Berkovits in [5, 6, 7, 8], and Brézis and Nirenberg in [12]. The case $g(x,t,s) = g(s)$, has been studied by Mustonen and Berkovits in [9]. The case $\beta = \gamma = 0$ and α is a eigenvalue of the D'Alembertian operator ($\Box$), has been studied in [7]. The case $\beta = 0$ and α is a eigenvalue of the operator T defined by $Tu = \Box u + \gamma u_t$, where $u_t = \frac{\partial u}{\partial t}$, has been studied in [5, 12]. In the general case, Anane, Chakrone and Ghanim in [2]. In the case of nonresonance, the problem (1.1) has been studied by Mustonen and Berkovits in [6] and [10], and by Brezis and Nirenberg in [12] but only in particular cases. The situation that we consider here is marked by the presence of one term of transportation $\beta\nabla u$, what constitutes an extension of the cases studied by Mustonen and Berkovits in [6, 10]. In our work, we show (see Corollary 3.2) while using homotopy argument given by Mustonen and Berkovits in [6], and of analogous techniques developed by Anane and Chakrone in [1] for the Laplacian (Δ), that the problem (1.1) has at least a solution for all $h \in L^2(Q)$.

2. Remarks and notation

Let $\delta, \mu \in \mathbb{R}$ such that $\delta < \mu$, we introduce the following general hypothesis For a.e. $(x,t) \in Q$, we have
$$\begin{aligned} \delta + \frac{\beta^2}{4} - \alpha \le\neq l(x,t) &:= \liminf_{|s|\to+\infty} \frac{g(x,t,s)}{s} \\ &\le \limsup_{|s|\to+\infty} \frac{g(x,t,s)}{s} := k(x,t) \\ &\le\neq \mu + \frac{\beta^2}{4} - \alpha \end{aligned} \tag{2.1}$$
The notation $\le\neq$ means that one has an large inequality on Q and strict on a set of measure different from zero.

Remark 2.1. (1) We denote by $Tu = \Box u + \gamma u_t$. Then

(i) T is a densely defined closed linear operator with closed range.
(ii) $\mathrm{Im}(T) = [\ker(T)]^{\perp}$.
(ii) λ is a eigenvalue of the D'Alembertian if and only if $\lambda - \frac{\gamma^2}{4}$ is a eigenvalue of T
(iii) If T_0 is the restriction of the operator T on $\mathrm{Im}(T) = T(D(T))$, with $D(T)$ is the domain of the operator T, then T_0 has compact inverse.

For the proof of the remarks (i)–(iii), see [4].
(2) We put $\tilde{g}(x,t,s) = (\alpha - \frac{\beta^2}{4})s + e^{\frac{\beta}{2}x} g(x,t,e^{-\frac{\beta}{2}x}s)$ and $\tilde{h}(x,t) = e^{\frac{\beta}{2}x}h(x,t)$. Let $N : L^2(Q) \to L^2(Q)$,

$$N(u) = \tilde{g}(x,t,u)$$

be the Nemytskii operator generated by the function $\tilde{g}$. For $r \in [0,1]$, consider the operator $H_r : D(T) \subset L^2(Q) \to L^2(Q)$,

$$H_r(u) = Tu - r(N(u) + \tilde{h}) - (1-r)\lambda u,$$

where $\delta < \lambda < \mu$.

$$\begin{gathered}\text{If there exists } R > 0, \text{ for all } r \in [0,1] \text{ and all } u \in D(T),\\ \text{with } \|u\| = \Big(\int_Q |u|^2\Big)^{1/2} = R, \text{ then } H_r(u) \neq 0.\end{gathered} \tag{2.2}$$

(3) If (C1) and (C2) are verified, then $\tilde{g}(x,t,s)$ is nondecreasing in s, thus the operator N is monotone. This statement and the following are easy to prove.
(4) Condition (C3) implies that for all $R > 0$ there exists $\tilde{\phi_R} \in L^2(Q)$ such that for a.e. $(x,t) \in Q$ we have

$$\max_{|s| \le R} |\tilde{g}(x,t,s)| \le \tilde{\phi_R}(x,t).$$

(5) If (C4) is verified, then for a.e. $(x,t) \in Q$, we have

$$\lambda_k - \frac{\gamma^2}{4} < \tilde{l}(x,t) := \liminf_{|s|\to+\infty} \frac{\tilde{g}(x,t,s)}{s} \le \limsup_{|s|\to+\infty} \frac{\tilde{g}(x,t,s)}{s} := \tilde{k}(x,t) < \lambda_{k+1} - \frac{\gamma^2}{4}$$

(6) If (2.1) is satisfied, then for a.e. $(x,t) \in Q$, we have

$$\delta \le\neq \tilde{l}(x,t) := \liminf_{|s|\to+\infty} \frac{\tilde{g}(x,t,s)}{s} \le \limsup_{|s|\to+\infty} \frac{\tilde{g}(x,t,s)}{s} := \tilde{k}(x,t) \le\neq \mu$$

i.e. for all $\varepsilon > 0$ there exists $a_\varepsilon \in L^2(Q)$ such that for a.e. $(x,t) \in Q$, and all $s \in \mathbb{R}$, we have

$$(\tilde{l}(x,t) - \varepsilon)s^2 - a_\varepsilon(x,t)|s| \le s\tilde{g}(x,t,s) \le (\tilde{k}(x,t) + \varepsilon)s^2 + a_\varepsilon(x,t)|s|.$$

(7) Under hypothesis (C3) and (2.1), there exists $\theta > 0$ and $\eta \in L^2(Q)$ such that a.e. $(x,t) \in Q$, and all $s \in \mathbb{R}$, we have

$$|\tilde{g}(x,t,s)| \le \theta|s| + \eta(x,t). \tag{2.3}$$

Proposition 2.2. *The problem* (1.1) *is equivalent to the problem*

$$\begin{gathered}Tv = \tilde{g}(x,t,v) + \tilde{h}(x,t) \quad \text{in } Q,\\ v(x,t+2\pi) = v(x,t) \quad \text{in }]0,\pi[\times\mathbb{R},\\ v(0,t) = v(\pi,t) = 0 \quad \forall t \in \mathbb{R},\end{gathered} \tag{2.4}$$

Proof. Assume that u is a solution of the problem (1.1). Let $v = e^{\frac{\beta}{2}x}u$, it is clear that v is 2π-periodic in t and $v(0,t) = v(\pi,t) = 0$ $\forall t \in \mathbb{R}$. On the other hand, we have

$$\begin{gathered}
v_x = \frac{\partial v}{\partial x} = \frac{\beta}{2} e^{\frac{\beta}{2}x} u + e^{\frac{\beta}{2}x} u_x, \\
v_{xx} = \frac{\partial^2 v}{\partial x^2} = \beta e^{\frac{\beta}{2}x} u_x + \frac{\beta^2}{4} e^{\frac{\beta}{2}x} u + e^{\frac{\beta}{2}x} u_{xx}, \\
v_t = \frac{\partial v}{\partial t} = e^{\frac{\beta}{2}x} u_t, \quad v_{tt} = \frac{\partial^2 v}{\partial t^2} = e^{\frac{\beta}{2}x} u_{tt};
\end{gathered}$$

thus

$$\begin{aligned}
\Box v &= v_{tt} - v_{xx} \\
&= e^{\frac{\beta}{2}x} u_{tt} - \beta e^{\frac{\beta}{2}x} u_x - \frac{\beta^2}{4} e^{\frac{\beta}{2}x} u - e^{\frac{\beta}{2}x} u_{xx} \\
&= -\frac{\beta^2}{4} v + e^{\frac{\beta}{2}x} (\Box u - \beta u_x) \\
&= (\alpha - \frac{\beta^2}{4}) v - \gamma v_t + e^{\frac{\beta}{2}x} (g(x,t,e^{-\frac{\beta}{2}x} v) + h(x,t)).
\end{aligned}$$

Hence $Tv = \tilde{g}(x,t,v) + \tilde{h}(x,t)$, and v is a solution of the problem (2.4). The reciprocal implication is demonstrated by an identical calculation. $\square$

3. Main results

Theorem 3.1. *Assume (C1), (C2), (C3) and* (2.1). *If* (H_r) *does not satisfy* (2.2), *then there exists* $m(x,t) \in L^\infty(Q)$, $v \in L^2(Q) \setminus \{0\}$ *and* $(u_n) \subset L^2(Q)$ *such that* v *is the nontrivial solution of the problem*

$$\begin{gathered}
Tu = m(x,t) u \quad \text{in } Q, \\
u(x, t + 2\pi) = u(x,t) \quad \text{in }]0,\pi[\times\mathbb{R}, \\
u(0,t) = u(\pi,t) = 0 \quad \forall t \in \mathbb{R}
\end{gathered} \tag{3.1}$$

and

$$\begin{gathered}
\|u_n\| \to +\infty, \quad \frac{u_n}{\|u_n\|} \to v \quad \text{in } L^2(Q), \\
\delta \leq\neq m(x,t) \leq\neq \mu \quad \text{a.e. in } Q.
\end{gathered}$$

Corollary 3.2. *Assume (C1), (C2), (C3) and* (2.1). *If there exist two consecutive eigenvalues of the D'Alembertian* λ_k *and* λ_{k+1} *such that* $0 \leq \lambda_k - \frac{\gamma^2}{4} < \delta < \mu < \lambda_{k+1} - \frac{\gamma^2}{4}$, *then problem* (1.1) *has at least one solution for all* $h \in L^2(Q)$.

Proof of theorem 3.1. As the proof is relatively long, we organize it in several lemmas. Suppose that (H_r) does not satisfy the estimate (2.2), then $\forall n \in \mathbb{N}$, there exist $r_n \in [0,1]$, and $u_n \in D(T)$ with $\|u_n\| = n$ such that

$$Tu_n - r_n(N(u_n) + \tilde{h}) - (1 - r_n)\lambda u_n = 0 \tag{3.2}$$

Let

$$v_n = \frac{u_n}{\|u_n\|}, \quad g_n(x,t) = \frac{\tilde{g}(x,t,u_n)}{\|u_n\|} \quad \text{a.e. in } Q.$$

The sequence (v_n) is bounded in $L^2(Q)$, then for subsequence $v_n \to v$ weakly in $L^2(Q)$.

Lemma 3.3. *Assume* (2.3) *and* (3.2). *(1) For a subsequence* $g_n \to f$ *weakly in* $L^2(Q)$. *(2)* $v_n \to v$ *strongly in* $L^2(Q)$, *in particular,* $\|v\| = 1$, *thus* $v \neq 0$.

Proof. (1) Dividing (2.3) by $\|u_n\|$, we have

$$|g_n(x,t)| \le \theta |v_n| + \frac{\eta(x,t)}{n},$$

thus

$$\|g_n\| \le \theta \|v_n\| + \frac{\|\eta\|}{n} \le \theta + \frac{\|\eta\|}{n},$$

hence g_n is bounded in $L^2(Q)$, one deduces that for a subsequence $g_n \to f$ weakly in $L^2(Q)$.
(2) Dividing by $\|u_n\|$ in (3.2), we have

$$Tv_n = r_n g_n + (1 - r_n)\lambda v_n + r_n \frac{\tilde{h}}{n}.$$

Which implies

$$v_n = (T_0^{-1})[r_n g_n + (1 - r_n)\lambda v_n + r_n \frac{\tilde{h}}{n}].$$

Since $g_n \to f$ weakly in $L^2(Q)$ and $v_n \to v$ weakly in $L^2(Q)$, then

$$r_n g_n + (1 - r_n)\lambda v_n + r_n \frac{\tilde{h}}{n} \to rf + (1-r)\lambda v \quad \text{weakly in } L^2(Q),$$

where $r = \lim_n r_n$. The operator T_0^{-1} is compact, thus

$$v_n = (T_0^{-1})[r_n g_n + (1 - r_n)\lambda v_n + r_n \frac{\tilde{h}}{n}] \to (T_0^{-1})[rf + (1-r)\lambda v] \quad \text{strongly in } L^2(Q).$$

Therefore, $v_n \to (T_0^{-1})[rf + (1-r)\lambda v] = v$ strongly in $L^2(Q)$. □

Lemma 3.4. *Assume* (2.3) *and (III). Then* $f(x,t) = 0$ *a.e. in* $A = \{(x,t) \in Q : v(x,t) = 0 \text{ a.e. in } Q\}$.

Proof. Let ψ be the function

$$\psi(x,t) = \operatorname{sign}(f(x,t))\chi_A(x,t) \text{ a.e. in } Q,$$

where χ_A is the indicatrice function. Since $g_n \to f$ weakly in $L^2(Q)$, we have $\int_Q g_n\psi \to \int_Q f\psi = \int_A |f(x,t)|$. On the other hand, as $v_n \to v$, and using (2.3), we have

$$\Big|\int_Q g_n\psi\Big| \le \int_Q |g_n\psi| \le \theta \int_Q |v_n\chi_A| + \int_Q \frac{\eta(x,t)\chi_A}{n} \to \theta \int_Q |v|\chi_A = 0,$$

thus $\int_A |f(x,t)| = 0$ and $f = 0$ a.e. in A. □

We define the function

$$d(x,t) = \begin{cases} \frac{f(x,t)}{v(x,t)} & \text{a.e. in } Q \setminus A, \\ \lambda & \text{a.e. in } A. \end{cases}$$

Lemma 3.5. *If one supposes* (2.3) ,(3.2) *and* (2.1)*, then* $\delta \le d(x,t) \le \mu$ *a.e. in* Q.

Proof. We prove that $\delta \le d(x,t)$ a.e. in Q. We denote $B = \{(x,t) \in Q : \delta(v(x,t))^2 > v(x,t)f(x,t) \text{ a.e.}\}$. It is sufficient to prove that $\operatorname{meas} B = 0$. Under condition (2.1) (cf. Remark 2.1. No. 5), we have

$$(\delta - \varepsilon)u_n^2 - a_\varepsilon(x,t)|u_n| \le u_n \tilde{g}(x,t,u_n).$$

Dividing by $\|u_n\|^2$, we get

$$(\delta-\varepsilon)v_n^2 - a_\varepsilon(x,t)\frac{|v_n|}{n} \le v_n g_n(x,t).$$

Multiplying by χ_B and integrating, we get

$$(\delta-\varepsilon)\int_Q v_n^2\chi_B - \int_Q \frac{a_\varepsilon(x,t)}{n}|v_n|\chi_B \le \int_Q v_n\chi_B g_n(x,t). \tag{3.3}$$

Under conditions (2.3) and (3.2), $g_n \to f$ weakly in $L^2(Q)$ and $v_n \to v$ strongly in $L^2(Q)$. Going to the limit in (IV), we have

$$(\delta-\varepsilon)\int_Q |v(x,t)|^2\chi_B \le \int_Q v(x,t)f(x,t)\chi_B.$$

Since ε is arbitrary, one concludes that

$$\int_Q [v(x,t)f(x,t) - \delta|v(x,t)|^2]\chi_B \ge 0.$$

Therefore, by the definition of B, $\operatorname{meas} B = 0$. By an analogous method, we prove that $d(x,t) \le \mu$ a.e. in Q. □

Lemma 3.6. *If one supposes* (2.3)*, (III) and* (2.1)*, then*

$$\begin{gathered} Tv = m(x,t)v \quad \text{in } Q,\\ v(x,t+2\pi) = v(x,t) \quad \text{in }]0,\pi[\times\mathbb{R},\\ v(0,t) = v(\pi,t) = 0 \quad \forall t\in\mathbb{R}, \end{gathered}$$

where $m(x,t) = rd(x,t) + (1-r)\lambda$ *and* $r = \lim_n r_n$.

Remark 3.7. It is easy to see that $m(x,t)$ is 2π-periodic in t, and $\delta \le m(x,t) \le \mu$ a.e. in Q.

Proof. In the proof of the Lemma 3.3, we have $rf + (1-r)\lambda v = Tv$. From the definition of the function m, we have $Tv = mv$. □

It remains to prove only the following lemma.

Lemma 3.8. *If one supposes* (2.3)*,* (3.2) *and* (2.1)*, then*

$$\delta \le\neq m(x,t) \le\neq \mu \quad \text{a.e. in } Q.$$

Proof. We prove that $m(x,t) \le\neq \mu$ a.e. in Q. (By analogous method, we prove that $\delta \le\neq m(x,t)$ a.e. in Q). Suppose by contradiction that $m(x,t) = \mu$ a.e. in Q. Under assumption (2.1), we have

$$v_n g_n \le (\tilde{k}(x,t)+\varepsilon)v_n^2 + \frac{a_\varepsilon|v_n|}{n}, \tag{3.4}$$

where $\tilde{k}(x,t) \in L^\infty(Q)$ such that $\tilde{k}(x,t) \le\neq \mu$. By (V), we have

$$\begin{aligned} &\int_Q r_n g_n v_n + (1-r_n)\lambda v_n^2 + r_n\int_Q \frac{\tilde{h}v_n}{n}\\ &\le \int_Q [r_n(\tilde{k}(x,t)+\varepsilon) + (1-r_n)\lambda]v_n^2 + r_n\int_Q (\tilde{h}\frac{v_n}{n} + a_\varepsilon\frac{|v_n|}{n}). \end{aligned} \tag{3.5}$$

Under conditions (2.3) and (3.2), $g_n \to f$ weakly in $L^2(Q)$ and $v_n \to v$ strongly in $L^2(Q)$. Going to the limit in (V), we get
$$\int_Q [r(fv) + (1-r)\lambda v^2] \le \int_Q [r(\tilde{k}(x,t)+\varepsilon) + (1-r)\lambda]v^2.$$
By the definition of m, we have
$$\int_Q [r(fv) + (1-r)\lambda v^2] = \int_Q m(x,t)v^2 = \int_Q \mu v^2.$$
Thus
$$\int_Q \mu v^2 \le \int_Q [r(\tilde{k}(x,t)+\varepsilon) + (1-r)\lambda]v^2.$$
Since ε is arbitrary, we have
$$\int_Q \mu v^2 \le \int_Q [r\tilde{k}(x,t) + (1-r)\lambda]v^2.$$
Hence
$$\int_Q [\mu - r\tilde{k}(x,t) - (1-r)\lambda]v^2 \le 0.$$
Since $\tilde{k}(x,t) \le \mu$ a.e. in Q and $\lambda < \mu$, we have $\mu - r\tilde{k}(x,t) - (1-r)\lambda \ge 0$, then
$$\int_Q [\mu - r\tilde{k}(x,t) - (1-r)\lambda]v^2 = 0.$$
Therefore, $[\mu - r\tilde{k}(x,t) - (1-r)\lambda]v^2 = 0$ a.e. in Q. Since $m(x,t) = \mu$ a.e. in Q, by the definition of the function of d, $(d(x,t) \neq \lambda)$, we have $\operatorname{meas} A = 0$ (i.e. $v(x,t) \neq 0$ a.e. in Q). Thus, $\mu = r\tilde{k}(x,t) + (1-r)\lambda$ a.e. in Q, this contradiction completes the proof. □

For the proof of Corollary 3.2 we will need the following two lemmas.

Lemma 3.9 ([6])**.** *Assume (C1), (C2),* (2.3)*, $\lambda \in \sigma(T)$ and $\lambda \ge 0$. Let $\tilde{h} \in L^2(Q)$, if there exist $R > 0$ such that*
$$Tu - r(N(u) + \tilde{h}) - (1-r)\lambda u \neq 0, \quad \forall u \in D(T),\ \|u\| = R,\ 0 \le r \le 1,$$
then problem (2.4) *admits at least one solution $u \in D(T)$ with $\|u\| < R$.*

Proof. By (2.3), (C1) and (C2), N is continuous and monotone; therefore the result ensues while using by the homotopy studied in [6]. □

Lemma 3.10. *If there exists two reals δ and μ such that*
$$\delta \le m(x,t) \le \mu \quad \text{a.e. in } Q \text{ with } [\delta,\mu] \cap \sigma(T) = \emptyset, \tag{3.6}$$
then the problem (3.1) *has only the trivial solution.*

Proof. Let $c \in [\delta,\mu]$ be arbitrary with
$$\frac{\max(|\mu - c|, |\delta - c|)}{\operatorname{dist}(c,\sigma(T))} < 1$$
(for example, $c = (\delta+\mu)/2$). Then the operator $T - cI$ is invertible and
$$\|(T - cI)^{-1}\| = \frac{1}{\operatorname{dist}(c,\sigma(T))}.$$

Hence for all $u \in D(T)$,

$$\|Tu - cu\| \geq dis(c, \sigma(T))\|u\|.$$

Assume now that $u \in D(T)$ is a solution of the problem (3.1). Then $\|Tu - cu\| = \|mu - cu\|$. Therefore,

$$\|u\| \leq \frac{\|mu - cu\|}{\operatorname{dist}(c, \sigma(T))}.$$

On the other hand, by (3.6),

$$|mu - cu| = |m - c||u| \leq \max(|\mu - c|, |c - \delta|)|u|$$

which implies $\|mu - cu\| \leq \max(|\mu - c|, |c - \delta|)\|u\|$. Thus

$$\|u\| \leq \frac{\max(|\mu - c|, |c - \delta|)}{\operatorname{dist}(c, \sigma(T))}\|u\|.$$

Since $\max(|\mu - c|, |\delta - c|)/\operatorname{dist}(c, \sigma(T)) < 1$, it follows that $u = 0$. ☐

Proof of corollary 3.2. Suppose by contradiction that the problem (1.1) does not admit a solution. Thus by proposition 2.2, (2.4) does not admit a solution. Hence by lemma 3.9, the homotopy (H_r) does not satisfy the estimate (2.2). And by Theorem 3.1, there exists $m(x,t) \in L^\infty(Q), v \in L^2(Q) \setminus \{0\}$ such that v is the nontrivial solution of the problem (3.1) and $\delta \leq\neq m(x,t) \leq\neq \mu$ a.e. in Q. Since $0 \leq \lambda_k - \frac{\gamma^2}{4} < \delta < \mu < \lambda_{k+1} - \frac{\gamma^2}{4}$, where $\lambda_k - \frac{\gamma^2}{4}$ and $\lambda_{k+1} - \frac{\gamma^2}{4}$ are two positive consecutive eigenvalues of T (cf. Remark 2.1 No. 1.ii), what is in contradiction with Lemma 3.10. Thus the proof is complete. ☐

Remark 3.11. (1) We have an analogous result, if in Corollary 3.2 λ_k and λ_{k+1} are two consecutive eigenvalues of the D'Alembertian such that $\lambda_k - \frac{\gamma^2}{4} < \delta < \mu < \lambda_{k+1} - \frac{\gamma^2}{4} \leq 0$, while replacing the operator T by $(-T)$.
(2) Note that if $\mu = 0$, $\delta = 0$ and $\gamma = 0$, we recover a result on the existence of the periodic solutions with conditions of non resonance of the problem

$$\begin{gathered}
\Box u = g(x,t,u) + h(x,t) \quad \text{in } Q, \\
u(x, t + 2\pi) = u(x,t) \quad \text{in }]0, \pi[\times\mathbb{R}, \\
u(0,t) = u(\pi,t) = 0 \quad \forall t \in \mathbb{R}
\end{gathered}$$

(3) Note that if $\mu = 0$, and $\delta = 0$, we recover a result on the existence of the periodic solutions with conditions of non resonance of the telegraph equation

$$\begin{gathered}
\Box u = \gamma u_t + g(x,t,u) + h(x,t) \quad \text{in } Q, \\
u(x, t + 2\pi) = u(x,t) \quad \text{in }]0, \pi[\times\mathbb{R}, \\
u(0,t) = u(\pi,t) = 0 \quad \forall t \in \mathbb{R}
\end{gathered}$$

References

[1] A. Anane and O. Chakrone, *Sur un théorème de point critique et application à un problème de non-résonance entre deux valeurs propres du p-Laplacien*, Annales de la faculté des sciences de Toulouse, Vol. IX, no 1, pp. 5-30, 2000.
[2] A. Anane, O. Chakrone and M. Ghanim, *Spectre d'ordre un pour un problme hyperbolique et applications.* Proyecciones revista de matematica, vol 21, No. 2,pp 109-125, 2002.

[3] A. K. Bennaoum, *On the Dirichlet problem for the nonlinear wave equation in bounded domains with corner points*, institut de mathématique pure et appliquée université catholique de Louvain-Belgique, recherches de mathématique, 51, 1996.

[4] A. K. Bennaoum and J. Mawhin, *Periodic solutions of some semilinear wave equation on balls and on spheres*, institut de mathématique pure et appliquée université catholique de Louvain-Belgique, recherches de mathématique, 15, 1992.

[5] Juha Berkovits and Vesa Mustonen, *On the resonance for semilinear equations with normal linear part*, J. Math. Maroc2, pp. 115-123, 1994.

[6] Juha Berkovits and Vesa Mustonen, *An extension of Leray-Schauder degree and applications to nonlinear wave equations* , Differentiel Integral Equations No. 3, 945-963, 1990.

[7] Juha Berkovits and Vesa Mustonen, *On semilinear wave equations at resonance*, proceedings of the first world congress of nonlinear analysts, 1992. Editor: Lakshmilkantham, Berlin, New York, 1996.

[8] Juha Berkovits and Vesa Mustonen, *An application of topologique degree to semilinear equations with nonlinearities strongly monotones, Bull*, London Math. Soc. 23, pp. 470-476, 1991.

[9] Juha Berkovits and Vesa Mustonen, *On multiple solutions for a class of semilinear wave equations*, Nonlinear Analysis: theory and applications, Vol. 16, No. 5, pp. 421-434, 1991.

[10] Juha Berkovits and Vesa Mustonen, *On nonresonance for systems of semilinear wave equations*, Nonlinear analysis: theory and applications, Vol.29, pp.627-638, 1997.

[11] Juha Berkovits and Vesa Mustonen, *Multipliciy results for some wave equations*, Nonlinear Vibration Problems, Zagadnienia Drgan Nieliniowych, 1993.

[12] H. Brezis and L. Nirenberg, *Characterization of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa (4) 5, pp. 225-326, 1978.

Aomar Anane
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: `anane@sciences.univ-oujda.ac.ma`

Omar Chakrone
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: `chakrone@sciences.univ-oujda.ac.ma`

Abdellah Zerouali
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: `abdellahzerouali@hotmail.com`