2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A SOLUTION METHOD OF THE SIGNORINI PROBLEM

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Abstract. In this note, we are interested in the variational formulation of an unilateral contact problem, the so-called Signorini problem. We show that the normal derivative of the solution, can be computed according to the data of the problem. What permits the determination of the solution by solving a Neumann problem.

1. Introduction

In this work, we are interested in the Signorini problem that consists in finding u such that

$$\begin{gathered} -\Delta u + u = f \quad \text{in } \Omega, \\ u \geq 0, \frac{\partial u}{\partial n} \geq 0, \quad u\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^2$, which boundary Γ is regular enough, and $f \in L^2(\Omega)$. Another unknown of the problem is the coincidence set on the border of Ω.

It is known that the variational principle applied to (1.1) produces the variational inequality that consists in finding u such that

$$a(u, v - u) \geq \int_\Omega f.(v - u) \quad \forall v \in K, \tag{1.2}$$

where a is the scalar product of $H^1(\Omega)$ and $K = \{v \in H^1(\Omega) : v \geq 0 \text{ on } \Gamma\}$.

As in [3], we introduce the function ψ, a solution of the Dirichlet problem,

$$\begin{gathered} -\Delta\psi + \psi = f \quad \text{in } \Omega \\ \psi = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.3}$$

and we show that the solution of (1.2) is the solution of the variational inequality

$$a(u, v - u) \geq \int_\Omega f.(v - u) \quad \forall v \in K_\psi, \tag{1.4}$$

where $K_\psi = \{v \in H^1(\Omega) : v \geq \psi \text{ in } \Omega\}$.

2000 *Mathematics Subject Classification.* 35J85, 35J35.
Key words and phrases. Signorini problem; variational inequality.

Published September 20, 2006.

To solve problem (1.4), we propose a method that consists in determining first $\frac{\partial u}{\partial n}$, with the help of the data of the problem, then solving a Neumann problem to determine u.

2. Formulation of the problem

Let Ω be a bounded open subset of $\mathbb{R}^2$, with boundary Γ is regular enough. The Sobolev space $H^1(\Omega)$ is equipped with its scalar product

$$a(u,v)=\int_\Omega \nabla u.\nabla v dx+\int_\Omega u.v dx \quad \forall u,v\in H^1(\Omega).$$

We denote by K the closed convex cone of $H^1(\Omega)$ defined as

$$K=\{v\in H^1(\Omega): v\geq 0 \text{ on } \Gamma\}.$$

Then we consider the variational inequality problem: Find $u\in K$ such that

$$a(u,v-u)\geq \int_\Omega f.(v-u), \quad \forall v\in K, \tag{2.1}$$

where $f\in L^2(\Omega)$.

In the sequel, we use of the following notations:

$$v^+=\max(v,0), \quad v^-=\min(v,0)$$

such that $v=v^++v^-$ for all v in $L^2(\Omega)$ or in $L^2(\Gamma)$.

We will note indifferently a function v of $H^1(\Omega)$ and its restriction to Γ.

Proposition 2.1. *Problem* (2.1) *is equivalent to the problem: Find $u\in K_\psi$ such that*

$$a(u,v-u)\geq \int_\Omega f.(v-u), \quad \forall v\in K_\psi, \tag{2.2}$$

where ψ is given by (1.3) *and $K_\psi=\{v\in H^1(\Omega): v\geq\psi \text{ in } \Omega\}$.*

Proof. It suffices to show that the solution u of (2.1) is in K_ψ, i.e: $u\geq\psi$.

Let u be the solution of (2.1). With $v=u-(u-\psi)^-\in K$ in (2.1), and while taking account of (1.3), one has

$$a((u-\psi)^-,(u-\psi)^-)\leq 0.$$

So $(u-\psi)^-=0$, which implies $u\geq\psi$ in Ω. □

Using Green's formula, (2.1) can be written as: Find $u\in K_\psi$ such that

$$a(u-\psi,v-u)+\langle\frac{\partial\psi}{\partial n},v-u\rangle\geq 0, \quad \forall v\in K_\psi, \tag{2.3}$$

where $\langle .,.\rangle$ designates the scalar product of $L^2(\Gamma)$, and $\frac{\partial\psi}{\partial n}$ the normal derivative of ψ.

To have $\frac{\partial\psi}{\partial n}$ at the same time as ψ, we propose the resolution of (1.3) by enforcing the boundary condition with Lagrange multipliers (see [4]), then the variational formulation of (1.3) reads: Find $(\psi,\lambda)\in H^1(\Omega)\times H^{-\frac12}(\Gamma)$ such that

$$\begin{gathered} a(\psi,v)-\int_\Gamma \lambda v=\int_\Omega f v, \quad \forall v\in H^1(\Omega),\\ \int_\Gamma \mu\psi=0, \quad \forall\mu\in H^{-\frac12}(\Gamma), \end{gathered} \tag{2.4}$$

whose solution is the saddle point of the Lagrangian $\mathcal{J}(v,\mu) = \frac{1}{2}a(v,v) - \int_\Omega fv - \int_\Gamma \mu v$.

It is known (see [4]) that problem (2.4) has a unique solution (ψ,λ), verifying

$$\begin{gathered} -\Delta\psi + \psi = f \quad \text{in } \Omega, \\ \lambda = \frac{\partial\psi}{\partial n} \quad \text{on } \Gamma, \\ \psi = 0 \quad \text{on } \Gamma. \end{gathered} \tag{2.5}$$

Note that f being in $L^2(\Omega)$, implies ψ in $H^2(\Omega)$ which implies $\frac{\partial\psi}{\partial n}$ in $H^{1/2}(\Gamma)$.

3. Transformation of the problem

Hereafter we deal with the problem (2.3), and we consider the function $\varphi : L^2(\Gamma) \to \mathbb{R}$ defined as

$$\nu \mapsto \langle (\frac{\partial\psi}{\partial n})^+, \nu^+ \rangle.$$

Note that φ is convex and continuous on $L^2(\Gamma)$.

Proposition 3.1. *A function u is a solution of (2.3) if and only if $w = u - \psi$ is the solution of the problem: Find $w \in H^1(\Omega)$ such that*

$$a(w, v-w) + \varphi(v) - \varphi(w) + \langle (\frac{\partial\psi}{\partial n})^-, v - w\rangle \geq 0, \quad \forall v \in H^1(\Omega). \tag{3.1}$$

Proof. Since the two problems (2.3) and (3.1) have an unique solution, it suffices to show that the solution w of (3.1) is non negative in Ω. Let w be the solution of (3.1), with $v = w^+$ in (3.1), we have

$$a(w, -w^-) + \langle (\frac{\partial\psi}{\partial n})^-, -w^-\rangle \geq 0$$

then $a(w^-, w^-) \leq \langle (\frac{\partial\psi}{\partial n})^-, -w^-\rangle \leq 0$, so $w^- = 0$. □

We remark that problem (3.1) differs from (2.3) by the fact that it is without constraints.

Proposition 3.2. *A function w is solution of problem (3.1) if and only if (w,μ) is solution of the problem: Find $(w,\mu) \in H^1(\Omega) \times L^2(\Gamma)$ such that*

$$\begin{gathered} a(w,v) + \langle \mu, v\rangle + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle = 0 \quad \forall v \in H^1(\Omega), \\ \mu \in \partial\varphi(w), \end{gathered} \tag{3.2}$$

where $\partial\varphi(w) = \{\lambda \in L^2(\Gamma) : \forall \nu \in L^2(\Gamma), \langle \lambda, \nu - w\rangle \leq \varphi(\nu) - \varphi(w)\}$ designates the subdifferential of φ at w.

Proof. Let w be the solution of (3.1). On the one hand, for all v in $H_0^1(\Omega)$, we have $a(w,v) = 0$, hence the mapping

$$v \mapsto a(w,v) + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle$$

is well defined on Γ.

On the other hand, for all v in $H^1(\Omega)$, one has

$$
\begin{aligned}
a(w,v) + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle &\geq \varphi(w) - \varphi(v+w) \\
&\geq \langle (\frac{\partial\psi}{\partial n})^+, w^+ - (v+w)^+\rangle \\
&\geq -\langle (\frac{\partial\psi}{\partial n})^+, v^+\rangle \\
&\geq -\|(\frac{\partial\psi}{\partial n})^+\|_{L^2(\Gamma)} \|v_{|\Gamma}\|_{L^2(\Gamma)}.
\end{aligned}
$$

Using the same argument, and with $-v$ one has

$$a(w,-v) + \langle (\frac{\partial\psi}{\partial n})^-, -v\rangle \geq -\|(\frac{\partial\psi}{\partial n})^+\|_{L^2(\Gamma)} \|v_{|\Gamma}\|_{L^2(\Gamma)}.$$

Hence for all v in $H^1(\Omega)$, we have

$$|a(w,v) + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle| \leq \|(\frac{\partial\psi}{\partial n})^+\|_{L^2(\Gamma)} \|v_{|\Gamma}\|_{L^2(\Gamma)}.$$

Therefore, the linear form $v \mapsto a(w,v) + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle$ is continuous on $H^{1/2}(\Gamma)$ for the norm of $L^2(\Gamma)$. Then because of the density of $H^{1/2}(\Gamma)$ in $L^2(\Gamma)$, there exists μ in $L^2(\Gamma)$ verifying the equality of (3.2).

Inversely, it is easy to see that if (w,μ) is solution of (3.2) then w is solution of (3.1). $\square$

To characterize $\partial\varphi(w)$, we consider the closed convex set of $L^2(\Gamma)$:

$$C = \{\lambda \in L^2(\Gamma) : \forall \nu \in L^2(\Gamma) : \langle \lambda, \nu\rangle \leq \varphi(\nu)\}.$$

It is easy to show that $C = \{\lambda \in L^2(\Gamma) : 0 \leq \lambda \leq (\frac{\partial\psi}{\partial n})^+ \text{ a.e in } \Gamma\}$.

Lemma 3.3. *for all μ in $L^2(\Gamma)$ and w in $H^1(\Omega)$, one has $\mu \in \partial\varphi(w)$ if and only if*

$$\mu \in C \quad \text{and} \quad \langle \lambda - \mu, w\rangle \leq 0, \quad \forall \lambda \in C.$$

In particular $(\frac{\partial\psi}{\partial n})^+ . \chi_{[w_{|\Gamma} > 0]} \in \partial\varphi(w)$.

Proof. Let $\mu \in \partial\varphi(w)$, i.e,

$$\forall v \in H^1(\Omega), \quad \langle \mu, v - w\rangle \leq \varphi(v) - \varphi(w). \tag{3.3}$$

With $v = 2w$ and $v = 0$ in (3.3), and while φ is positively homogeneous, one has

$$\langle \mu, w\rangle = \varphi(w).$$

Therefore, while taking back (3.3), one deducts that $\mu \in C$. Inversely, let $\mu \in C$ and $\langle \lambda - \mu, w\rangle \leq 0$, for all $\lambda \in C$. For λ in $\partial\varphi(w) \subset C$, we have

$$\langle \lambda, w\rangle = \varphi(w) \text{ and } \langle \lambda - \mu, w\rangle = 0.$$

So that for all $v \in H^1(\Omega)$, $\langle \mu, v - w\rangle = \langle \mu, v\rangle - \varphi(w) \leq \varphi(v) - \varphi(w)$. $\square$

Taking into account lemma 3.3, the problem (3.2) can be written as: Find $(w,\mu) \in H^1(\Omega) \times C$ such that

$$
\begin{gathered}
a(w,v) + \langle \mu, v\rangle + \langle (\frac{\partial\psi}{\partial n})^-, v\rangle = 0 \quad \forall v \in H^1(\Omega), \\
\langle \lambda - \mu, w\rangle \leq 0, \quad \forall \lambda \in C.
\end{gathered}
\tag{3.4}
$$

Remark 3.4. The bilinear form a is symmetric, then the problem (3.4) is equivalent to the saddle point problem [2]: Find $(w,\mu) \in H^1(\Omega) \times C$ such that

$$\mathcal{L}(w,\lambda) \le \mathcal{L}(w,\mu) \le \mathcal{L}(v,\mu) = 0 \quad \forall (v,\lambda) \in H^1(\Omega) \times C,$$

where the Lagrangian $\mathcal{L}$ is defined on $H^1(\Omega) \times L^2(\Gamma)$ by

$$\mathcal{L}(v,\lambda) = \frac{1}{2} a(v,v) + \langle \lambda, v\rangle + \langle (\frac{\partial \psi}{\partial n})^-, v\rangle$$

The study of the mixed formulation (3.4), is in preparation by the authors. However, it is clear that the determination, a priori, of μ, permits to solve the problem (3.4) as being the Neumann problem

$$\begin{gathered} -\Delta w + w = 0 \quad \text{in } \Omega, \\ \frac{\partial w}{\partial n} = -\mu - (\frac{\partial \psi}{\partial n})^- \quad \text{on } \Gamma. \end{gathered}$$

Which defines u as a solution of the problem

$$\begin{gathered} -\Delta u + u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = (\frac{\partial \psi}{\partial n})^+ - \mu \quad \text{on } \Gamma, \end{gathered}$$

What we propose in this work is to uncouple the problem (3.4), by showing that μ can be computed according to the data of the initial problem. We consider, then, the linear mapping $A : H^{-\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ defined by $g \mapsto v$, where v is a solution to

$$\begin{gathered} -\Delta v + v = 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial n} = g \quad \text{on } \Gamma, \end{gathered}$$

which is continuous, more precisely, one has the following result [4, theorem 2.7].

Lemma 3.5. *There exist two positive constants c_1 and c_2, such that*

$$c_1 \|Ag\|_{H^1(\Omega)} \le \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \le c_2 \|Ag\|_{H^1(\Omega)},$$

for all g in $H^{-1/2}(\Gamma)$.

Using the set $M = A(C) = \{v \in H^1(\Omega) : -\Delta v + v = 0 \text{ in } \Omega \text{ and } \frac{\partial v}{\partial n} \in C\}$ which is closed, bounded and convex in $H^1(\Omega)$, problem (3.4), can be written as: Find $(w,z) \in H^1(\Omega) \times M$ such that

$$\begin{gathered} a(w+z+t, v) = 0 \quad \forall v \in H^1(\Omega), \\ a(s-z, w) \le 0, \quad \forall s \in M, \end{gathered}$$

where $t = A\big((\frac{\partial \psi}{\partial n})^-\big)$ and $z = A\mu$. What defines z as being the projection of $-t$ on M; i.e,

$$t = A\big((\frac{\partial \psi}{\partial n})^-\big), z = \operatorname{proj}_M(-t) \quad \text{and} \quad w = -z - t.$$

4. A projection algorithm

For the determination of $z = \operatorname{proj}_M(-t)$, and therefore μ, according to the data, we propose a projection algorithm inspired of the one of Degueil [1].

Description of the algorithm. For an initial guess $\mu_0 \in C$ (for example $\mu_0 = 0$), we compute z_0 such that $z_0 = A\mu_0$ then we construct the sequences (μ_n) and (z_n), as follows:

(1) Given z_n and μ_n such that $z_n = A\mu_n$ and $\mu_n \in C$, we compute $v_n = Ag_n$ where
$$g_n = (\frac{\partial\psi}{\partial n})^+ . \chi_{[(-t-z_n)_{|\Gamma}>0]}.$$
By lemma 3.3, we see that $g_n \in \partial\varphi(-t-z_n) \subset C$, and then:
$$a(v - v_n, z_n + t) \geq 0, \ \forall v \in M. \tag{4.1}$$

(2) We compute $z_{n+1} = \text{proj}_{[z_n,v_n]}(-t)$ (the projection of $-t$ on the segment $[z_n, v_n]$), i.e,
$$\begin{gathered} z_{n+1} = \lambda_n v_n + (1-\lambda_n) z_n, \\ \mu_{n+1} = \lambda_n g_n + (1-\lambda_n)\mu_n \\ \lambda_n = \min\Big(1, \frac{a(z_n - v_n, z_n + t)}{\|v_n - z_n\|^2_{H^1(\Omega)}} = \frac{\langle \mu_n - g_n, z_n + t\rangle}{\langle \mu_n - g_n, v_n - z_n\rangle}\Big) \end{gathered} \tag{4.2}$$
if $v_n = z_n$ then $z_{n+1} = z_n$, the algorithm stops.

Convergence result. As in [1], we have the following convergence result.

Theorem 4.1. *With the hypothesis and notation above, one has*
$$\lim_{n\to+\infty} \|z_n - z\|_{H^1(\Omega)} = \lim_{n\to+\infty} \|\mu_n - \mu\|_{H^{-\frac12}(\Gamma)} = 0.$$

Proof. Let us, first, show that $\lim_{n\to+\infty} \|z_n - z\|_{H^1(\Omega)} = 0$. For all n, one has:
$$\begin{aligned} \|z_n + t\|^2_{H^1(\Omega)} &= \|z_n - z_{n+1}\|^2_{H^1(\Omega)} + \|z_{n+1} + t\|^2_{H^1(\Omega)} + 2a(z_n - z_{n+1}, z_{n+1} + t) \\ &\geq \|z_n - z_{n+1}\|^2_{H^1(\Omega)} + \|z_{n+1} + t\|^2_{H^1(\Omega)} \end{aligned}$$
Then $\|z_n - z_{n+1}\|^2_{H^1(\Omega)} \leq \|z_n + t\|^2_{H^1(\Omega)} - \|z_{n+1} + t\|^2_{H^1(\Omega)}$, what implies that
$$\lim_{n\to+\infty} \|z_n - z_{n+1}\|_{H^1(\Omega)} = 0. \tag{4.3}$$
On the other hand, $z = \text{proj}_M(-t)$, then
$$a(v - z, z + t) \geq 0, \quad \forall v \in M. \tag{4.4}$$
With $v = z_{n+1}$ in (4.4) and while taking account of ((4.1) and (4.2)), one has
$$\begin{aligned} &\|z_{n+1} - z\|^2_{H^1(\Omega)} \\ &= a(z_{n+1} - z, z_{n+1} - z_n) + a(z_{n+1} - z, z_n + t) - a(z_{n+1} - z, z + t) \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + a(z_{n+1} - z, z_n + t) \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + a(z_{n+1} - z, z_n + t) + a(z - v_n, z_n + t) \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + a(z_{n+1} - v_n, z_n + t) \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + (1-\lambda_n) a(z_n - v_n, z_n + t) \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + (1-\lambda_n)\lambda_n \|z_n - v_n\|^2_{H^1(\Omega)} \\ &\leq a(z_{n+1} - z, z_{n+1} - z_n) + (1-\lambda_n) \|z_{n+1} - z_n\|_{H^1(\Omega)} \|z_n - v_n\|_{H^1(\Omega)} \\ &\leq \|z_{n+1} - z_n\|_{H^1(\Omega)} \big\{ \|z_{n+1} - z\|_{H^1(\Omega)} + \|z_n - v_n\|_{H^1(\Omega)} \big\}. \end{aligned}$$
We conclude using (4.3) and the fact that M is bounded. To show that $\lim_{n\to+\infty} \|\mu_n - \mu\|_{H^{-\frac12}(\Gamma)} = 0$, it suffices to use Lemma 3.5. □

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