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ON A PROBLEM OF LOWER LIMIT IN THE STUDY OF NONRESONANCE WITH LERAY-LIONS OPERATOR

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Abstract. We prove the solvability of the Dirichlet problem

$$\begin{aligned} Au &= f(u) + h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

for a given h, under a condition involving only the asymptotic behaviour of the potential F of f, where A is a Leray-Lions operator.

1. Introduction and statement of results

This paper concerns the existence of solutions to the problem

$$\begin{aligned} Au &= f(u) + h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, A is an operator of the form $A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(\nabla u)$, f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and h is a given function on Ω. Also we consider the problem

$$\begin{aligned} -\Delta_p u &= f(u) + h \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

where Δ_p denotes the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$.

A classical result, essentially due to Hammerstein [9] asserts that if f satisfies a suitable polynomial growth restriction connected with the Sobolev imbeddings and if

$$\limsup_{x \to \pm\infty} \frac{2F(s)}{|s|^2} < \lambda_1, \tag{1.3}$$

then problem (1.2) with $p = 2$ is solvable for any h. Here F denotes the primitive $F(s) = \int_0^s f(t)dt$ and λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. Several improvements of this result have been considered in the recent years.

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In 1989, the case $N = 1$ and $p = 2$ was considered in [7]. It was shown there that (1.2) with $p = 2$ is solvable for any $h \in L^\infty(\Omega)$ if

$$\liminf_{x\to\pm\infty} \frac{2F(s)}{|s|^2} < \lambda_1. \tag{1.4}$$

When $N \geq 1$ and $p = 2$, showed later in [8] that (1.2) is solvable for any $h \in L^\infty(\Omega)$ if

$$\liminf_{s\to\pm\infty} \frac{2F(s)}{|s|^2} < \big(\frac{\pi}{2R(\Omega)}\big)^2, \tag{1.5}$$

where $R(\Omega)$ denotes the radius of the smallest open ball $B(\Omega)$ containing Ω. This result was extended to the p-Laplacian case in [5] where solvability of (1.2) was derived under the condition

$$\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < (p-1)\left\{\frac{1}{R(\Omega)}\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\right\}^p. \tag{1.6}$$

Note that this condition reduces to (1.5) when $p = 2$.

The question now naturally arises whether $(p-1)\big\{\frac{1}{R(\Omega)}\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\big\}^p$ can be replaced by λ_1 in (1.6), where λ_1 denotes the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ (cf[1]).

Observe that for $N > 1$ and $p = 2$, $(\frac{\pi}{2R(\Omega)})^2 < \lambda_1$, and a similar strict inequality holds when $1 < p < \infty$. In [2], it was showed that the constants in (1.5) and (1.6) can be improved a little bit.

Denote by $l(\Omega)$ the length of the smallest edge of an arbitrary parallelepiped containing Ω. If

$$\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < C_p(l) \tag{1.7}$$

where $C_p(l) = (p-1)\big\{\frac{2}{l(\Omega)}\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\big\}^p$ then for any $h \in L^\infty(\Omega)$ the problem (1.2) has a solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$.

Observe that for $N = 1$, $C_p = \lambda_1$ the first eigenvalue of $-\Delta$ on $\Omega =]0, l(\Omega)[$.

In particular, $C_2 = \big(\frac{\pi}{l}\big)^2$, and it recovers the result of [7]. It is clear that (1.7) is a weaker hypothesis than (1.6). The difference between (1.7) and (1.6) is particularly important when Ω is a rectangle or a triangle. However $C_p(l) < \lambda_1$ when $N > 1$, and the question raised above remains open.

In this paper we investigate the question of replacing Δ_p by the operator of the form

$$A(u) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} A_i(\nabla u).$$

We assume the following hypotheses:

- (A0) For all $i \in \{1, 2, \dots, N\}$, $A_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is continuous.
- (A1) there exists $(c, k) \in (]0, +\infty[)^2$ such that $|A_i(\xi)| \leq c|\xi|^{p-1} + K$ for all $\xi \in \mathbb{R}^{\mathbb{N}}$, and all $i \in \{1, 2, \dots, N\}$.
- (A2)
  - (a) $\sum_{i=1}^N (A_i(\xi) - A_i(\xi'))(\xi_i - \xi'_i) > 0$ for all $\xi \neq \xi' \in \mathbb{R}^N$;
  - (b) for all $i \in \{1, 2, \dots, N\}$, the function defined by $r_i(s) = A_i(0, \dots, 0, s, 0, \dots, 0)$ for $s \in \mathbb{R}$ is odd;
  - (c) for each $i \in \{1, 2, \dots, N\}$, there exists $a_i \in]0, +\infty[$ such that $\lim_{s\to+\infty} r_i(s)/s^{p-1} = a_i$;
  - (d) for each $i \in \{1, 2, \dots, N\}$, $r_i \in C^1(\mathbb{R}^*)$ and $\lim_{s\to 0} s r'_i(s) = 0$;

(e) for all $i \in \{1,2,\dots,N\}$, $A_i(\xi) = 0$ for all $\xi \in \mathbb{R}^N$ such that $\xi_i = 0$.

Remark 1.1. *(1) The hypothesis (A2)(d) is in particular satisfied if we suppose that for $i \in \{1,\dots,N\}$, $r_i \in C^1(\mathbb{R}^*)$ and there exists q_i, $1 < q_i < \infty$, there exists $\eta_i > 0$, there exists $(a,b) \in \mathbb{R}^2$, such that for all $|s| < \eta_i$, $|r_i'(s)| \le a|s|^{q_i-2} + b$.*
(2) The assumption (A2)(d) is an hypothesis of homogenization at infinity for the operator A.

Definition 1.2. For $i \in \{1,2,\dots,N\}$, we define

$$l_i(s) = \frac{1}{p-1}[sr_i(s) - \int_0^s r_i(t)dt] \quad \forall s \in \mathbb{R}.$$

Proposition 1.3. *Assume (A0), (A1) and (A2). Then: (1) The operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined, strictly monotone and*

$$\langle Au, v\rangle = \sum_{i=1}^N \int_\Omega A_i(\nabla u)\frac{\partial v}{\partial x_i}dx \quad \forall u, v \in W_0^{1,p}(\Omega).$$

(2) For each $i \in \{1,2,\dots,N\}$, the function $r_i : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and $r_i(0) = 0$.
(3) For each $i \in \{1,2,\dots,N\}$, the function l_i satisfies

(i) *l_i is even, continuous and $l_i(0) = 0$;*
(ii) $\lim_{s\to+\infty} \frac{l_i(s)}{s^p} = \frac{a_i}{p}$
(iii) *$l_i \in C^1(\mathbb{R})$ and* $l_i'(s) = \begin{cases} \frac{sr_i'(s)}{p-1} & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases}$
(iv) *l_i is strictly increasing in $\mathbb{R}^+$.*

Proof. (1) By (A0), (A1), it is clear that the operator A is defined from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, we have

$$\langle Au, v\rangle = \sum_{i=1}^N \int_\Omega A_i(\nabla u)\frac{\partial v}{\partial x_i}dx \quad \forall u, v \in W_0^{1,p}(\Omega)$$

and by (A1)(a), we verify easily that A is strictly monotone.

(2) Let $i \in \{1,\dots,N\}$. By (A0) and (A2)-(b), r_i is continuous and $r_i(0) = 0$, in the end r_i is strictly increasing. Indeed, let $(s,s') \in \mathbb{R}^2$ such that $s \neq s'$, we have

$$(r_i(s) - r_i(s'))(s - s') = \sum_{i=1}^N (A_i(\xi) - A_i(\xi'))(\xi_i - \xi_i') > 0$$

where $\xi = (0,\dots,s,\dots 0)$ and $\xi' = (0,\dots,s',\dots 0)$

(3)(i) By the foregoing, the function l_i is even, continuous and $l_i(0) = 0$ for every $i \in \{1,\dots,N\}$

(3)(ii) We show first that

$$\lim_{s\to+\infty} \frac{1}{s^p}\int_0^s r_i(t)dt = \frac{a_i}{p}. \tag{1.8}$$

Let $\varepsilon > 0$, by (A2)(c), there exists $\eta_\varepsilon = \eta$ such that $|r_i(s) - a_i s^{p-1}| \le \varepsilon s^{p-1}$ for all $s \ge \eta$.

Integrating from η to s, we obtain

$$\Big|\int_0^s r_i(t)dt - \int_0^\eta r_i(t)dt - \frac{a_i}{p}[s^p - \eta^p]\Big| \le \frac{\varepsilon}{p}[s^p - \eta^p].$$

Dividing by s^p and letting $n \to +\infty$, we obtain

$$\lim_{s\to+\infty} \big|\frac{1}{s^p}\int_0^s r_i(t)dt - \frac{a_i}{p}\big| = 0$$

i.e (1.8) holds. Writing

$$\frac{l_i(s)}{s^p} = \frac{1}{p-1}\big\{\frac{r_i(s)}{s^{p-1}} - \frac{1}{s^p}\int_0^s r_i(t)dt\big\}.$$

By (1.8) and (A2)(c), we have $\lim_{s\to+\infty}\frac{l_i(s)}{s^p} = \frac{a_i}{p}$

(3)(iii) Since $r_i \in C^1(\mathbb{R}^*)$, we have $l_i \in C^1(\mathbb{R}^*)$ and $l_i'(s) = \frac{1}{p-1}sr_i'(s)$ for every $s \neq 0$. On the other hand, for $s \neq 0$, since r_i is increasing and odd, we have

$$\big|\frac{l_i(s)}{s}\big| = \frac{1}{p-1}\big|r_i(s) - \frac{1}{s}\big|\int_0^s r_i(t)dt \le \frac{2}{p-1}r_i(|s|).$$

It results that $l_i'(0)$ exists and $l_i'(0) = 0$. By (A2)-(d) we obtain $\lim_{s\to0} l_i'(s) = \lim_{s\to0} sr_i'(s)$. This proves that $l_i \in C^1(\mathbb{R})$.

(3)(iv) is a consequence of (3)(iii) □

Example 1.4. We give at first some examples for operators A satisfying the hypothesis (A0), (A1) and (A2). (1) Let

$$Au = -\Delta_p u = -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i})$$

Then we have $A_i(\xi) = |\xi|^{p-2}\xi_i$ for every $\xi = (\xi_i) \in \mathbb{R}^N$.
$r(s) = r_i(s) = |s|^{p-2}s$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.
$l(s) = l_i(s) = \frac{1}{p}|s|^p$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.

(2) Let

$$Au = -\Delta_p u - \Delta_q u = -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i} + |\nabla u|^{q-2}\frac{\partial u}{\partial x_i})$$

where $1 < q < p < +\infty$. Then we have $A_i(\xi) = |\xi|^{p-2}\xi_i + |\xi|^{q-2}\xi_i$ for every $\xi = (\xi_i) \in \mathbb{R}^N$.
$r(s) = r_i(s) = |s|^{p-2}s + |s|^{q-2}s$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.
$l(s) = l_i(s) = \frac{1}{p}|s|^p + \frac{q-1}{q(p-1)}|s|^q$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.

(3) Let

$$Au = -\Delta_{p,\varepsilon} u = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\big[\varepsilon + |\nabla u|^2\big)^{\frac{p-2}{2}}\frac{\partial u}{\partial x_i}\big],$$

where $\varepsilon > 0$. Then we have $A_i(\xi) = (\varepsilon + |\xi|^2)^{\frac{p-2}{2}}\xi_i$ for every $\xi = (\xi_i) \in \mathbb{R}^N$.
$r(s) = r_i(s) = (\varepsilon + |s|^2)^{\frac{p-2}{2}}s$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.
$l(s) = l_i(s) = (\varepsilon + |s|^2)^{\frac{p-2}{2}}\big(\frac{s^2}{p} - \frac{\varepsilon}{p(p-1)}\big) + \frac{1}{p(p-1)}\varepsilon^{\frac{p}{2}}$ for every $s \in \mathbb{R}$ and every $i \in \{1,\dots,N\}$.

2. Proof of Main Theorem

We consider the Dirichlet problem (1.1) where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and $h \in L^\infty(\Omega)$.

Denote by $[AB]$ the smallest edge of an arbitrary parallelepiped containing Ω. Making an orthogonal change of variables, we can always suppose that AB is parallel to one of the axis of $\mathbb{R}^N$. So $\Omega \subset P = \prod_{j=1}^N [a_j, bj]$ with, for some i, $|AB| = b_i - a_i = \min_{1\leq j\leq N}\{b_j - a_j\}$, a quantity which we denote by $b - a$.

Denote by $l = l_i$, $r = r_i$, F the primitive $F(s) = \int_0^s f(t)dt$ and

$$C_p = (p-1)\big\{\frac{2}{b-a}\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\big\}^p.$$

Theorem 2.1. *Assume*

$$\liminf_{s\to\pm\infty} \frac{F(s)}{l(s)} < C_p. \tag{2.1}$$

Then for any $h \in L^\infty(\Omega)$, the problem (1.1) has a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the weak sense; i.e

$$\sum_{i=1}^N \int_\Omega A_i(\nabla u)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega f(u)\varphi + \int_\Omega h\varphi \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Definition 2.2. An upper solution for (1.1) is defined as a function $\beta : \overline{\Omega} \to \mathbb{R}$ such that

- $\beta \in C^1(\overline{\Omega})$
- $A(\beta) \in C(\overline{\Omega})$
- $A(\beta)(x) \geq f(\beta(x)) + h(x)$ a e x in Ω.

A lower solution α is defined by reversing the inequalities above.

Lemma 2.3. *Assume that (1.1) admits an upper solution β and a lower solution α with $\alpha(x) \leq \beta(x)$ in Ω. Then (1.1) admits a solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$, with $\alpha(x) \leq u(x) \leq \beta(x)$ in Ω.*

Proof. Let

$$\widetilde{f}(x,s) = \begin{cases} f(\beta(x)) & \text{if } s \geq \beta(x), \\ f(s) & \text{if } \alpha(x) \leq s \leq \beta(x), \\ f(\alpha(x)) & \text{if } s \leq \alpha(x) \end{cases}$$

for every $(x,s) \in \overline{\Omega} \times \mathbb{R}$ such that $\widetilde{f}$ is bounded and continuous in $\overline{\Omega} \times \mathbb{R}$, then the problem

$$\begin{gathered} Au = \widetilde{f}(x,u) + h(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

admits a solution $u \in W_0^{1,p}(\Omega)$ in the weak sense, indeed the operator A is strictly monotone, so we can use the result of Lions [10] concerned the pseudomonotones operators.

We claim that $\alpha(x) \leq u(x) \leq \beta(x)$ in Ω, which clearly implies the conclusion.

To prove the first inequality, one multiplies (2.2) by $w = u - u_\alpha$, where $u_\alpha(x) = \max(u(x), \alpha(x))$, integrates by parts and uses the fact that α is a lower solution we obtain $\langle A(u) - A(u-w), w\rangle \leq 0$, which implies $w = 0$ (since A is strictly monotone). $\square$

Lemma 2.4. *Let $a < b$ and $M > 0$, and assume*

$$\liminf_{s\to+\infty} \frac{F(s)}{l(s)} < C_p. \tag{2.3}$$

then there exists $\beta_1 \in C^1(I)$ such that $(r(\beta_1'(t)))' \in C(I)$ and

$$-(r(\beta_1'(t)))' \geq f(\beta_1(t)) + M \quad \forall t \in I,$$
$$\beta_1(t) \geq 0 \quad \forall t \in I$$

where $I = [a, b]$.

Lemma 2.5. *Assume*

$$\liminf_{s\to-\infty} \frac{F(s)}{l(s)} < C_p. \tag{2.4}$$

then there exists $\alpha_1 \in C^1(I)$ such that $(r(\alpha_1'(t)))' \in C(I)$ and

$$-(r(\alpha_1'(t)))' \leq f(\alpha_1(t)) - M \quad \forall t \in I$$
$$\alpha_1(t) \leq 0 \quad \forall t \in I$$

where $I = [a, b]$.

Accepting for a moment the conclusion of these two lemmas, let us turn to the Proof of Theorem 2.1. By Lemma 2.3 it suffices to show the existence of an upper solution and a lower solution for (1.1). Let us describe the construction of the upper solution (that of the lower solution is similar).

Let $M > \|h\|_\infty$ and $i \in \{1, 2, \dots, N\}$ such that $b = b_i$, $a = a_i$. By Lemma 2.4 there exists $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I)$, $(r(\beta_1'(t)))' \in C(I)$ and

$$-(r(\beta_1'(t)))' \geq f(\beta_1(t)) + M \quad \forall t \in I,$$
$$\beta_1(t) \geq 0 \quad \forall t \in I.$$

Writing $\beta(x) = \beta_1(x_i)$ for all $x = (x_i) \in \overline{\Omega}$, it is clear that $\beta \in C^1(\overline{\Omega})$, $A(\beta(x)) = A(\beta_1(x_i)) \in C(\overline{\Omega})$, and we have by (A2)(e):

$$\begin{aligned}
A(\beta(x)) &= -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} A_j(\nabla\beta(x)) \\
&= -\frac{\partial}{\partial x_i}(r_i(\beta_1'(x_i))) \\
&= -(r(\beta_1'(x_i)))' \\
&\geq f(\beta_1(x_i)) + M \\
&= f(\beta(x)) + M \\
&\geq f(\beta(x)) + h(x) \quad \text{a.e.} x \in \Omega
\end{aligned}$$

The proof of Theorem 2.1 is thus complete.

Next, we present the proof of Lemma 2.4. The proof of Lemma 2.5 follows similarly.

First case. Suppose $\inf_{s\geq 0} f(s) = -\infty$. Then there exists $\beta \in \mathbb{R}^*+$ such that $f(\beta) < -M$, and the constant function β provides a solution to the problem in Lemma 2.4.

Second case. Suppose now $\inf_{s\geq 0} f(s) > -\infty$. Let $K > M$ such that $\inf_{s\geq 0} f(s) > -K+1$. Thus $f(s)+K \geq 1$ for all $s \geq 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s)+K & \text{if } s \geq 0 \\ f(0)+K & \text{if } s < 0 \end{cases}$$

and denote $G(s) = \int_0^s g(t)dt$ for all s in $\mathbb{R}$. It is easy to see that $g(s) \geq 1$ for all s in $\mathbb{R}$ and that

$$0 \leq \liminf_{s\to+\infty} \frac{G(s)}{l(s)} = \liminf_{s\to+\infty} \frac{F(s)}{l(s)} < C_p.$$

Now it is clearly sufficient to prove the existence of a function $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I)$, $(r(\beta_1'(t)))' \in C(I)$ and

$$-(r(\beta_1'(t)))' = g(\beta_1(t)) \quad \forall t \in I$$
$$\beta_1(t) \geq 0 \quad \forall t \in I$$

For this purpose we will need the following four Lemmas.

Lemma 2.6. *Let $0 < c < \infty$ and $t \in]0,1[$, then*

$$\lim_{\alpha\to+\infty} \frac{\alpha}{l^{-1}(c(l(\alpha)-l(\alpha t)))} = \frac{1}{c^{1/p}(1-t^p)^{\frac{1}{p}}}$$

In particular by, Fatou Lemma,

$$\frac{1}{c^{1/p}} \int_0^1 \frac{dt}{(1-t^p)^{1/p}} \leq \liminf_{\alpha\to+\infty} \int_0^1 \frac{\alpha dt}{l^{-1}(c(l(\alpha)-l(\alpha t)))}$$

Proof. Denote $s(\alpha) = \frac{\alpha}{l^{-1}(c(l(\alpha)-l(\alpha t)))}$ and $\frac{a_i}{p} = d$. By Proposition 1.3 (3)(ii), we have

$$\lim_{s\to+\infty} \frac{s^{1/p}}{l^{-1}(s)} = d^{1/p}.$$

On the other hand,

$$\lim_{\alpha\to+\infty} [c(l(\alpha)-l(\alpha t))] = +\infty,$$

and more generally,

$$\lim_{\alpha\to+\infty} \frac{l(\alpha)-l(\alpha t)}{\alpha^p} = d(1-t^p) > 0\,.$$

Writing

$$s(\alpha) = \frac{1}{[\frac{c(l(\alpha)-l(\alpha t))}{\alpha^p}]^{1/p}} \frac{[c(l(\alpha)-l(\alpha t))]^{1/p}}{l^{-1}(c(l(\alpha)-l(\alpha t)))}$$

Letting $n \to +\infty$ and by the three limits above, we have

$$\lim_{\alpha\to+\infty} s(\alpha) = \frac{1}{c^{1/p}(1-t^p)^{1/p}}$$

□

Lemma 2.7. *For $d > 0$, define*

$$\tau_G(d) = \int_0^d \frac{ds}{l^{-1}[\frac{G(d)-G(s)}{p-1}]}\,.$$

Then

$$\limsup_{d\to+\infty} \tau_G(d) \geq \Big(\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\Big)\Big(\frac{1}{p-1} \liminf_{s\to+\infty} \frac{G(s)}{l(s)}\Big)^{1/p}.$$

In particular (2.3) *implies* $\limsup_{d\to+\infty}\tau_G(d) > (b-a)/2$.

Proof. Let ρ be a positive number such that $\liminf_{s\to+\infty}\frac{G(s)}{l(s)} < \rho < C_l$. Then $\limsup_{s\to+\infty}[\rho l(s) - G(s)] = +\infty$. Let w_n be the smallest number in $[0,n]$ such that $\max_{0\le s\le n} K(s) = K(w_n)$ where $K(s) = \rho l(s) - G(s)$; it is easily seen that (w_n) is increasing with respect to n. Since $\rho l(s) - G(s) < \rho l(w_n) - G(w_n)$ for all $s \in [0, w_n[$, we have $\frac{G(w_n)-G(s)}{p-1} < \frac{\rho}{p-1}(l(w_n) - l(s))$ for all $s \in [0, w_n[$, since $l : [0,+\infty[\to [0,+\infty[$ is an increasing *homeomorphism*, we have

$$\frac{1}{l^{-1}\big[\frac{\rho}{p-1}(l(w_n)-l(s))\big]} < \frac{1}{l^{-1}\big[\frac{1}{p-1}(G(w_n)-G(s))\big]}.$$

Integrating from 0 to w_n and changing variable $s = uw_n$ in the first member of inequality, we obtain

$$\int_0^1 \frac{w_n}{l^{-1}[\frac{\rho}{p-1}(l(w_n)-l(w_n s))]}ds \le \tau_G(w_n).$$

Letting $n\to+\infty$, we obtain

$$\liminf_{n\to+\infty}\int_0^1 \frac{w_n}{l^{-1}\big[\frac{\rho}{p-1}(l(w_n)-l(w_n s))\big]}ds \le \limsup_{n\to+\infty}\tau_G(w_n).$$

By Lemma 2.6, it results

$$\limsup_{d\to+\infty}\tau_G(d) \ge \Big[\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\Big]\Big[\frac{\rho}{p-1}\Big]^{\frac{-1}{p}}.$$

Letting $\rho \to \liminf_{s\to+\infty}\frac{G(s)}{l(s)}$, the Lemma is proved. □

Lemma 2.8. *Let $d > 0$ and consider the mapping T_d defined by*

$$T_d(u) = d - \int_a^t r^{-1}\Big(\Big[\int_a^\tau g(u(s))ds\Big]^{1/(p-1)}\Big)d\tau$$

in the Banach space $C(I)$. Then T_d has a fixed point.

Proof. Clearly by Ascoli's theorem T_d is compact. The proof of Lemma 2.8 uses an homotopy argument based on the Leray Schauder topological degree. So T_d will have a fixed point if the following condition holds:

There exists $\rho > 0$ such that $(I - \lambda T_d)(u) \ne 0$ for all $u \in \partial B(0,\rho)$ for all $\lambda \in [0,1]$, where $\partial B(0,\rho) = \{u \in C(I); \|u\|_\infty = \rho\}$.

To prove that this condition holds, suppose by contradiction that for all $n = 1, 2, \dots$ there exists $u_n \in \partial B(0,n)$, $\lambda_n \in [0,1]$ such that: $u_n = \lambda_n T_d(u_n)$. The latter relation implies

$$u_n = \lambda_n d - \lambda_n \int_a^t r^{-1}\Big(\Big[\int_a^\tau g(u(s))ds\Big]^{\frac{1}{p-1}}\Big)d\tau \tag{2.5}$$

Therefore, $u_n \in C^1(I)$ and we have successively

$$\begin{gathered} u_n'(t) = -\lambda_n r^{-1}\Big(\Big[\int_a^\tau g(u(s))ds\Big]^{\frac{1}{p-1}}\Big) < 0 \quad \forall t \in]a,b], \\ u_n'(a) = 0, \end{gathered} \tag{2.6}$$

$\big(r[\frac{u_n'(t)}{\lambda_n}]\big)' \in C(I)$ and

$$-\Big(r\Big(\frac{u_n'(t)}{\lambda_n}\Big)\Big)' = g(u_n(t)) \quad \forall t \in I. \tag{2.7}$$

Note that by (2.6), $u_n'(t) < 0$ in $]a,b]$, so that u_n is decreasing. Hence, for $n > d$, $u_n(b) = -n$. Multiplying the equation (2.7) by $u_n'(t)$, we obtain

$$-\lambda_n\Big(l\Big(\frac{u_n'(t)}{\lambda_n}\Big)\Big)' = \frac{1}{p-1}\frac{d}{dt}G(u_n(t)). \tag{2.8}$$

Indeed

$$\begin{aligned}\Big(l\Big(\frac{u_n'(t)}{\lambda_n}\Big)\Big)' &= \Big[l\Big(r^{-1}\Big(r\big(\frac{u_n'(t)}{\lambda_n}\big)\Big)\Big)\Big]' \\ &= \big(l\circ r^{-1}\big)'\Big(r\big(\frac{u_n'(t)}{\lambda_n}\big)\Big)\Big(r\big(\frac{u_n'(t)}{\lambda_n}\big)\Big)' \\ &= \frac{1}{p-1}\frac{u_n'(t)}{\lambda_n}\Big(r\big(\frac{u_n'(t)}{\lambda_n}\big)\Big)'\end{aligned}$$

By (2.8), we have

$$\lambda_n\Big(l\big(\frac{u_n'(t)}{\lambda_n}\big)\Big) = \frac{1}{p-1}(G(\lambda_n d) - G(u_n(t))$$

and

$$-\frac{u_n'(t)}{\lambda_n l^{-1}\big[\frac{G(\lambda_n d)-G(u_n(t))}{(p-1)\lambda_n}\big]} = 1.$$

Integrating from a to b and changing variable $s = u_n(t)$ $(u_n(a) = \lambda_n d$ and $u_n(b) = -n)$, we obtain

$$\int_{-n}^{\lambda_n d}\frac{ds}{\lambda_n l^{-1}\big[\frac{G(\lambda_n d)-G(s)}{(p-1)\lambda_n}\big]} = b-a$$

i.e.

$$\int_0^{\lambda_n d}\frac{ds}{\lambda_n l^{-1}\big[\frac{G(\lambda_n d)-G(s)}{(p-1)\lambda_n}\big]} = b-a+\int_0^{-n}\frac{ds}{\lambda_n l^{-1}\big[\frac{G(\lambda_n d)-G(s)}{(p-1)\lambda_n}\big]} \geq 0$$

Since $G(s) = sg(0)$ for $s \leq 0$ and changing variable $s = -u$, we obtain

$$0 \leq (b-a) - \int_0^n \frac{ds}{\lambda_n l^{-1}\big[\frac{G(\lambda_n d)-sg(0)}{(p-1)\lambda_n}\big]} \tag{2.9}$$

Denote by $l(u) = \frac{G(\lambda_n d)-G(s)}{(p-1)\lambda_n}$ such that $l'(u)du = \frac{g(0)}{(p-1)\lambda_n}ds$ and $ds = \frac{\lambda_n}{g(0)}r'(u)udu$ for $u \neq 0$ and denote $\alpha_n = l^{-1}\big[\frac{G(\lambda_n d)}{(p-1)\lambda_n}\big]$ and $\beta_n = l^{-1}\big[\frac{(G(\lambda_n d)+ng(0))}{(p-1)\lambda_n}\big]$. By (2.9), we obtain

$$\begin{aligned}0 &\leq (b-a) - \int_{\alpha_n}^{\beta_n}\frac{r'(u)}{g(0)}du \\ &= (b-a) - \frac{1}{g(0)}r\big\{l^{-1}\big[\frac{G(\lambda_n d)-ng(0)}{(p-1)\lambda_n}\big]\big\} + \frac{1}{g(0)}r\big\{l^{-1}\big[\frac{G(\lambda_n d)}{(p-1)\lambda_n}\big]\big\}.\end{aligned}$$

Since

$$\frac{G(\lambda_n d)-ng(0)}{(p-1)\lambda_n} \geq \frac{ng(0)}{(p-1)}, \quad \frac{G(\lambda_n d)}{(p-1)\lambda_n} \leq \frac{d}{p-1}\max_{0\leq s\leq d}|g(s)|$$

and $r \circ l^{-1}$ is increasing, it results that

$$0 \leq (b-a) - \frac{1}{g(0)} r\big\{l^{-1}\big[\frac{ng(0)}{(p-1)\lambda_n}\big]\big\} + \frac{1}{g(0)} r\big\{l^{-1}\big[\frac{d}{p-1}\max_{0\leq s\leq d}|g(s)|\big]\big\}.$$

Letting $n \to +\infty$, we get a contradiction. Let us denote by $u_d \in C(I)$ a fixed point of the mapping T_d of Lemma 2.8 □

Lemma 2.9. *There exists $d > 0$ such that $u_d(t) \geq 0$ for all $t \in [a, \frac{a+b}{2}[$.*

Proof. We know that u_d is decreasing and that $u_d(a) = d$ for all $d > 0$. Let us distinguish two cases.

First if there exists $d > 0$ such that $u_d(b) \geq 0$, then the conclusion of Lemma 2.9 clearly follows. So we can assume that $u_d(b) < 0$ for every $d > 0$. Since $u_d(a) = d > 0$, there exists $\delta_d \in]a, b[$ such that $u_d(\delta_d) = 0$. It is clear that $u_d(t) \geq 0$ for all $t \in [a, \delta_d[$, and so it is sufficient to show that $\limsup_{d\to+\infty} \delta_d > \frac{a+b}{2}$. Processing as in the proof of Lemma 2.8, we obtain

$$-u_d'(t)\big\{l^{-1}\big(\frac{G(d) - G(u_d(t))}{p-1}\big)\big\}^{-1} = 1.$$

Integrating from a to δ_d and changing variable $s = u_d(t)$, one gets

$$\tau_G(d) = \int_0^d \frac{ds}{l^{-1}\big[\frac{G(d)-G(s)}{p-1}\big]} = \delta_d - a,$$

consequently

$$\limsup_{d\to+\infty} \delta_d > a + \frac{b-a}{2} = \frac{a+b}{2}$$

□

Proof of Lemma 2.4 continued. . Denoting $u_d(t)$ by $u(t)$, we have $u \in C^1(I)$, $(r(u'))' \in C(I)$ and

$$-(r(u'))' = g(u(s)) \quad \forall t \in I,$$
$$u(t) \geq 0 \quad \forall t \in [a, \frac{a+b}{2}[,$$
$$u'(a) = 0.$$

Define a function β_1 from $[a, b]$ to $\mathbb{R}$ by

$$\beta_1(t) = \begin{cases} u(\frac{3a+b}{2} - t) & \text{if } t \in [a, \frac{a+b}{2}], \\ u(t - \frac{b-a}{2}) & \text{if } t \in [\frac{a+b}{2}, b]. \end{cases}$$

We will show that this function β fulfills the conditions of Lemma 2.4. To see this it is sufficient to show that

(a) β_1 is nonnegative in $[a, b]$,
(b) $\beta_1 \in C^1([a, b])$,
(c) $(r(\beta_1'(t)))' \in C([a, b])$ and $-(r(\beta_1'(t)))' = g(\beta_1(t))$ for all $t \in [a, b]$.

Proof of (a). If $a \leq t \leq \frac{a+b}{2}$, then $a \leq \frac{3a+b}{2} - t \leq \frac{a+b}{2}$, and if $\frac{a+b}{2} \leq t \leq b$, then $a \leq t - \frac{b-a}{2} \leq \frac{a+b}{2}$, so that the conclusion follows from the sign of u on $[a, \frac{a+b}{2}]$.

Proof of (b). $\beta_1 \in C^1([a, \frac{a+b}{2}[)$, $\beta_1 \in C^1(]\frac{a+b}{2}, b])$, and moreover $\frac{d}{dt^+}\beta_1(\frac{a+b}{2}) = u'(a) = 0$ and $\frac{d}{dt^-}\beta_1(\frac{a+b}{2}) = u'(a) = 0$.

Proof of (c). We know that $-(r(u'(t)))' = g(u(t))$ for $t \in [a,b]$, therefore

$$-(r(u'(t)) = \int_a^t g(u(s))ds.$$

If $a \le t \le \frac{a+b}{2}$ then $a \le \frac{3a+b}{2} - t \le \frac{a+b}{2}$, which gives

$$\beta_1(t) = u\big(\frac{3a+b}{2} - t\big) \quad \text{and} \quad \beta_1'(t) = -u'\big(\frac{3a+b}{2} - t\big).$$

We obtain

$$-(r(u'(\frac{a+b}{2} - t)) = r(\beta_1'(t)).$$

The change of variable $u = \frac{3a+b}{2} - s$ yields

$$\int_a^{\frac{3a+b}{2}-t} g(u(s))ds = \int_t^{\frac{a+b}{2}} g(u(\frac{3a+b}{2} - s))ds,$$

hence

$$r(\beta_1'(t)) = \int_t^{\frac{a+b}{2}} g(\beta_1(s))ds \quad \forall t \in [a, \frac{a+b}{2}]$$

and

$$-(r(\beta_1'(t)))' = g(\beta_1(t)) \quad \forall t \in [a, \frac{a+b}{2}]$$

The proof is similar for all $t \in [\frac{a+b}{2}, b]$. □

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