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THE DIVISION METHOD FOR SUBSPECTRA OF SELF-ADJOINT DIFFERENTIAL VECTOR-OPERATORS

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ABSTRACT. We discuss the division method for subspectra which appears to be one of the key approaches in the study of spectral properties of self-adjoint differential vector-operators, that is operators generated as a direct sum of self-adjoint extensions on an Everitt-Markus-Zettl multi-interval system. In the current work we show how the division method may be applied to obtain the ordered spectral representation and Fourier-like decompositions for self-adjoint differential vector-operators, after which we obtain the analytical decompositions for the measurable (relative to a spectral parameter) generalized eigenfunctions of a self-adjoint differential vector-operator.

1. INTRODUCTION

Problem Overview. We begin with a physical example of a Schrödinger vector-operator. Gesztesy and Kirsch [6] in particular considered a Schrödinger operator generated by the Hamiltonian

$$H = -\frac{d^2}{dx^2} + \big(s^2 - \frac{1}{4}\big)\frac{1}{\cos^2 x}\,, \quad s > 0\,. \tag{1.1}$$

Since the potential in Hamiltonian has a countable number of singularities on a discreet set X in $\mathbb{R}$, leading to spoiling of the local integrability, it is impossible to apply the standard methods of the theory of ordinary differential operators. In order to proceed and build a self-adjoint extension of a minimal operator generated by (1.1) on $\mathbb{R} \setminus X$, one may take self-adjoint extensions T_i, generated by the same Hamiltonian (1.1) in the coordinate spaces $L^2(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi)$, $i \in \mathbb{Z}$, and then consider the direct sum operator $\oplus_{i\in\mathbb{Z}} T_i$ in the space

$$\oplus_{i\in\mathbb{Z}} L^2\big(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\big)\,.$$

This direct sum operator appears to be one of the possible self-adjoint extensions of the minimal operator considered on $\mathbb{R} \setminus X$. Moreover in the case $s \geqslant 1$ the minimal

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on $\mathbb{R} \setminus X$ operator appears to be essentially self-adjoint and its only self-adjoint extension is a direct sum operator.

This physical example gave birth to the theory of general differential direct sum operators, or in the text below vector-operators. Beginning from 1992, the theory of differential vector-operators has been investigated in connection with their non-spectral properties in a Hilbert space ([1], [2, 3] and in complete locally convex spaces [4, 5]. The interest in such a theory is explained by its numerous applications in theoretical physics and pure mathematics. Thus, physical applications may be found in a single or a multi-particle quantum mechanics, especially in problems where a quantum system is split into a number of disconnected subsystems under the influence of a potential. For applications in quantum mechanics see also the respective references in [3].

As it was shown in the fundamental works [2] and [3], a differential vector-operator is an object which resembles an ordinary differential operator by its general properties, but in fact it has much more complicated structure.

Although the bigger part of studies concerned only non-spectral properties of differential vector-operators, there has been some development of their spectral theory recently. Some results describing position of spectra of Schrödinger vector-operators were presented in 1985 in [6] and the most recent results for general quasi-differential vector-operators belong to Sobhy El-Sayed Ibrahim [7, 8].

The internal spectral structure of abstract self-adjoint vector-operators was first investigated in [9], for which see also [10, 11]. The structure of coordinate operators as differential operators played the key role in [12] where the unitary transformation making the ordered representation was described in terms of generalized eigenfunctions of a differential vector-operator. These generalized eigenfunctions appear to be only measurable relative to the spectral parameter, therefore it is an essential problem to obtain their decomposition over some set of analytical kernels. This problem is positively solved by Theorem 2.4 of the current work.

Mathematical background. Basic concepts of quasi-differential operators are well described in [2, 3]. A good reference for operators with real coefficients is the book of Naimark [13].

Let Ω be a finite or a countable set of indices. On Ω, we have a multi-interval differential Everitt-Markus-Zettl system $\{I_i, \tau_i\}_{i\in\Omega}$, where I_i are arbitrary intervals of the real line and τ_i are formally self-adjoint differential expressions of a finite order. This EMZ system generates a family of Hilbert spaces $\{L^2(I_i) = L^2_i\}_{i\in\Omega}$ and families of minimal $\{T_{min,i}\}_{i\in\Omega}$ and maximal $\{T_{max,i}\}_{i\in\Omega}$ differential operators. Consider a respective family $\{T_i\}_{i\in\Omega}$ of self-adjoint extensions. Further, we introduce a system Hilbert space $\mathbf{L}^2 = \oplus_{i\in\Omega} L^2_i$, consisting of the vectors $\mathbf{f} = \oplus_{i\in\Omega} f_i$ such that $f_i \in L^2_i$ and

$$\|\mathbf{f}\|^2 = \sum_{i\in\Omega} \|f_i\|_i^2 = \sum_{i\in\Omega} \int_{I_i} |f_i|^2 \, dx < \infty \, .$$

In the space $\mathbf{L}^2$ consider the operator $T : D(T) \subseteq \mathbf{L}^2 \to \mathbf{L}^2$, defined on the domain

$$D(T) = \big\{ \mathbf{f} \subseteq \mathbf{L}^2 : \sum_{i\in\Omega} \|T_i f_i\|_i^2 < \infty \big\}$$

by $T\mathbf{f} = \oplus_{i\in\Omega} T_i f_i$.

The operator T is called a *differential vector-operator* generated by the self-adjoint extensions T_i, or a *self-adjoint differential vector-operator.* If Ω is infinite, the vector-operator T is called *infinite*. The operators T_i are called *coordinate* operators.

The abstract preliminaries for this work may be found, for instance, in books [14, 15].

Fix $i \in \Omega$. For each T_i there exists a unique resolution of the identity E^i_λ and a unitary operator U_i, making the isometrically isomorphic mapping of the Hilbert space L^2_i onto the space $L^2(M_i, \mu_i)$, where the operator T_i is represented as a multiplication operator. Below, we remind the structure of the mapping U_i.

We call $\phi \in L^2_i$ a *cyclic vector* if for each $z \in L^2_i$ there exists a Borel function f, such that $z = f(T_i)\phi$. Generally, there is no a cyclic vector in L^2_i but there is a collection $\{\phi^k\}$ of them in L^2_i, such that $L^2_i = \oplus_k L^2_i(\phi^k)$, where $L^2_i(\phi^k)$ are T_i-invariant subspaces in L^2_i generated by the cyclic vectors ϕ^k. That is $L^2_i(\phi^k) = \overline{\{f(T_i)\phi^k\}}$, for a varying Borel function f, such that $\phi^k \in D(f(T_i))$.

A vector $\phi \in L^2_i$ is called *maximal* relative to the operator T_i, if each measure $(E^i(\cdot)x, x)_i$, $x \in L^2_i$, is absolutely continuous with respect to the measure $(E^i(\cdot)\phi, \phi)_i$.

For each Hilbert space L^2_i, there exist a unique (up to unitary equivalence) decomposition $L^2_i = \oplus_k L^2_i(\varphi^k_i)$, where φ^1_i is maximal in L^2_i relative to T_i, and a decreasing set of multiplicity sets e^i_k, where e^i_1 is the whole line, such that $\oplus_k L^2_i(\varphi^k_i)$ is equivalent with $\oplus_k L^2(e^i_k, \mu_i)$, where the measure of the ordered representation is defined as $\mu_i(\cdot) = (E^i(\cdot)\varphi^1_i, \varphi^1_i)_i$. A spectral representation of T_i in $\oplus_k L^2(e^i_k, \mu_i)$ is called the *ordered representation* and it is unique, up to a unitary equivalence. Two operators are called *equivalent*, if they create the same ordered representation of their spaces.

For $i \in \Omega$, we introduce a *sliced union* of sets M_i (see also preliminaries) as a set M, containing all M_i on different copies of $\cup_{i\in\Omega} M_i$. The sets M_i do not intersect in M, but they can *superpose*, i.e. two sets M_i and M_j superpose, if their projections in the set $\cup_{i\in\Omega} M_i$ intersect.

For $z_i \in L^2_i$, $i \in \Omega$, define $\widehat{\mathbf{z_i}} = \{0, \dots, 0, z_i, 0, \dots, 0\} \in \mathbf{L}^2$, where z_i is on the i-th place.

For each $i \in \Omega$, let $\delta(T_i)$ denote the *subspectrum* of the operator T_i, i.e. $\delta(T_i) = \sigma_{pp}(T_i) \cup \sigma^*_{cont}(T_i)$, where $\sigma_{pp}(T_i)$ is the set of eigenvalues which may be open and $\sigma^*_{cont}(T_i)$ is the continuous spectrum with a removed set of spectral measure zero. $\sigma^*_{cont}(T_i)$ may be also open. Note that $\overline{\delta(T_i)} = \sigma(T_i)$. For instance, the subspectrum of an operator having the complete system of eigenfunctions with eigenvalues being the rational numbers of $[0,1]$ equals to $\mathbb{Q} \cap [0,1]$; the subspectrum of an operator having the continuous spectrum [0,1] is assumed to equal to (0,1) without loss of generality. The notion of the subspetrum arises quite naturally. Indeed, let we are given a self-adjoint operator A with a simple spectrum $\sigma(A) = [a,b]$. Choosing any point $\xi \in \sigma(A)$ we can obtain $\sigma(A) = [a,\xi) \cup [\xi, b]$. If we are interested in obtaining a formula $A_1 \oplus A_2 = A$, where $\sigma(A_1) = [a,\xi]$ and $\sigma(A_2) = [\xi, b]$, we have to suppose that $\xi \notin \sigma_{pp}(A)$. But if we pass to subspectra, we will not need to care about inessential points appearing as limit points.

Consider a projecting mapping $P : M \to \cup_{i\in\Omega} M_i$ such that $P(\delta(T_i)) = \delta(T_i)$. Let $\Omega = \cup_{k=1}^K A_k$, $A_k \cap A_s = \emptyset$ for $k \neq s$ and

$$A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l\,, P(\delta(T_s)) \cap P(\delta(T_l)) = B_{sl}\,,$$

where $\|E^t(B_{sl})\varphi_t\|_t^2 = 0$ for any cyclic $\varphi_t \in L_t^2$, $t = s, l\}$. From all such divisions of Ω we choose and fix the one, which contains the minimal number of A_k. In case when all the coordinate spectra $\sigma(T_i)$ are simple, we define the number $\Lambda = \min\{K\}$ as the *spectral index* of the vector-operator T.

The following two lemmas were proved in [9].

Lemma 1.1. *The identity resolution $\{E_\lambda\}$ of the vector-operator T equals to the direct sum of the coordinate identity resolutions $\{E_\lambda^i\}$, that is $\{E_\lambda\} = \oplus_{i\in\Omega}\{E_\lambda^i\}$*

Lemma 1.2. *Let each T_i have a cyclic vector a_i in L_i^2. Then the vector-operator T has minimum Λ cyclic vectors $\{\mathbf{a}_k\}_{k=1}^{\Lambda}$, having the form $\mathbf{a}_k = \sum_{i\in A_k} \widehat{\mathbf{a_i}}$.*

In the next section we will see what a spectral multiplicity of a vector-operator is. Nevertheless, this notation is intuitively clear. Running ahead, let us present here two examples, which will show the difference between the spectral index and the spectral multiplicity of the vector-operator T.

Example 1. We have a three-interval EMZ system $\{I_i, \tau_i, 1\}_{i=1}^3$ (a kinetic energy, a mirror kinetic energy, an impulse):

$$I_1 = [0, +\infty), \quad \tau_1 = -\big(\frac{d}{dt}\big)^2,$$
$$D(T_1) = \{f \in D(T_{max,1}) : f(0) + kf'(0) = 0, -\infty < k \leqslant \infty\};$$
$$I_2 = [0, +\infty), \quad \tau_2 = \big(\frac{d}{dt}\big)^2,$$
$$D(T_2) = \{f \in D(T_{max,2}) : f(0) + sf'(0) = 0, -\infty < s \leqslant \infty\};$$
$$I_3 = [0, 1], \quad \tau_3 = \frac{1}{i}\frac{d}{dt},$$
$$D(T_3) = \{f \in D(T_{max,3}) : f(0) = e^{i\alpha} f(1),\, \alpha \in [0, 2\pi]\}.$$

(a) If $k, s \in (-\infty, 0] \cup \{+\infty\}$ then

$$\delta(T_1) = (0, +\infty), \quad \delta(T_2) = (-\infty, 0), \quad \delta(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

For this system we have: $\{1, 2, 3\} = \cup_{k=1}^2 A_k$ and $A_1 = \{1, 2\}$, $A_2 = \{3\}$. Thus, here the spectral index does not coincide with the spectral multiplicity (which is 1) and equals 2.

(b) The case $0 < k, s < +\infty$ leads to

$$\delta(T_1) = \{-\frac{1}{k^2}\} \cup (0, +\infty), \quad \delta(T_2) = (-\infty, 0) \cup \{\frac{1}{s^2}\}, \quad \delta(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

If

$$\alpha \notin \Big[\bigcup_{n=-\infty}^{\infty} (2\pi n + \frac{1}{k^2})\Big] \bigcup \Big[\bigcup_{n=-\infty}^{\infty} (2\pi n - \frac{1}{s^2})\Big],$$

we have $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$. That is $\Lambda = 3$ but $\oplus_{i=1}^3 T_i$ has a simple spectrum.

Example 2. Let us have a vector-operator $\oplus_{i=1}^3 T_i$ with

$$\delta(T_1) = \bigcup_{n\in\mathbb{Z}, n\geqslant 0} n, \quad \delta(T_2) = \bigcup_{n\in\mathbb{Z}, n\leqslant 0} n, \quad \delta(T_3) = \bigcup_{n\in\mathbb{Z}, n\neq 0} n.$$

The spectral index is 3 but spectral multiplicity is 2.

Definition 1.3. A vector-operator $T = \oplus_{i\in\Omega} T_i$ with the simple coordinate spectra $\sigma(T_i)$ is called *distorted* if its spectral index does not equal its spectral multiplicity.

Generally it is not possible to build a spectral representation for a distorted vector-operator without applying the division method, but in some cases (Example 1) Theorem 1.2 may lead to the construction of a spectral representation even for some distorted vector-operators. If we want to obtain an ordered spectral representation for any self-adjoint vector-operator, only implementation of the division method can achieve this.

2. The division method for subspectra (DMS)

Below we present the three theorems (2.1, 2.2 and 2.3) without their complete proofs. Only the structural parts of the proofs essential for the demonstration of the DMS are presented. The complete proof of Theorem 2.1 may be found in [11, 10] and refer to [12] for the proofs of Theorems 2.2 and 2.3.

Theorem 2.1. *If θ_i and $\{e^i_n\}_{n=1}^{m_i}$ are measures and multiplicity sets of ordered representations for coordinate operators T_i, $i \in \Omega$, then there exist processes Pr_1 and Pr_2, such that the measure*

$$\theta = Pr_1(\{\theta_i\}_{i\in\Omega})$$

is the measure of an ordered representation and the sets

$$s_n = Pr_2(\{e^i_k\}_{i\in\Omega;\, k=\overline{1,m_i}})$$

are the canonical multiplicity sets of the ordered representation of the operator T. Thus, the unitary representation of the space $\mathbf{L}^2$ on the space $\oplus_n L^2(s_n,\theta)$ is the ordered representation and it is unique up to unitary equivalence.

Proof. We divide the proof into units for convenience. Parts **(A)** and **(B)** represent the DMS.
(A) Let a_i be maximal vectors relative to the operators T_i in L^2_i. We want to find a maximal vector relative to the vector-operator T. We know, that the vector $\oplus_{i\in\Omega} a_i$ does not give a single measure, if a set $P(\delta(T_i)) \cap P(\delta(T_j))$ has a non-zero spectral measure for $i \neq j$. Consider restrictions $T_i|_{L^2_i(a_i)} = T'_i$. Since all the operators T'_i have single cyclic vectors a_i, we can divide Ω into A_k, $k = \overline{1,\Lambda}$ and apply Lemma 1.2 for the operator $\oplus_{i\in\Omega} T'_i$. Then we have derive Λ vectors $\mathbf{a}^k = \oplus_{j\in A_k} a_j$, which are maximal in the respective spaces $\mathbf{L}^2(\mathbf{a}^k) = \oplus_{j\in A_k} L^2_j(a_j)$.
(B) Let now $1 < \Lambda < \infty$. Define $T^k = \oplus_{j\in A_k} T'_j$. For any two operators T^k and T^s, $k \neq s$, let us introduce the sets $\delta_{k,s} = P(\delta(T^k)) \cap P(\delta(T^s))$ and $\delta_k = P(\delta(T^k)) \setminus \delta_{k,s}$. There exist unitary representations

$$U^k : \mathbf{L}^2(\mathbf{a}^k) \to L^2(\mathbb{R}, \mu_{\mathbf{a}^k}).$$

Consider measures μ_k and $\mu_{k,s}$, defined as

$$\mu_{k,s}(e) = \mu_{\mathbf{a}^k}(e \cap \delta_{k,s})$$

and $\mu_k(e) = \mu_{\mathbf{a}^k}(e \cap \delta_k)$, for any measurable set e. For any operator T^k (with respect to T^s), measures μ_k and $\mu_{k,s}$ are mutually singular and $\mu_k + \mu_{k,s} = \mu_{\mathbf{a}^k}$; therefore,

$$L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) = L^2(\mathbb{R}, \mu_k) \oplus L^2(\mathbb{R}, \mu_{k,s}).$$

This means that (according to our designations):

$$U^{k^{-1}}: L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) \longrightarrow \mathbf{L}^2(\mathbf{a}_k^k) \oplus \mathbf{L}^2(\mathbf{a}_{k,s}^k)$$

and

$$\mathbf{a}^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k, \tag{2.1}$$

where $\mathbf{a}_k^k$ and $\mathbf{a}_{k,s}^k$ form the measures μ_k and $\mu_{k,s}$ respectively. Define also as $\max\{w, \psi\}$ the vector, which is maximal of the two vectors in the brackets (Note that this designation is valid only for vectors, considered on the same set. In order not to complicate the investigation we assume here that any two vectors are comparable in this sense. In order to achieve this, it is enough to decompose each coordinate operator T_i into the direct sum $T_i^{pp} \oplus T_i^{cont}$, where the operators have respectively pure point and continuous spectra. Then after redesignation we obtain the equivalent vector-operator to the initial vector-operator $\oplus T_i$).

Consider first two operators T^1 and T^2. It is clear, that the vector

$$\mathbf{a}^{1\oplus 2} = \mathbf{a}_1^1 \oplus \mathbf{a}_2^2 \oplus \max\left\{\mathbf{a}_{1,2}^1, \mathbf{a}_{2,1}^2\right\}$$

is maximal in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2)$. Note that $\mathbf{a}_1^1$ and $\mathbf{a}_2^2$ and they both may equal zero. The maximal vector in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2) \oplus \mathbf{L}^2(\mathbf{a}^3)$ will have the form:

$$\mathbf{a}^{1\oplus 2\oplus 3} = \mathbf{a}_{1\oplus 2}^{1\oplus 2} \oplus \mathbf{a}_3^3 \oplus \max\left\{\mathbf{a}_{1\oplus 2,3}^{1\oplus 2}, \mathbf{a}_{3,1\oplus 2}^3\right\},$$

where $\mathbf{a}_{1\oplus 2}^{1\oplus 2}$ is the narrowed vector $\mathbf{a}^{1\oplus 2}$, corresponding to the set which is free from the superposition with $\delta(T_3)$, as shown in (2.1).

Continuing this process, we obtain a maximal vector in the main space $\mathbf{L}^2$:

$$\mathbf{a}^{1\oplus\dots\oplus\Lambda} = \mathbf{a}_{1\oplus\dots\oplus\Lambda-1}^{1\oplus\dots\oplus\Lambda-1} \oplus \mathbf{a}_\Lambda^\Lambda \oplus \max\left\{\mathbf{a}_{1\oplus\dots\oplus\Lambda-1,\Lambda}^{1\oplus\dots\oplus\Lambda-1}, \mathbf{a}_{\Lambda,1\oplus\dots\oplus\Lambda-1}^\Lambda\right\}. \tag{2.2}$$

Let $\Lambda = \infty$. We obtain $\mathbf{a}^{1\oplus\dots\oplus\Lambda}$ as a vector which satisfies the following equality:

$$\left\|[\oplus_{i\in\Omega} E^i(\cdot)]\,\mathbf{a}^{1\oplus\dots\oplus\Lambda}\right\|^2 = \lim_{L\to\infty} \left\|[\oplus_{j=1}^L E^j(\cdot)]\,\mathbf{a}^{1\oplus\dots\oplus L}\right\|^2, \tag{2.3}$$

since the limit on the right side exists.

(C) The next step is to build the measure of the ordered representation for the vector-operator. From Lemma 1.1 and the reasonings above, it follows that such a measure will be

$$\theta(\cdot) = \left([\oplus_{i\in\Omega} E^i(\cdot)]\,\mathbf{a}^{1\oplus\dots\oplus\Lambda}, \mathbf{a}^{1\oplus\dots\oplus\Lambda}\right).$$

(D) The canonical multiplicity sets s_n of the vector-operator have the form:

$$s_n = \Big[\bigcup_i P(e_n^i)\Big] \bigcup \Big[\bigcup_{\sum m_i \geqslant n} \bigcap P\left(e_{m_i}^i \backslash e_{m_i+1}^i\right)\Big]. \tag{2.4}$$

□

Example 3. Let us have a vector-operator $T = \oplus_{i=1}^5 T_i$, generated by five coordinate operators with simple continuous spectra:

$$\delta(T_1) = (0,2), \quad \delta(T_2) = (1,3), \quad \delta(T_3) = (2,4), \quad \delta(T_4) = (3,6), \quad \delta(T_5) = (0,4).$$

Divide $\Omega = \{1,2,3,4,5\}$ into A_k:

$$A_1 = \{1,3\}, \quad A_2 = \{2,4\}, \quad A_3 = \{5\}.$$

The spectral index Λ is 3 and we pass to the three operators

$$T^1 = T_1 \oplus T_3, \quad T^2 = T_2 \oplus T_4, \quad T^3 = T_5,$$

with maximal elements respectively:

$$\mathbf{a}^1 = a_1 \oplus a_3, \quad \mathbf{a}^2 = a_2 \oplus a_4, \quad \mathbf{a}^3 = a_5.$$

Consider first two sub-vector-operators T^1 and T^2. Find elements $\mathbf{a}_1^1$, $\mathbf{a}_{1,2}^1$, $\mathbf{a}_2^2$, $\mathbf{a}_{2,1}^2$. Since the spectra are continuous, we may assume that

$$\max\{\mathbf{a}_{1,2}^1, \mathbf{a}_{2,1}^2\} = \mathbf{a}_{1,2}^1.$$

We derive a maximal vector $\mathbf{a}^{1\oplus 2}$ in the space $\oplus_{i=1}^4 L_i^2$:

$$\mathbf{a}^{1\oplus 2} = \mathbf{a}_1^1 \oplus \mathbf{a}_{1,2}^1 \oplus \mathbf{a}_2^2 = \mathbf{a}^1 \oplus \mathbf{a}_2^2.$$

Apply the DMS to the vectors $\mathbf{a}^{1\oplus 2}$ and $\mathbf{a}^3$. We obtain the vecotrs

$$\mathbf{a}_{1\oplus 2}^{1\oplus 2}, \ \mathbf{a}_{1\oplus 2,3}^{1\oplus 2}, \ \mathbf{a}_{3,1\oplus 2}^3 = \mathbf{a}^3, \ \ \mathbf{a}_3^3 = 0.$$

Eventually, we find the maximal element in $\oplus_{i=1}^5 L_i^2 = \oplus_{k=1}^3 L^2(\mathbf{a}^k)$:

$$\mathbf{a}^{1\oplus 2\oplus 3} = \mathbf{a}_{1\oplus 2}^{1\oplus 2} \oplus \max\{\mathbf{a}_{1\oplus 2,3}^{1\oplus 2}, \mathbf{a}^3\} \oplus 0 = \mathbf{a}_{1\oplus 2}^{1\oplus 2} \oplus \mathbf{a}_{1\oplus 2,3}^{1\oplus 2} \oplus 0 = \mathbf{a}^{1\oplus 2} \oplus 0 = \mathbf{a}^1 \oplus \mathbf{a}_2^2 \oplus 0.$$

It is easy to see that the multiplicity sets for the initial vector-operator are: $s_1 = \mathbb{R}$, $s_2 = (0, 4)$, $s_3 = (1, 4)$.

Let us return to Examples 1 and 2. For the distorted vector-operator $T_1 \oplus T_2 \oplus T_3$ from Example 1, a spectral measure will be constructed on the maximal vector $\mathbf{a}^{1\oplus 2\oplus 3}$. The multiplicity sets s_i, $i \geqslant 2$ have measures zero. For the vector-operator from Example 2 two spectral measures are constructed on $\mathbf{a}^{1\oplus 2\oplus 3}$ and

$$\min\{a_{1,2}^1, a_{2,1}^2\} \oplus \min\{a_{2,3}^2, a_{3,2}^3\} \oplus \min\{a_{3,1}^3, a_{1,3}^1\},$$

where the sense of the minimums is clear. The multiplicity set s_2 will be

$$[P(\delta(T_1)) \cap P(\delta(T_2))] \cup [P(\delta(T_2)) \cap P(\delta(T_3))] \cup [P(\delta(T_3)) \cap P(\delta(T_1))].$$

Now the term 'distorted vector-operator' is clearly explained by the form of the cyclic vectors for such an operator.

Let $I = \bigvee_{i\in\Omega} I_i$ denote the sliced union of intervals I_i. Similarly, $I^k = \bigvee_{j\in A_k} I_j$. If x_i are variables on I_i, then $\vee x_i$ will designate a variable either on I or I^k depending on the context. This notation shows, that a vector-function

$$z = \{z_1(x_1), \dots, z_n(x_n), \dots\}$$

on I or I^k may be written as $z(\vee x_i)$. In particular, we may also write $\mathbf{z}(\vee x_i)$ instead of $\mathbf{z} = \oplus_{i\in\Omega} z_i$.

Let us introduce the space $\oplus_{i\in\Omega} L^\infty(I_i^n)$. Here, $\mathbf{z}(\vee x_i) \in \oplus_{i\in\Omega} L^\infty(I_i^n)$ means that

$$\sup_{i\in\Omega} \big\{ \operatorname{ess} \sup_{x_i \in I_i^n} |z_i(x_i)| \big\} < \infty,$$

where for each i, families $\{I_i^n\}_{n=1}^\infty$ represent compact subintervals of I_i, such that $\cup_{n=1}^\infty I_i^n = I_i$. In [4, Lemma 2.1], it was shown that $\oplus_{i\in\Omega} L^\infty(I_i^n) = (\oplus_{i\in\Omega} L^1(I_i^n))^*$, where the space of Lebesgue-integrable vector-functions $\oplus_{i\in\Omega} L^1(I_i^n)$ is defined analogously to $\mathbf{L}^2$.

We also need to introduce a symbolic integral $\int_{\bigvee J_i} f(\vee x_i)\, d(\vee x_i)$ defined by:

$$\int_{\bigvee J_i} f(\vee x_i)\, d(\vee x_i) = \oplus_i \int_{J_i} f_i(x_i)\, dx_i,$$

where $f(\vee x_i)$ is understood to be measurable relative to $d(\vee x_i)$, if $f_i(x_i)$ are measurable relative to Lebesgue measures dx_i.

Theorem 2.2. *Let T be a self-adjoint vector-operator, generated by an EMZ system $\{I_i, \tau_i\}_{i\in\Omega}$. Let U be an ordered representation of the space $\mathbf{L}^2 = \oplus_{i\in\Omega} L^2(I_i)$ relative to T with the measure θ and the multiplicity sets s_k, $k = \overline{1,m}$. Then there exist kernels $\Theta_k(\vee x_i, \lambda)$, measurable relative to $d(\vee x_i)\times\theta$, such that $\Theta_k(\vee x_i, \lambda) = 0$ for $\lambda \in \mathbb{R} \setminus s_k$ and $(\oplus_{i\in\Omega}\tau_i - \lambda)\Theta_k(\vee x_i, \lambda) = 0$ for each fixed λ. Moreover for any bounded Borel set Δ,*

$$\int_\Delta |\Theta_k(\vee x_i, \lambda)|^2 \, d\theta(\lambda) \in \oplus_{i\in\Omega} L^\infty(I_i^n) \quad \forall n \in \mathbb{N}. \tag{2.5}$$

$$(U\mathbf{w})^k(\lambda) = \lim_{n\to\infty} \int_{I^n} \mathbf{w}(\vee x_i)\, \overline{\Theta_k(\vee x_i, \lambda)}\, d(\vee x_i), \quad \mathbf{w} \in \mathbf{L}^2, \tag{2.6}$$

where the limit exists in $L^2(s_k, \theta)$. The kernels $\{\Theta_k(\vee x_i, \lambda)\}_{k=1}^n$, $n \leqslant m$, are linearly independent as vector-functions of the first variable almost everywhere relative to the measure θ on s_n.

Proof. We again need the DMS to prove this theorem. Fix i. If θ_i and $\{e_p^i\}_{p=1}^{m_i}$ are respectively the measure and the multiplicity sets of an ordered representation for T_i, then there exists the decomposition $L_i^2 = \oplus_{p=1}^{m_i} L^2(e_p^i, \theta_i)$, which implies $T_i = \oplus_{p=1}^{m_i} T_i^p$ and $L^2(e_p^i, \theta_i)$ are T_i^p-invariant. For vector-operator $(\oplus_{i\in\Omega}\oplus_{p=1}^{m_i} T_i^p) \to$ redesignate $\to \oplus_s T_s$, $s = \{i, p\} \in \Omega_1$, we may write $\Omega_1 = \cup_{k=1}^{\Lambda} A_k$.

Let us separate the proof into units for convenience.

(A) For each T_j, $j \in A_k$ and $k = \overline{1,\Lambda}$, there exists a single cyclic vector $a_j \in L_j^2$ and [15, XII.3, Lemma 9 and XIII.5, Theorem 1(I)] a function $W_j(x_j, \lambda)$ defined on $I_j \times e_j$ (note, that for a fixed $i \in \Omega$, $I_j = I_i$ for all $p = \overline{1, m_i}$) and measurable relative to $dx_j \times \mu_{a_j}$, such that $W_j(x_j, \lambda) = 0$, $\lambda \in \mathbb{R} \setminus e_j$ and for any bounded $\Delta \subset e_j$: $\int_\Delta |W_j(x_j, \lambda)|^2 \, d\mu_{a_j}(\lambda) \in L^\infty(I_j^n)$, $n \in \mathbb{N}$. Also

$$\big(E^j(\Delta) F_j(T_j) a_j\big)(x_j) = \int_\Delta W_j(x_j, \lambda) F_j(\lambda) \, d\mu_{a_j}(\lambda), \tag{2.7}$$

for any $F_j \in L^2(e_j, \mu_{a_j})$. On $I^k = \bigvee_{j\in A_k} I_j$, we construct the vector-function

$$W^k(\vee x_j, \lambda) = \{W_1(x_1, \lambda), \dots, W_n(x_n, \lambda), \dots\},$$

which is obviously measurable relative to $d(\vee x_j)\times \sum \mu_{a_j}$. Separate arguments show that this vector-function is a correctly constructed generalized eigenfunction and thus satisfies the statement of the theorem within each A_k.

Note that since for all $p = \overline{1, m_i}$ there exists the equality $(\tau_i - \lambda) W_i^p = 0$ (see [15, XIII.5, Theorem 1]), it is obvious that $(\oplus_{j\in A_k}\tau_j - \lambda) W^k = 0$, where $\tau_j = \tau_i$ for a fixed i and all $p = \overline{1, m_i}$. If $P(\delta(T_i)) \cap P(\delta(T_j))$ has zero spectral measures for all $i, j \in \Omega$, then $A_k : \Omega_1 = \cup_{k=1}^{\Lambda_1} A_k$ may be constructed such that A_k contains of indices $\{i, k\}$, $i \in \Omega$, $k = \overline{1, \max_i\{m_i\}}$.

(B) Consider the set of indices $\Omega_2 = \{j \in \Omega_1 : j = \{i, 1\}, i \in \Omega\}$. Construct $A_k : \Omega_2 = \cup_{k=1}^{\Lambda_2} A_k$. Apply the reasonings used in (A), considering everywhere Ω_2 instead of Ω_1. Hence, for each A_k and we find a vector-function $W_1^k(\vee x_j, \lambda)$ which is the solution of the equation $(\oplus_{j\in A_k}\tau_j - \lambda)\mathbf{y} = 0$. Consider W_1^k and W_1^s for $s \neq k$. For $\mathbf{a}^k$ there exists the decomposition $\mathbf{a}^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k$ (see the proof of Theorem 2.1). This fact induces the decomposition for W_1^k: $W_1^k = W_{1,k}^k \oplus W_{1,k,s}^k$. It is clear that being the restrictions of W_1^k, the vector-functions $W_{1,k}^k$ and $W_{1,k,s}^k$ are also the solutions of the equation $(\oplus_{j\in A_k}\tau_j - \lambda)\mathbf{y} = 0$. They, along with $\mathbf{a}_k^k$ and $\mathbf{a}_{k,s}^k$

define unitary transformations U_k^k and $U_{k,s}^k$, such that: $U_k^k : \mathbf{L}^2(\mathbf{a}_k^k) \to L^2(\mathbb{R}, \mu_k)$ and $U_{k,s}^k : \mathbf{L}^2(\mathbf{a}_{k,s}^k) \to L^2(\mathbb{R}, \mu_{k,s})$ (see the proof of Theorem 2.1). This implies, that the decomposition $W^k = W_{1,k}^k \oplus W_{1,k,s}^k$ is correct.

Define as $\max\{W_{1,k,s}^k, W_{1,s,k}^s\}$ the vector-function, which corresponds to the vector $\max\{\mathbf{a}_{k,s}^k, \mathbf{a}_{s,k}^s\}$, respectively $\min\{W_{1,k,s}^k, W_{1,s,k}^s\}$ as the vector-function which corresponds to that $\mathbf{a}_{k,s}^k$ or $\mathbf{a}_{s,k}^s$, which is not maximal of the two.

(C) Without loss of generality, suppose that $k = 1$ and $s = 2$. From the reasonings presented in Part (A) of this proof, it follows that

$$\Theta_1^{1\oplus 2} = W_{1,1}^1 \oplus W_{1,2}^2 \oplus \max\left\{W_{1,1,2}^1, W_{1,2,1}^2\right\}$$

is correctly constructed vector-function satisfying the statement of the theorem for the case $T = [\oplus_{j\in A_1} T_j] \oplus [\oplus_{q\in A_2} T_q]$. Apply the above described process to $\Theta_1^{1\oplus 2}$ and W_1^3 to obtain the correctly constructed vector-function:

$$\Theta_1^{1\oplus 2\oplus 3} = \Theta_{1,1\oplus 2}^{1\oplus 2} \oplus W_{1,3}^3 \oplus \max\left\{\Theta_{1,1\oplus 2,3}^{1\oplus 2}, W_{1,3,1\oplus 2}^3\right\}.$$

Continuing this process, we finally obtain:

$$\begin{aligned} \Theta_1(\vee x_i, \lambda) &= \Theta_1^{1\oplus\cdots\oplus\Lambda_2} \\ &= \Theta_{1,1\oplus\cdots\oplus\Lambda_2-1}^{1\oplus\cdots\oplus\Lambda_2-1} \oplus W_{1,\Lambda_2}^{\Lambda_2} \oplus \max\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1}, W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\}, \end{aligned}$$

where in the case of $\Lambda_2 = \infty$, $\Theta_1^{1\oplus\cdots\oplus\Lambda_2}$ is the function which satisfies (analogously to (2.3)):

$$\Big\|[\oplus_{i\in\Omega} E^i(\Delta)] \int_\Delta \Theta_1^{1\oplus\cdots\oplus\Lambda_2} d\theta(\lambda)\Big\|^2 = \lim_{L\to\infty} \Big\|[\oplus_{j=1}^L E^j(\Delta)] \int_\Delta \Theta_1^{1\oplus\cdots\oplus L}\, d\theta_L(\lambda)\Big\|^2, \tag{2.8}$$

for any bounded Borel set Δ, where

$$\theta_L(\cdot) = \big([\oplus_{j=1}^L E^j(\cdot)]\, \mathbf{a}^{1\oplus\cdots\oplus L}, \mathbf{a}^{1\oplus\cdots\oplus L}\big)$$

is the measure of the ordered representation of the space $\oplus_{j=1}^L L_j^2$. The limit on the right side exists and in fact it appears that

$$\int_\Delta \Theta_1^{1\oplus\cdots\oplus L}\, d\theta_L(\lambda) \to \int_\Delta \Theta_1^{1\oplus\cdots\oplus\Lambda_2} d\theta(\lambda),$$

as $L \to \infty$.

(D) Define $\Omega_3 = \{j \in \Omega_1 : j = \{i, 2\}, i \in \Omega\}$. Construct $A_k : \Omega_3 = \cup_{k=1}^{\Lambda_3} A_k$. Apply processes (B) and (C) of this proof, substituting everywhere Ω_3 instead of Ω_2. We obtain a vector-function $\Theta_2^{1\oplus\cdots\oplus\Lambda_3}$, which is defined on the set $\cup_i P(e_2^i)$. But, as we know (see (2.4)), the set s_2 also includes the sets where there are non-empty superpositions of $\delta(T_i)$. Therefore, designating

$$\begin{gathered} \Theta_2^1 = \Theta_2^{1\oplus\cdots\oplus\Lambda_3}, \quad \Theta_2^2 = \min\{W_{1,1,2}^1, W_{1,2,1}^2\}, \dots, \\ \Theta_2^{\Lambda_2+1} = \min\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1}, W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\}, \end{gathered}$$

we may again use the process (C) to build the vector-function $\Theta_2(\vee x_i, \lambda)$ defined on s_2 and $\Theta_2(\vee x_i, \lambda) = 0$ for $\lambda \in \mathbb{R} \setminus s_2$. Using processes (B), (C), (D) and formula (2.4), we finally obtain $\Theta_m(\vee x_i, \lambda)$.

(E) The linear independence is proved by separate arguments. □

Theorem 2.3 (Eigenfunction expansions)**.** *For any $\mathbf{w} \in \mathbf{L}^2$, there exists a decomposition*
$$\mathbf{w} = \sum_{k=1}^{m} \lim_{n\to\infty} \int_{-n}^{+n} (U\mathbf{w})^k(\lambda)\Theta_k(\vee x_i, \lambda)\, d\theta(\lambda),$$

Since the kernels from Theorem 2.2 are only measurable relative to λ, the following theorem is important:

Theorem 2.4. *Each kernel $\Theta_k(\vee x_i, \lambda)$, $k = \overline{1, m}$, may be decomposed as*
$$\Theta_k(\vee x_i, \lambda) = \sum_{s=1}^{M_k} \gamma_{sk}(\lambda)\sigma_{sk}(\vee x_i, \lambda), \tag{2.9}$$
where the M_k are finite for each k and $\sigma_{sk}(\vee x_i, \lambda)$ depend analytically on λ.

Proof. We separate the proof in parts which will correspond to the analogous parts of the proof of Theorem 2.2.
(A*) Each kernel $W_j(x_j, \lambda)$ from the part (A) of the proof of Theorem 2.2 may be decomposed:
$$W_j(\cdot, \lambda) = \sum_{s=1}^{n_j} \alpha_{js}(\lambda)\sigma_{js}(\cdot, \lambda),$$
where α_{js} are supposed to equal zero on $\mathbb{R} \setminus e_j$, see [15, p. 1351]. Supplementing the defining systems with zeros where necessary, we obtain:
$$\begin{aligned} W^k(\vee x_j, \lambda) &= \oplus_{j\in A_k} W_j(x_j, \lambda) \\ &= \oplus_{j\in A_k} \sum_{s=1}^{n_j} \alpha_{js}(\lambda)\sigma_{js}(x_j, \lambda) \\ &= \sum_{q=1}^{N_k} \oplus_{j\in A_k} \alpha_{jq}(\lambda)\sigma_{jq}(x_j, \lambda) \\ &= \sum_{q=1}^{N_k} \alpha_q^k(\lambda)\sigma_q^k(\vee x_j, \lambda), \end{aligned}$$
where
$$N_k = \max_{j\in A_k} n_j, \quad \alpha_q^k(\lambda) = \sum_{j\in A_k} \alpha_{jq}(\lambda), \quad \sigma_q^k(\vee x_j, \lambda) = \oplus_{j\in A_k} \sigma_{jq}(x_j, \lambda).$$
Since e_j and e_k do not intersect almost everywhere for $j, k \in \Omega_2$, $j \neq k$, the series $\sum_{j\in A_k} \alpha_{jq}(\lambda)$ converges almost everywhere on $\cup_{j\in\Omega_2} P(e_j)$.

(B*) and (C*) Now pass to the part (B). There we obtained the decompositions $W_1^k = W_{1,k}^k \oplus W_{1,k,s}^k$ and $W_1^s = W_{1,s}^s \oplus W_{1,s,k}^s$. Let us totally order the set $\{T^j\}_{j=1}^{\Lambda_2}$ saying that $T^k \preceq T^s$ if $\max\{W_{1,k,s}^k, W_{1,s,k}^s\} = W_{1,k,s}^k$. At that, $T^k \simeq T^s$ if and only if $T^k \preceq T^s$ and $T^s \preceq T^k$. According to this, we build $\oplus_{j=1}^{\Lambda_2} T^j$, where $T^j \preceq T^{j+1}$, $j = \overline{1, \Lambda_2 - 1}$ if $\Lambda_2 \geqslant 2$. The obtained vector-operator is obviously equivalent to the initial vector-operator (comprising unordered operators). Note that
$$W_1^k(\vee x_i, \lambda) = \sum_{q=1}^{N_k} \alpha_{1q}^k(\lambda)\sigma_{1q}^k(\vee x_j, \lambda)$$

and analogously

$$W_1^s(\vee x_i,\lambda)=\sum_{p=1}^{N_s}\alpha_{1p}^s(\lambda)\sigma_{1p}^s(\vee x_j,\lambda).$$

All the above leads to the following equality:

$$\begin{aligned}\Theta_1^{1\oplus 2}&=W_{1,1}^1\oplus W_{1,2}^2\oplus\max\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}=W_1^1\oplus W_{1,2}^2\\&=\Big(\sum_{q=1}^{N_1}\alpha_{1q}^1(\lambda)\sigma_{1q}^1(\vee x_j,\lambda)\Big)\oplus\Big(\sum_{p=1}^{N_2}\alpha_{1p}^2(\lambda)\chi_{\delta_2}(\lambda)\sigma_{1p}^2(\vee x_j,\lambda)\Big)\\&=\sum_{s=1}^{N^{1\oplus 2}}\alpha_{1s}^{1\oplus 2}(\lambda)\sigma_{1s}^{1\oplus 2}(\vee x_j,\lambda)\,,\end{aligned}$$

where $N^{1\oplus 2}=\max\{N_1,N_2\}$; $\alpha_{1s}^{1\oplus 2}(\lambda)=\alpha_{1s}^1(\lambda)+\alpha_{1s}^2(\lambda)\chi_{\delta_2}(\lambda)$,

$$\sigma_{1s}^{1\oplus 2}(\vee x_j,\lambda)=\sigma_{1s}^1(\vee x_j,\lambda)\oplus\big(\sigma_{1s}^2(\vee x_j,\lambda)\chi_{\delta_2}(\lambda)\big),$$

$s=\overline{1,N^{1\oplus 2}}$.

Continuing this process until the finite Λ_2, we obtain:

$$\Theta_1(\vee x_i,\lambda)=\Theta_1^{1\oplus\dots\oplus\Lambda_2}=\sum_{s=1}^{N^{1\oplus\dots\oplus\Lambda_2}}\alpha_{1s}^{1\oplus\dots\oplus\Lambda_2}(\lambda)\sigma_{1s}^{1\oplus\dots\oplus\Lambda_2}(\vee x_j,\lambda),\tag{2.10}$$

where $N^{1\oplus\dots\oplus\Lambda_2}=\max\{N_1,N_2,\dots N_{\Lambda_2}\}$ and for $s=\overline{1,N^{1\oplus\dots\oplus\Lambda_2}}$:

$$\begin{gathered}\alpha_{1s}^{1\oplus\dots\oplus\Lambda_2}(\lambda)=\alpha_{1s}^1(\lambda)+\sum_{i=2}^{\Lambda_2}\alpha_{1s}^i(\lambda)\chi_{\delta_i}(\lambda);\\\sigma_{1s}^{1\oplus\dots\oplus\Lambda_2}(\vee x_j,\lambda)=\sigma_{1s}^1(\vee x_j,\lambda)\oplus\Big(\oplus_{i=2}^{\Lambda_2}\sigma_{1s}^i(\vee x_j,\lambda)\chi_{\delta_i}(\lambda)\Big)\,.\end{gathered}\tag{2.11}$$

In the case of infinite Λ_2, $N^{1\oplus\dots\oplus\Lambda_2}$ is clearly finite. The series in the right side of (2.11) pointwise converges, since it consists of items defined on non-intersecting sets. $\sigma_{1s}^{1\oplus\dots\oplus\Lambda_2}(\vee x_j,\lambda)$ is defined by induction.

(D*) Borrowing the designations from (D) and using processes described in (A*) and (C*), we shall come to the decomposition of $\Theta_2^{1\oplus\dots\oplus\Lambda_3}$:

$$\Theta_2^{1\oplus\dots\oplus\Lambda_3}=\sum_{s=1}^{N^{1\oplus\dots\oplus\Lambda_3}}\alpha_{2s}^{1\oplus\dots\oplus\Lambda_3}(\lambda)\sigma_{2s}^{1\oplus\dots\oplus\Lambda_3}(\vee x_j,\lambda).$$

To obtain $\Theta_2(\vee x_i,\lambda)$, as in (D), we repeat part (C*) for

$$\Theta_2^1=\Theta_2^{1\oplus\dots\oplus\Lambda_3},\quad\Theta_2^2=W_{1,2,1}^2,\quad\dots,\quad\Theta_2^{\Lambda_2+1}=W_{1,\Lambda_2,1\oplus\dots\oplus\Lambda_2-1}^{\Lambda_2}.$$

Finally, the same way we obtain decompositions for all $\Theta_k(\vee x_i,\lambda)$, $k=\overline{1,m}$, which will have the form (2.9). □

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