2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 95–107.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MAXIMUM AND ANTI-MAXIMUM PRINCIPLES FOR THE P-LAPLACIAN WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper we study the maximum and the anti-maximum principles for the problem $\Delta_p u = |u|^{p-2}u$ in the bounded smooth domain $\Omega \subset \mathbb{R}^N$, with $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + h$ as a non linear boundary condition on $\partial\Omega$ which is supposed $C^{2\beta}$ for some β in $]0, 1[$, and where $h \in L^\infty(\partial\Omega)$. We will also examine the existence and the non existence of the solutions and their signs.

1. Introduction

In this work we consider the problem

$$\begin{gathered} \Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + h \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a $C^{2,\beta}$ boundary where $\beta \in]0, 1[$, $h \in L^\infty(\partial\Omega)$ and $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

For $h \equiv 0$ in $\partial\Omega$, Fernandez Bonder, Pinasco and Rossi [1] proved that the problem

$$\begin{gathered} \triangle_p u = |u|^{p-2}u \quad \text{on } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u \quad \text{on } \partial\Omega \end{gathered}$$

admits an infinite sequence of eigenvalues (λ_n) such that $\lambda_n \to \infty$ as $n \to \infty$. Martinez and Rossi [4] showed that the first eigenvalue given by

$$\lambda_1 = \inf \Big\{ \int_\Omega |\nabla u|^p + \int_\Omega |u|^p : u \in W^{1,p}(\Omega) \text{ and } \int_{\partial\Omega} |u|^p = 1 \Big\}$$

is simple and isolated with the eigenfunctions do not change sign in Ω.

We will be interested in he case where $h \not\equiv 0$ in $\partial\Omega$. The case where $h \equiv 0$, is treated by Godoy, Gossez and Paczka [3]; they have proved in that

2000 *Mathematics Subject Classification.* 35J65, 35J25.
Key words and phrases. Anti-maximum; p-laplacian; non linear boundary condition.

Published Septebmer 20, 2006.

(i) The maximum principle holds if and only if $\lambda_{-1}(m) < \lambda < \lambda_1(m)$ with $\lambda_{-1}(m)$ and$\lambda_1(m)$ are the principal eigenvalues of the p-Laplacien with the weight m, for the Dirichlet problem, (respectively $0 < \lambda < \lambda^*(m)$, for the Neumann problem, with $\lambda^*(m)$ is the nontrivial principal eigenvalue).
(ii) The anti-maximum principle holds at the right of $\lambda_1(m)$ and at the left of $\lambda_{-1}(m)$ (resp, at the right of $\lambda^*(m)$ and at the left of 0. Moreover it is nonuniform when $p \leq N$ and uniform when $p > N$).

In what follows one supposes that any solution of (1.1) is in $C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in]0,1[$.

2. The maximum principle

The following result will be proven.

Theorem 2.1. *The maximum principle holds for problem* (1.1) *if and only if* $\lambda \leq \lambda_1$.

Proof. (i) Given $\lambda \leq \lambda_1$, $0 \lneqq h \in L^\infty(\partial\Omega)$ and u be a solution of (1.1), let us show that $u \geq 0$ in Ω. We recall that u is a solution of(1.1) if and only if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v + \int_\Omega |u|^{p-2}uv = \lambda\int_{\partial\Omega}|u|^{p-2}uv + \int_{\partial\Omega} hv \tag{2.1}$$

for all $v \in W^{1,p}(\Omega)$. Applying this equality to u^-, one finds

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u^- + \int_\Omega |u|^{p-2}uu^- = \lambda\int_{\partial\Omega}|u|^{p-2}uu^- + \int_{\partial\Omega} hu^-.$$

Then

$$\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p = \lambda\int_{\partial\Omega}|u^-|^p - \int_{\partial\Omega} hu^-,$$

and

$$\lambda\int_{\partial\Omega}|u^-|^p - \Big(\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p\Big) = \int_{\partial\Omega} hu^-.$$

However, $\lambda \leq \lambda_1 = \inf\{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p : u \in W^{1,p}(\Omega) \text{ and } \int_{\partial\Omega}|u|^p = 1\}$, so

$$\begin{aligned}\int_{\partial\Omega} hu^- &= \lambda\int_{\partial\Omega}|u^-|^p - \Big(\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p\Big)\\ &\leq \lambda_1\int_{\partial\Omega}|u^-|^p - \Big(\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p\Big) \leq 0.\end{aligned}$$

Moreover $h \gneqq 0$ and $u^- \geq 0$ thus $\int_{\partial\Omega} hu^- \geq 0$ from where $\int_{\partial\Omega} hu^- = 0$. Consequently $\lambda\int_{\partial\Omega}|u^-|^p = \int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p$; i-e, u^- is a positive eigenfunction associated with λ.

If $\lambda \neq \lambda_1$ thenu^- change sign on $\partial\Omega$ by [4, lemma 2.4], which is not possible, so $u^- \equiv 0$ on $\partial\Omega$. Hence $0 = \lambda\int_{\partial\Omega}|u^-|^p = \int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p$ and thus $u^- \equiv 0$ in Ω.

If $\lambda = \lambda_1$ then u^- is a positive eigenfunction associated to λ_1 and it is in $C^{1,\alpha}(\overline{\Omega})$, so $u^- > 0$ or $u^- \equiv 0$ in$\overline{\Omega}$ by the following maximum principle.

Theorem 2.2 (Vasquez [5])**.** *Let* $u \in C^1(\Omega)$ *be such that* $\Delta_p u \in L^2_{\rm loc}(\Omega)$, $u \geq 0$ *a.e in* Ω *and* $\Delta_p u \leq \beta(u)$ *a.e in* Ω, *with* $\beta : [0,+\infty[\to \mathbb{R}$ *is a increasing continuous function,* $\beta(0) = 0$ *and either* $\beta(s) = 0$ *for some* $s > 0$ *or* ($\beta(s) > 0$ *for all* $s > 0$ *and*

$\int_0^1 (j(s))^{-\frac{1}{p}} ds = \infty$ with $j(s) = \int_0^s \beta(t)dt$). Then if u does not vanish identically on Ω, it is positive everywhere in Ω.

Moreover if $u \in C^1(\Omega \cup \{x_0\})$ for an $x_0 \in \partial\Omega$ that satisfies an interior sphere condition and $u(x_0) = 0$ then $\frac{\partial u}{\partial n}(x_0) > 0$ where n is an interior normal at x_0.

Indeed, ($u^- > 0$ or $u^- = 0$) in Ω, and if there exists $x_0 \in \partial\Omega$ such that $u^-(x_0) = 0$, then $\frac{\partial u^-}{\partial \nu}(x_0) < 0$. However $|\nabla u^-|^{p-2}\frac{\partial u^-}{\partial \nu}(x_0) = \lambda_1 |u^-(x_0)|^{p-2}u^-(x_0) = 0$ and thus $\nabla u^- \equiv 0$ in $\overline{\Omega}$; i.e., $u^- = 0$ in $\overline{\Omega}$. Consequently $u \geq 0$ in$\overline{\Omega}$.
(ii) Given $\lambda > \lambda_1$, and one supposes that the maximum principle is applicable to the problem (1.1); i.e., for all $h \in L^\infty(\partial\Omega)$, if $h \gneqq 0$ then any solution is positive on Ω.

Considering Φ a positive eigenfunction associated to the first eigenvalue λ_1, and $h = (\lambda - \lambda_1)|\Phi|^{p-2}\Phi \gneqq 0$, one checks that $\Psi = -\Phi \leq 0$ is a solution of (1.1), but this contradicted the maximum principle. □

3. Existence and nonexistence of solutions for (1.1)

In this paragraph one shows stronger results existence and nonexistence. They are stated in the following theorem.

Theorem 3.1. *Given $h \in L^\infty(\partial\Omega)$ such that $h \geq 0$ in $\partial\Omega$.*

(i) *If$\lambda < \lambda_1$ and $h \gneqq 0$ in $\partial\Omega$, then the problem admits an unique solution u which is in $C^1(\overline{\Omega})$ and$u > 0$ on $\overline{\Omega}$.*
(ii) *If $\lambda = \lambda_1$ and $h \gneqq 0$ in $\partial\Omega$, then the problem has no solution.*
(iii) *If $\lambda > \lambda_1$ then the problem does not have any solution u such that $u \gneqq 0$ on $\overline{\Omega}$.*

For the proof of this theorem we need the following lemma.

Lemma 3.1. *Let u be a solution of (1.1) with $u > 0$ on $\overline{\Omega}$, and $h \geq 0$ in $\partial\Omega$ (or $h \leq 0$ in $\partial\Omega$). Then $\forall \varphi \in C^1(\overline{\Omega})$ with $\varphi \geq 0$ in $\overline{\Omega}$, one has*
$$\lambda \int_{\partial\Omega} \varphi^p + \int_{\partial\Omega} h \frac{\varphi^p}{u^{p-1}} \leq \int_\Omega |\nabla\varphi|^p + \int_\Omega \varphi^p.. \tag{3.1}$$
Moreover, the equality holds if and only if φ is a multiple of u.

Proof. Given u and φ in $C^1(\overline{\Omega})$ with $u > 0$ and $\varphi \geq 0$ on $\overline{\Omega}$, we denote $R(\varphi, u) = |\nabla\varphi|^p - |\nabla u|^{p-2}\nabla u \nabla(\frac{\varphi^p}{u^{p-1}})$, and we prove that $R(\varphi, u) \geq 0$ and that $R(\varphi, u) = 0$ if and only if there exists $c \geq 0$ such that $\varphi = cu$.

One has $R(\varphi, u) = |\nabla\varphi|^p + (p-1)\frac{\varphi^p}{u^p}|\nabla u|^p - p\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla u \nabla\varphi$. Applying Minkovsky inequality to $|\nabla\varphi|$ and $\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-1}$, one obtains
$$\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-1}|\nabla\varphi| \leq \frac{1}{p}|\nabla\varphi|^p + \frac{1}{q}(\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-1})^q \tag{3.2}$$
where $\frac{1}{q} + \frac{1}{p} = 1$. This implies $\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-1}|\nabla\varphi| \leq \frac{1}{p}|\nabla\varphi|^p + \frac{p-1}{q}(\frac{\varphi^p}{u^p}|\nabla u|^{p-1})$. And it follows that
$$R(\varphi, u) = |\nabla\varphi|^p + (p-1)\frac{\varphi^p}{u^p}|\nabla u|^p - p\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla u \nabla\varphi \geq 0.$$
The inequality (3.2) becomes an equality if and only if
$$|\nabla\varphi| = \frac{\varphi}{u}|\nabla u|. \tag{3.3}$$

Let $\psi = \frac{\varphi}{u} \geq 0$. Then we have $\varphi(x) = \psi(x)u(x)$ which implies $\nabla\varphi = \psi\nabla u + u\nabla\psi$. Then (3.3) is equivalent to $|\psi\nabla u + u\nabla\psi| = \psi|\nabla u|$, which is true if and only if $u^2|\nabla\psi|^2 = -2u\psi\nabla u\nabla\psi$. We have also:

$$\begin{aligned}(R(\varphi, u) = 0) &\Leftrightarrow |\nabla\varphi|^p = |\nabla u|^{p-2}|\nabla u|\|\nabla(\frac{\varphi}{u})^p u| \\ &\Leftrightarrow |\nabla\varphi|^p = |\nabla u|^p(\frac{\varphi}{u})^p + u|\nabla u|^{p-2}|\nabla u|\nabla(\psi^p).\end{aligned}$$

Since $|\nabla\varphi| = \psi|\nabla u|$, one has

$$R(\varphi, u) = 0 \Leftrightarrow |\nabla u|^{p-2}|\nabla u|\nabla(\psi^p) = |\nabla u|^{p-2}(\nabla u\nabla\psi)p\psi^{p-1} = 0$$

If $|\nabla u| \equiv 0$, then (3.3) implies that $|\nabla\varphi| = 0$. And in this case u and φ are constants; therefore there exists $c \geq 0$ such that $\varphi = cu$.

If $|\nabla u| \neq 0$, on the set $\{|\nabla u| \neq 0\}$, (3.3) is equivalent to $\nabla u\nabla\psi = 0$. Therefore, one has also (3.3) is equivalent to $-2\psi\nabla u\nabla(\varphi\psi) = u|\nabla\psi|^2$ Thus $\nabla u\nabla\psi = 0$ if and only if $|\nabla\psi| = 0$, which is equivalent to ψ equals a constant.

Consequently $R(\varphi, u) = 0$ if and only there exists $c \geq 0$ such that $\varphi = cu$. Conclusions:

(i) $0 \leq \int_\Omega R(\varphi, u) = \int_\Omega |\nabla\varphi|^p - \int_\Omega |\nabla u|^{p-2}\nabla u\nabla(\frac{\varphi^p}{u^{p-1}})$ for all$(u, \varphi) \in (C^1(\overline{\Omega}))^2$ with $u > 0$ and $\varphi \geq 0$ on $\overline{\Omega}$.

(ii) $\int_\Omega R(\varphi, u) = \int_\Omega |\nabla\varphi|^p - \int_\Omega |\nabla u|^{p-2}\nabla u\nabla(\frac{\varphi^p}{u^{p-1}}) = 0$ if and only there exists $c \geq 0$ such that $\varphi = cu$.

Then if u is a solution of (1.1) and $\varphi \in C^1(\overline{\Omega})$ such that $\varphi \geq 0$ on $\overline{\Omega}$, we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(\frac{\varphi^p}{u^{p-1}}) + \int_\Omega (u)^{p-1}(\frac{\varphi^p}{u^{p-1}}) = \lambda\int_{\partial\Omega}(u)^{p-1}(\frac{\varphi^p}{u^{p-1}}) + \int_{\partial\Omega} h\frac{\varphi^p}{u^{p-1}}.$$

Consequently,

$$0 \leq \int_\Omega |\nabla\varphi|^p - \lambda\int_{\partial\Omega}(u)^{p-1}(\frac{\varphi^p}{u^{p-1}}) - \int_{\partial\Omega} h\frac{\varphi^p}{u^{p-1}} + \int_\Omega (u)^{p-1}(\frac{\varphi^p}{u^{p-1}}),$$

then

$$0 \leq \int_\Omega |\nabla\varphi|^p - \lambda\int_{\partial\Omega}\varphi^p - \int_{\partial\Omega} h\frac{\varphi^p}{u^{p-1}} + \int_\Omega \varphi^p,$$

and

$$\int_\Omega |\nabla\varphi|^p + \int_\Omega \varphi^p \geq \lambda\int_{\partial\Omega}\varphi^p + \int_{\partial\Omega} h\frac{\phi^p}{u^{p-1}}. \tag{3.4}$$

If there is equality in (3.4) then $\int_\Omega R(\varphi, u) = 0$ which is equivalent to $R(\varphi, u) = 0$ almost everywhere in Ω. So there exists $c \geq 0$ such that $\varphi = cu$. □

Proof of Theorem 3.1. The case of $\lambda < \lambda_1$: (i) Existence of the solutions for the problem (1.1): One considers the function $\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(v) = \frac{1}{p}(\int_\Omega |\nabla v|^p + \int_\Omega |v|^p) - \frac{\lambda}{p}\int_{\partial\Omega}|v|^p - \int_{\partial\Omega} hv\,.$$

(a) Φ is C^1 and weakly lower semi-continuous.

(b) Let us show that Φ is coercive: Let $v \in W^{1,p}(\Omega)$ such that $\|v\|_{W^{1,p}(\Omega)} \neq 0$, and we consider $u = \frac{v}{\|v\|_{W^{1,p}(\Omega)}}$ and $s = \|v\|_{W^{1,p}(\Omega)}$. So we have

$$\begin{aligned}
\Phi(v) &= \Phi(su) \\
&= \frac{1}{p} s^p \Big[\Big(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p \Big) - \lambda \int_{\partial\Omega} |u|^p \Big] - s \int_{\partial\Omega} hu \\
&\geq \frac{1}{p} s^p (1 - \frac{\lambda}{\lambda_1}) \|u\|^p_{W^{1,p}(\Omega)} - s \|h\|_{L^q(\partial\Omega)} \|u\|_{L^p(\partial\Omega)} \\
&\geq \frac{1}{p} s^p (1 - \frac{\lambda}{\lambda_1}) - \frac{s}{(\lambda_1)^{\frac{1}{p}}} \|h\|_{L^q(\partial\Omega)}
\end{aligned}$$

Thus as $s \to +\infty$ one has $\Phi(su) \to +\infty$; i. e. $\Phi(v) \to +\infty$ as $\|v\|_{W^{1,p}(\Omega)} \to +\infty$, so Φ is coercive. The function Φ is weakly lower semi-continuous coercive, so it admits a critical point u which is solution of (1.1).
(ii) One shows that $u > 0$ on $\overline{\Omega}$. According to the regularity and the theorem 2.1, one has $u \in C^1(\overline{\Omega})$ and $u \geqslant 0$ on Ω. And with the maximum principle by Vazquez [5], one gets $u > 0$ on Ω. Moreover $\partial\Omega$ is $C^{2,\beta}$, so $u > 0$ on $\overline{\Omega}$, because if there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, then $\frac{\partial u}{\partial n}(x_0) > 0$; i.e. $\frac{\partial u}{\partial \nu}(x_0) < 0$. However, $|\nabla u|^{p-2} \frac{\partial u}{\partial \nu}(x_0) = \lambda |u|^{p-2} u(x_0) + h(x_0) = h(x_0)$, then $|\nabla u|^{p-2} \frac{\partial u}{\partial \nu}(x_0) = h(x_0) < 0$. That is impossible since $h \geqslant 0$ in $\partial\Omega$. So $u > 0$ on $\overline{\Omega}$.
(iii) Uniqueness of the solution: By contradiction one supposes that there exist two nontrivial solutions of (1.1) u and v with $h \gneqq 0$ in $\partial\Omega$. According to what precedes one has $u > 0$ and $v > 0$ on $\overline{\Omega}$. And Applying lemma 3.1 at u and v one gets

$$\lambda \int_{\partial\Omega} v^p + \int_{\partial\Omega} h(\frac{v^p}{u^{p-1}}) \leq \int_\Omega |\nabla v|^p + \int_\Omega |v|^p = \lambda \int_{\partial\Omega} v^p + \int_{\partial\Omega} hv. \tag{3.5}$$

Then

$$\lambda \int_{\partial\Omega} h(\frac{v^p}{u^{p-1}}) - \int_{\partial\Omega} hv \leq 0, \text{ i-e } \int_{\partial\Omega} hv(\frac{v^{p-1} - u^{p-1}}{u^{p-1}}) \leq 0. \tag{3.6}$$

By exchanging the roles of u and v one has

$$\int_{\partial\Omega} hu(\frac{u^{p-1} - v^{p-1}}{u^{p-1}}) \leq 0. \tag{3.7}$$

Inequalities (3.6) and (3.7) imply

$$\int_{\partial\Omega} h\big[u((\frac{u^{p-1} - v^{p-1}}{u^{p-1}}) + v((\frac{v^{p-1} - u^{p-1}}{u^{p-1}})\big] \leq 0;$$

i.e., $\int_{\partial\Omega} h\big[(u^{p-1} - v^{p-1})(\frac{v}{u^{p-1}} - \frac{u}{v^{p-1}})\big] \leq 0$. Consequently

$$\int_{\partial\Omega} h \frac{(v^{p-1} - u^{p-1})(v^p - u^p)}{u^{p-1} v^{p-1}} \leq 0.$$

However,

$$\frac{(v^{p-1} - u^{p-1})(v^p - u^p)}{u^{p-1} v^{p-1}} \geq 0$$

and $h \geq 0$ in $\partial\Omega$. Then (3.6) and (3.7) are equalities, and one has equality in (3.5); that is true if and only if u is a multiple of v. Consequently there exists $c \geq 0$ such that $u = cv$ on Ω and since $u > 0$ and$v > 0$ we have $c > 0$. Replacing u by cv in equation (1.1), one obtains

$$\Delta_p(cv) = |cv|^{p-2}cv \quad \text{in } \Omega,$$
$$|\nabla cv|^{p-2}\frac{\partial(cv)}{\partial\nu} = \lambda|cv|^{p-2}cv + h \quad \text{on } \partial\Omega.$$

This is equivalent to

$$\Delta_p(v) = |v|^{p-2}v \quad \text{in } \Omega,$$
$$|\nabla v|^{p-2}\frac{\partial(v)}{\partial\nu} = \lambda|v|^{p-2}v + \frac{h}{c^{p-1}} \quad \text{on } \partial\Omega.$$

But v is a solution of (1.1) then $\frac{h}{c^{p-1}} = h$, and since$h \neq 0$ one has $c = 1$ and $u = v$. So the uniqueness of the solution of (1.1) in the case $\lambda < \lambda_1$ and $h \gneqq 0$ is proved.

The case $\lambda = \lambda_1$: Suppose that there exists $u \in W^{1,p}(\Omega)$ a nontrivial solution of (1.1). One has $u \in C^1(\overline{\Omega})$ and $u > 0$ on $\overline{\Omega}$. And let φ_1 be a positive eigenfunction associated to λ_1, then $\varphi_1 \in C^1(\overline{\Omega})$ and $\varphi_1 > 0$ on $\overline{\Omega}$. Applying lemma 3.1 at u and φ_1 one gets

$$\lambda_1\int_{\partial\Omega}\varphi_1^p + \int_{\partial\Omega}h(\frac{\varphi_1^p}{u^{p-1}}) \leq \int_\Omega|\nabla\varphi_1|^p + \int_\Omega|\varphi_1|^p.$$

However $\int_\Omega|\nabla\varphi_1|^p + \int_\Omega|\varphi_1|^p = \lambda_1\int_{\partial\Omega}\varphi_1^p$, then $\int_{\partial\Omega}h\frac{\varphi_1^p}{u^{p-1}} = 0$ and $h\frac{\varphi_1^p}{u^{p-1}} = 0\sigma$ a.e. in $\partial\Omega$. But $\varphi_1 > 0$ and$u > 0$ on $\overline{\Omega}$ and $h \neq 0$ in $\partial\Omega$ a contradiction. Then if $\lambda = \lambda_1$ and $h \gneqq 0$ in $\partial\Omega$, problem (1.1) has no solution in $W^{1,p}(\Omega)$.

The case of $\lambda > \lambda_1$: By contradiction we suppose that there is $u \gneqq 0$ a solution of (1.1). Then $u \in C^1(\overline{\Omega})$ and $u > 0$ on $\overline{\Omega}$. Lemma 3.1 implies

$$\lambda\int_{\partial\Omega}\varphi^p + \int_{\partial\Omega}h\frac{\varphi^p}{u^{p-1}} \leq \int_\Omega|\nabla\varphi|^p + \int_\Omega|\nabla\varphi|^p$$

This implies that $\lambda\int_{\partial\Omega}\varphi^p \leq \int_\Omega|\nabla\varphi|^p + \int_\Omega|\nabla\varphi|^p$ for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geq 0$. Also by density $\lambda\int_{\partial\Omega}\varphi^p \leq \int_\Omega|\nabla\varphi|^p + \int_\Omega|\nabla\varphi|^p|^p$ for all $\varphi \in W^{1,p}(\Omega)$. Then

$$\lambda \leq \frac{\int_\Omega|\nabla\varphi|^p + \int_\Omega|\nabla\varphi|^p|^p}{\int_{\partial\Omega}\varphi^p} \quad \forall\varphi \in W^{1,p}(\Omega) \text{ with } \int_{\partial\Omega}\varphi^p \neq 0$$

Consequently $\lambda \leq \lambda_1$, which is a contradiction, and the result follows. □

4. The anti-maximum principle

In this part we study the anti-maximum principle for the problem (1.1).

Theorem 4.1. *Given $h \in L^\infty(\partial\Omega)$ with $h \gneqq 0$, there exists $\delta = \delta(h) > 0$ such that if $\lambda_1 < \lambda < \lambda_1 + \delta$, and u is a solution of (1.1), then*

$$u < 0 \quad \textit{in } \overline{\Omega} \quad \textit{and} \quad \frac{\partial u}{\partial v} < 0 \quad \textit{on } \partial\Omega \tag{4.1}$$

It will thus be said that the antimaximum principle (AMP) holds on the right of λ_1.

Proof. By contradiction we suppose that for all $k \in N^*$ there exists $(\mu_k)_k \subset \mathbb{R})$ such that

$$\lambda_1 < \mu_k < \lambda_1 + \frac{1}{k}$$

and there exists $(u_k)_k \subset W^{1,p}(\Omega)$ so that

$$
\begin{gathered}
\Delta_p u_k = |u_k|^{p-2}u_k + h \quad \text{in } \Omega, \\
|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial \nu} = \mu_k |u_k|^{p-2}u_k + h \quad \text{on } \partial\Omega
\end{gathered} \tag{4.2}
$$

and u_k does not check (4.1). One has according to the regularity result $u_k \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$. And $\|u_k\|_\infty$ does not remain bounded. Indeed, if there exists $M > 0$ such that $\|u_k\|_\infty \le M$ for all k, then $\|\Delta_p u_k\| = \||u_k|^{p-1}\|_\infty \le M^{p-1} = M'$, and we get also $\|u_k\|_{C^{1,\alpha}(\overline{\Omega})} \le K_1$ independently of k.

So since $(u_k) \subset C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$, compact, then for a subsequence $u_k \to u$ in $C^1(\overline{\Omega})$. Moreover, if

$$
\begin{gathered}
\Delta_p u_k = |u_k|^{p-2}u_k + h \quad \text{in}\Omega, \\
|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial \nu} = \mu_k |u_k|^{p-2}u_k + h \quad \text{on } \partial\Omega,
\end{gathered}
$$

then

$$
\begin{gathered}
\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda_1 |u|^{p-2}u + h \quad \text{on } \partial\Omega.
\end{gathered}
$$

This contradicts the result of theorem 3.1 which ensures the nonexistence of solution for (1.1) when $\lambda = \lambda_1$ and $h \gneqq 0$. Consequently $\|u_k\|_\infty \to +\infty$.

Let us consider $v_k = \frac{u_k}{\|u_k\|_\infty}$, so $\|v_k\|_\infty = 1$ and as previously for a subsequence $v_k \to v$ in $C^1(\overline{\Omega})$ with $\|v\|_\infty = 1$. Also v_k solves

$$
\begin{gathered}
\Delta_p v_k = \frac{\Delta_p u_k}{\|u_k\|_\infty^{p-1}} = |v_k|^{p-2}v_k \quad \text{in } \Omega, \\
|\nabla v_k|^{p-2}\frac{\partial v_k}{\partial \nu} = \frac{1}{\|u_k\|_\infty^{p-1}}|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial \nu} = \mu_k |v_k|^{p-2}v_k + \frac{h}{\|u_k\|_\infty^{p-1}} \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.3}
$$

Then

$$
\begin{gathered}
\Delta_p v = |v|^{p-2}v \quad \text{in } \Omega, \\
|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = \lambda_1 |v|^{p-2}v \quad \text{on } \partial\Omega.
\end{gathered}
$$

So v is a eigenfunction associated to λ_1 and $v \neq 0$. Applying again the maximum principle one has ($v > 0$ in $\overline{\Omega}$) or ($v < 0$ in $\overline{\Omega}$).
(i) If $v > 0$ in $\overline{\Omega}$, then for k sufficiently large we have $v_k > 0$ in$\overline{\Omega}$, but v_k is a solution of (1.1) with $\lambda = \mu_k > \lambda_1$ and $h' = \frac{h}{\|u_k\|_\infty^{p-1}} \gneqq 0$ on $\partial\Omega$ which leads to a contradiction with the theorem 3.1.
(ii) If $v < 0$ in $\overline{\Omega}$, one has $|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = \lambda_1 |v|^{p-2}v < 0$ in $\partial\Omega$, then $\frac{\partial v}{\partial \nu} < 0$ in $\partial\Omega$. So for k sufficiently large, ($v_k < 0$ in $\overline{\Omega}$ and $\frac{\partial v_k}{\partial \nu} < 0$ in $\partial\Omega$). This means that u_k checks (4.1). Contradiction with the assumption. This completes the proof of the anti-maximum principle. □

Now we study the uniformity of this principle. One will show that the AMP is nonuniform on the right of λ_1 when $p \le N$, and uniform when $p > N$. In the latter case one will characterize the interval of uniformity. Moreover one shows that the AMP still holds on the right of this interval but not uniformly. For this, one shows accesses the following result.

Lemma 4.1. *If $p \leq N$ then $\lambda_1 = \overline{\lambda_1}$. If $p > N$ then $\lambda_1 < \overline{\lambda_1}$, and the interval $]\lambda_1, \overline{\lambda_1}]$ does not contain any eigenvalue where $\overline{\lambda_1} = \inf_{u \in P}(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p)$ and*

$$P = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p = 1 \text{ and } u \text{ vanishes in some ball in } \overline{\Omega}\}.$$

Proof. It is clear that $\lambda_1 \leq \overline{\lambda_1}$.
(i) If $p < N$: As in Godoy, Gossez and Paczka [2], one defines a sequence $(v_k)_k$ as follows: For $k \in \mathbb{N}^*$,

$$\begin{cases} 1 & \text{if } |x| \geq 1/k \\ 2k|x| - 1 & \text{if } 1/(2k) < |x| < 1/k, \\ 0 & \text{if } |x| \leq 1/2k \end{cases}$$

Note that $v_k$converges to the constant function 1 as $k \to +\infty$ in $W^{1,p}_{\rm loc}(R^N)$. Given $x_0 \in \Omega$, and u a eigenfunction associated to λ_1; i.e., $\lambda_1 = \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p}$, then the sequence (w_k) defined by $w_k(x) = u(x)v_k(x - x_0)$ vanishes in the ball $B(x_0, \frac{1}{2k})$ and

$$\overline{\lambda_1} \leq \frac{\int_\Omega |\nabla w_k|^p + \int_\Omega |w_k|^p}{\int_{\partial\Omega} |w_k|^p} \to \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p} \text{as } k \to +\infty$$

So $\overline{\lambda_1} \leq \lambda_1$, then $\overline{\lambda_1} = \lambda_1$.
(ii) If $p = N$: One defines a sequence $(v_k)_k$ as follows: For $k \in \mathbb{N}^*$,

$$v_k = \begin{cases} 1 - \frac{1}{2k} & \text{if } |x| \geq \frac{1}{k} \\ |x|^{\delta_k} - \frac{1}{k} & \text{if } (\frac{1}{k})^{1/\delta_k} < |x| < \frac{1}{k} \\ 0 & \text{if } |x| \leq (\frac{1}{k})^{1/\delta_k}, \end{cases}$$

where δ_k satisfies $(\frac{1}{k})^{\delta_k} = 1 - \frac{1}{k}$. ($\delta_k = 1 - \frac{\ln(k+1)}{\ln(k)} \to 0$ as $k \to +\infty$). And (w_k) is as previously, and we show that

$$\frac{\int_\Omega |\nabla w_k|^p + \int_\Omega |w_k|^p}{\int_{\partial\Omega} |w_k|^p} \to \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p} \quad \text{as } k \to +\infty$$

Indeed

$$\int_\Omega |w_k|^p = \int_{|x-x_0| \geq \frac{1}{k}} |1 - \frac{1}{2k}|^p |u(x)|^p + \int_{(\frac{1}{k})^{\frac{1}{\delta_k}} \leq |x-x_0| \leq \frac{1}{k}} \||x - x_0|^{\delta_k} - \frac{1}{k}|^p |u(x)|^p,$$

and

$$\int_{(\frac{1}{k})^{\frac{1}{\delta_k}} \leq |x-x_0| \leq \frac{1}{k}} \||x - x_0|^{\delta_k} - \frac{1}{k}|^p |u(x)|^p \leq (1 - \frac{2}{k})^p \int_{(\frac{1}{k})^{\frac{1}{\delta_k}} \leq |x-x_0| \leq \frac{1}{k}} |u(x)|^p \to 0,$$

consequently

$$\overline{\lambda_1} \leq \frac{\int_\Omega |\nabla w_k|^p + \int_\Omega |w_k|^p}{\int_{\partial\Omega} |w_k|^p} \to \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p} = \lambda_1,$$

then $\overline{\lambda_1} = \lambda_1$.
(iii) When $p > N$, we have

$$\overline{\lambda_1} = \inf_{u \in G} \Big(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p \Big)$$

where $G = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p = 1 \text{ and } \exists x_0 \in \overline{\Omega} : u(x_0) = 0\}$. Since $W^{1,p}(\Omega) \underset{cpt}{\hookrightarrow} C(\overline{\Omega})$, the minimum is achieved in a certain function $\overline{u}$, that one can suppose positive on $\overline{\Omega}$. And there exists $x_0 \in \overline{\Omega}$ such that $\overline{u}(x_0) = 0$. Let us show that it vanishes only once on $\overline{\Omega}$.

One assumes, by contradiction, that there exists $x_1 \neq x_0 \in \overline{\Omega}$ with $\overline{u}(x_1) = 0$. Setting $F = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p = 1 \text{ and } u(x_0) = 0\}$ and $\mu = \inf_{u \in F}(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p)$, one gets $\overline{\lambda_1} \leq \mu$, $\overline{u} \in F$ and $\overline{\lambda_1} = \int_\Omega |\nabla \overline{u}|^p + \int_\Omega |\overline{u}|^p$ then $\overline{\lambda_1} = \mu$.

Applying the standard theorem on Lagrange multipliers, we ensure the existence of $(\alpha_1, \alpha_2) \in R^2$ such that for all $w \in W^{1,p}(\Omega)$ with $w(x_0) = 0$ one has

$$\Phi'(\overline{u}).w = \alpha_1 \Psi_1'(\overline{u}).w + \alpha_2 \Psi_2'(\overline{u}).w = \alpha_1 \Psi_1'(\overline{u}).w$$

where $\Psi_1(u) = \int_{\partial\Omega} |u|^p - 1$, $\Psi_2(u) = u(x_1)$ and $\Phi(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p$. Then for all $w \in W^{1,p}(\Omega) : w(x_0) = 0$ one has $\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla w + \int_\Omega \overline{u}^{p-1} w = \alpha_1 \int_{\partial\Omega} \overline{u}^{p-1} w$. Taking $w = \overline{u}$ one gets $\alpha_1 = \overline{\lambda_1}$ and

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla w + \int_\Omega \overline{u}^{p-1} w = \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} w \tag{4.4}$$

for all $w \in W^{1,p}(\Omega)$ with $w(x_0) = 0$. By the same process, one has

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla w + \int_\Omega \overline{u}^{p-1} w = \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} w$$

for all $w \in W^{1,p}(\Omega)$ with $w(x_1) = 0$. Knowing that for all $w \in W^{1,p}(\Omega)$, there exists $(w_0, w_1) \in (W^{1,p}(\Omega))^2$ such that $w = w_0 + w_1$, $w_0(x_0) = 0$ and $w_1(x_1) = 0$, one arrives at the result: For all $w \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla w + \int_\Omega \overline{u}^{p-1} w = \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} w \tag{4.5}$$

Then $\overline{\lambda_1}$ is a eigenvalue and $\overline{u}$ is a positive eigenfunction associated. However by [4, Lemma 2.4], $\overline{\lambda_1} = \lambda_1$ and $\overline{u} > 0$ on $\overline{\Omega}$ by the maximum principle. But this contradicted the assumption that $\overline{u}$ vanishes in $\overline{\Omega}$. Consequently $\overline{u}$ vanishes only once on $\overline{\Omega}$, and

$$\overline{\lambda_1} = \int_\Omega |\nabla \overline{u}|^p + \int_\Omega |\overline{u}|^p = \inf_{u \in F} \Big(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p \Big)$$

with $\overline{u} > 0$ in $\overline{\Omega} \backslash \{x_0\}$ and $\overline{u}(x_0) = 0$.

Now one considers $u_\epsilon(x) = \max(\overline{u}(x), \epsilon)$ and $B_\epsilon = \{x \in \overline{\Omega} : \overline{u}(x) < \epsilon\}$ where $\epsilon > 0$. One has $u_\epsilon \to \overline{u}$, as $\epsilon \to 0$, in $W^{1,p}(\Omega)$, and

$$\int_{\partial\Omega} |u_\epsilon|^p \partial\sigma = \int_{\partial\Omega \backslash B_\epsilon} |\overline{u}|^p \partial\sigma + \epsilon^p \int_{\partial\Omega \cap B_\epsilon} \partial\sigma \underset{\epsilon \to 0}{\to} \int_{\partial\Omega} |\overline{u}|^p \partial\sigma = 1$$

Then $\int_{\partial\Omega} |u_\epsilon|^p \partial\sigma > 0$ for ϵ enough small. One hopes to show that

$$\lambda_1 \leq \frac{\int_\Omega |\nabla u_\epsilon|^p + \int_\Omega |u_\epsilon|^p}{\int_{\partial\Omega} |u_\epsilon|^p} < \overline{\lambda_1} = \int_\Omega |\nabla \overline{u}|^p + \int_\Omega |\overline{u}|^p$$

for ϵ enough small. We notice that

$$\int_\Omega |u_\epsilon|^p = \int_\Omega \overline{u}^p - \int_{\Omega \cap B_\epsilon} \overline{u}^p + \epsilon^p \operatorname{meas}(\Omega \cap B_\epsilon),$$

$$\int_{\partial\Omega} |u_\epsilon|^p = \int_{\partial\Omega} \overline{u}^p - \int_{B_\epsilon \cap \partial\Omega} \overline{u}^p + \epsilon^p \operatorname*{meas}_\sigma(B_\epsilon \cap \partial\Omega),$$

$$\int_\Omega |\nabla u_\epsilon|^p = \int_\Omega |\nabla \overline{u}|^p - \int_{\Omega \cap B_\epsilon} |\nabla \overline{u}|^p$$

and we conclude that

$$\begin{aligned} A_\varepsilon &= \frac{\int_\Omega |\nabla u_\epsilon|^p + \int_\Omega |u_\epsilon|^p}{\int_{\partial\Omega} |u_\epsilon|^p} - \int_\Omega |\nabla \overline{u}|^p - \int_\Omega |\overline{u}|^p \\ &= \frac{\int_\Omega |\nabla u_\epsilon|^p + \int_\Omega |u_\epsilon|^p - \int_{\partial\Omega} |u_\epsilon|^p (\int_\Omega |\nabla \overline{u}|^p + \int_\Omega \overline{u}^p)}{\int_{\partial\Omega} |u_\epsilon|^p} \\ &= \Big(\int_\Omega |\nabla \overline{u}|^p + \int_\Omega |\overline{u}|^p \Big) C_\epsilon \end{aligned}$$

where

$$C_\epsilon = \frac{\big(\int_{B_\epsilon \cap \partial\Omega} \overline{u}^p - \epsilon^p \operatorname{meas}_\sigma(B_\epsilon \cap \partial\Omega)\big) - \big(\int_{\Omega \cap B_\epsilon} |\nabla \overline{u}|^p + \int_{\Omega\cap B_\epsilon} \overline{u}^p - \epsilon^p mes(B_\epsilon)\big)}{1 - \int_{B_\epsilon \cap \partial\Omega} |\overline{u}|^p + \epsilon^p \operatorname{meas}_\sigma(B_\epsilon \cap \partial\Omega)}.$$

We will prove that $A_\varepsilon/\varepsilon$ converges towards a non positive value. For that we recall that $\overline{u}$ checks the property (4.4), and considering the sequence $v_\epsilon = \min(\overline{u}, \epsilon)$ we get

$$\int_{\Omega\cap B_\epsilon} |\nabla \overline{u}|^p + \int_{\Omega \cap B_\epsilon} \overline{u}^p = \overline{\lambda_1} \int_{\partial\Omega \cap B_\epsilon} \overline{u}^p + \epsilon (P_\varepsilon),$$

where $P_\varepsilon = \overline{\lambda_1} \int_{\partial\Omega \setminus B_\epsilon} \overline{u}^{p-1} - \int_{\Omega\setminus B_\epsilon} \overline{u}^{p-1} \to \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} - \int_\Omega \overline{u}^{p-1}$ as $\varepsilon \to 0$.

If $\overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} = \int_\Omega \overline{u}^{p-1}$, then (4.4) holds for $w \equiv 1$; consequently it holds also for all φ in $W^{1,p}(\Omega)$ because $\varphi = \psi + \varphi(x_0) w$ with $\psi = \varphi - \varphi(x_0)$. So $\overline{u}$ is a positive solution of the problem

$$\begin{gathered} \Delta_p v = |v|^{p-2} v \quad \text{in } \Omega, \\ |\nabla v|^{p-2} \frac{\partial v}{\partial \nu} = \overline{\lambda_1} |v|^{p-2} v \quad \text{on } \partial\Omega, \end{gathered}$$

then $\overline{u} > 0$ on $\overline{\Omega}$, absurdity since $\overline{u}(x_0) = 0$. So we deduce that $\overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} \neq \int_\Omega \overline{u}^{p-1}$.

If $\overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} < \int_\Omega \overline{u}^{p-1}$, then since $\int_{\partial\Omega\cap B_\epsilon} \overline{u}^p = o(\epsilon^p)$ one has

$$\frac{1}{\epsilon} \Big(\int_{\Omega \cap B_\epsilon} |\nabla \overline{u}|^p + \int_{\Omega\cap B_\epsilon} \overline{u}^p \Big) \underset{\epsilon \to 0}{\to} \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} - \int_\Omega \overline{u}^{p-1} < 0.$$

This implies that for ϵ enough small $\int_{\Omega\cap B_\epsilon} |\nabla \overline{u}|^p + \int_{\Omega\cap B_\epsilon} \overline{u}^p < 0$, which is not true. Then $\overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} > \int_\Omega \overline{u}^{p-1}$. Moreover $1 - \int_{B_\epsilon \cap \partial\Omega} \overline{u}^p + \epsilon^p \operatorname{meas}_\sigma(B_\epsilon \cap \partial\Omega) \to 1$ as $\epsilon \to 0$ and $\int_{B_\epsilon \cap \partial\Omega} \overline{u}^p - \epsilon^p \operatorname{meas}_\sigma(B_\epsilon \cap \partial\Omega) = o(\epsilon^p)$, then

$$\frac{1}{\epsilon} \Big(\frac{\int_\Omega |\nabla u_\epsilon|^p + \int_\Omega |u_\epsilon|^p}{\int_{\partial\Omega} |u_\epsilon|^p} - \int_\Omega |\nabla \overline{u}|^p - \int_\Omega |\overline{u}|^p \Big) \underset{\epsilon \to 0}{\to} \overline{\lambda_1} \int_{\partial\Omega} \overline{u}^{p-1} - \int_\Omega \overline{u}^{p-1} < 0$$

So for ϵ enough small $\lambda_1 \leq \frac{\int_\Omega |\nabla u_\epsilon|^p + \int_\Omega |u_\epsilon|^p}{\int_{\partial\Omega} |u_\epsilon|^p} < \int_\Omega |\nabla \overline{u}|^p + \int_\Omega |\overline{u}|^p = \overline{\lambda_1}$. Consequently, $\lambda_1 < \overline{\lambda_1}$. To complete the proof, one shows that there is no eigenvalue in $]\lambda_1, \overline{\lambda_1}]$. Let us suppose by absurdity that there exists an eigenvalue μ in $]\lambda_1, \overline{\lambda_1}]$, with associated eigenfunction v. By [4, Lemma 2.4], v changes sign on $\partial\Omega$, consequently it vanishes somewhere in $\overline{\Omega}$. Then $v \in G$, so

$$\overline{\lambda_1} \int_{\partial\Omega} |v|^p \leq \|\nabla v\|^p_{L^p(\Omega)} + \|v\|^p_{L^p(\Omega)} = \mu \int_{\partial\Omega} |v|^p$$

Since $\int_{\partial\Omega} |v|^p \neq 0$, one concluded that $\overline{\lambda_1} = \mu$. In this case v satisfies (4.5), and it vanishes only once on $\overline{\Omega}$. Applying (4.5) to $w = v^-$ one obtains

$$\overline{\lambda_1} \int_{\partial\Omega} |v^-|^p = \|\nabla v^-\|^p_{L^p(\Omega)} + \|v^-\|^p_{L^p(\Omega)},$$

then v^- vanishes only once on $\overline{\Omega}$. Contradiction since v changes sign on $\partial\Omega$. $\square$

Theorem 4.2. *If $p > N$*

(i) *For all $h \in L^\infty(\partial\Omega)$, such that $h \gneqq 0$ in $\overline{\Omega}$, if $\lambda \in]\lambda_1, \overline{\lambda_1}]$, then any solution u of* (1.1) *satisfies $u < 0$ in $\overline{\Omega}$.*
(ii) *$\overline{\lambda_1}$ is the largest number such that the preceding implication holds.*
(iii) *Given $h \in L^\infty(\partial\Omega)$, $h \gneqq 0$ in $\partial\Omega$, there exists $\delta = \delta(h) > 0$ such that if $\overline{\lambda_1} < \lambda < \overline{\lambda_1} + \delta$, and u is a solution of* (1.1) *then $u < 0$ on $\overline{\Omega}$.*

Theorem 4.3. *The AMP is not uniform on the right of $\overline{\lambda_1}$ for all $1 < p < \infty$; i.e., there is no δ independent of h satisfying (i).*

The proof of this result is a consequence of the following theorem.

Theorem 4.4. *Given $\epsilon > 0$, there exists $h \in L^\infty(\partial\Omega)$, such that $h \gneqq 0$ in $\partial\Omega$ and if $\overline{\lambda_1} + \epsilon < \lambda$ then problem* (1.1) *does not admit any non-positive solution.*

Proof of theorem 4.2 (i) Given $h \in L^\infty(\partial\Omega)$, such that $h \gneqq 0$ in $\partial\Omega$, and $\lambda \in]\lambda_1, \overline{\lambda_1}]$, then if u is a solution of (1.1), it does not satisfy $u \gneqq 0$ on $\overline{\Omega}$ by the Theorem 3.1. Taking $u^- \neq 0$ as testing function in (1.1), one gets

$$\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p = \lambda \int_{\partial\Omega} |u^-|^p - \int_{\partial\Omega} hu^- \leq \lambda \int_{\partial\Omega} |u^-|^p.$$

If $\int_{\partial\Omega} |u^-|^p = 0$, then $\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p = 0$, consequently $u^- \equiv 0$, this is not true, so $\int_{\partial\Omega} |u^-|^p > 0$. One deduces that

$$\frac{\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p}{\int_{\partial\Omega} |u^-|^p} \leq \lambda.$$

If $\lambda < \overline{\lambda_1}$ then

$$\frac{\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p}{\int_{\partial\Omega} |u^-|^p} \leq \lambda < \overline{\lambda_1} = \inf_{u\in G}\Big(\int_\Omega |\nabla u|^p + \int_\Omega |u|^p\Big),$$

so $u^- \notin G$ i-e u^- does not vanish any where on $\overline{\Omega}$. Consequently $u < 0$ on $\overline{\Omega}$. If $\lambda = \overline{\lambda_1}$ then $\frac{\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p}{\int_{\partial\Omega} |u^-|^p} \leq \overline{\lambda_1}$. Two cases are distinguished: If $u^- \notin G$ then $u < 0$ on $\overline{\Omega}$. But if $u^- \in G$ then $\frac{\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p}{\int_{\partial\Omega} |u^-|^p} = \overline{\lambda_1}$, and u^- vanishes only once on $\overline{\Omega}$. In this case one has the equality

$$\int_{\partial\Omega} hu^- = \lambda \int_{\partial\Omega} |u^-|^p - \int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p = 0,$$

so u^- vanishes on the set where $h \neq 0$ which is of positive measure. Absurdity. So $u^- \notin G$ and as previously $u < 0$ on $\overline{\Omega}$.
(ii) This part is a consequence of Theorem 4.4.

(iii) This result is similar as Theorem 4.1. As in his proof we suppose, by contradiction, that for all $k \in N^*$ there exists $(\mu_k)_k \subset \mathbb{R})$ such that $\overline{\lambda_1} < \mu_k < \overline{\lambda_1} + \frac{1}{k}$, and there exists $(u_k)_k \subset W^{1,p}(\Omega)$ holding

$$\begin{gathered}\Delta_p u_k = |u_k|^{p-2}u_k + h \quad \text{in } \Omega, \\ |\nabla u_k|^{p-2}\frac{\partial u_k}{\partial \nu} = \mu_k |u_k|^{p-2}u_k + h \quad \text{in } \partial\Omega,\end{gathered}$$

where u_k is not non-positive on $\overline{\Omega}$, and $u_k \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$. One distinguishes two cases, either $\|u_k\|_\infty$ remains bounded, or $\|u_k\|_\infty \to +\infty$. In the first case, one has for a subsequence $u_k \to u$ in $C^1(\overline{\Omega})$, and u solves

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \overline{\lambda_1}|u|^{p-2}u + h \quad \text{on } \partial\Omega.\end{gathered}$$

By part (i) one gets $u < 0$ on $\overline{\Omega}$. Contradiction. In the second case one considers $v_k = \frac{u_k}{\|u_k\|_\infty}$, so $\|v_k\|_\infty = 1$ and as previously for a subsequence $v_k \to v$ in $C^1(\overline{\Omega})$ with $\|v\|_\infty = 1$, and v solves

$$\begin{gathered}\Delta_p v = |v|^{p-2}v \quad \text{in } \Omega, \\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = \overline{\lambda_1}|v|^{p-2}v \quad \text{on } \partial\Omega.\end{gathered}$$

But this yields a contradiction since by Lemma 4.1, $\overline{\lambda_1}$ is not an eigenvalue.

Proof of theorem 4.4. Assume by contradiction that there exists $\varepsilon > 0$ such that for any $h \gneqq 0$ there exists $\lambda(h)$ with $\lambda(h) \geq \overline{\lambda_1} + \varepsilon$ such that (1.1) has a solution $u(h) < 0$ in $\overline{\Omega}$. We consider $\varphi \in C^1(\overline{\Omega})$ satisfying $\int_{\partial\Omega} |\varphi|^p \neq 0$ and φ vanishes in some ball in $\overline{\Omega}$, and we choose $h \in L^\infty(\partial\Omega)$, such that $h \gneqq 0$ in $\partial\Omega$ and $\operatorname{supp}\varphi \cap \operatorname{supp} h \cap \partial\Omega = \emptyset$. Applying Lemma 3.1 to $|\varphi|$ and $v = -u(h) > 0$ in $\overline{\Omega}$, we obtain

$$\lambda(h) \int_{\partial\Omega} |\varphi|^p \leq \int_\Omega |\nabla \varphi|^p + \int_\Omega |\varphi|^p,$$

Then

$$\overline{\lambda_1} + \varepsilon \leq \lambda(h) \leq \frac{\int_\Omega |\nabla\varphi|^p + \int_\Omega |\varphi|^p}{\int_{\partial\Omega} |\varphi|^p},$$

for all $\varphi \in W^{1,p}(\Omega)$ satisfying $\int_{\partial\Omega} |\varphi|^p \neq 0$ and φ vanishes in some ball in $\overline{\Omega}$, which contradicts the definition of $\overline{\lambda_1}$. $\square$

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