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SOLUTION OF A SPECTRAL PROBLEM FOR THE CURL OPERATOR ON A CYLINDER

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Abstract. In this work, we give method to construct an explicit solutions to one spectral problem with the curl operator on a bounded cylinder. The eigen-values of this operator are square roots of eigen-values of Laplace operator (with Dirichlet boundary condition) and zero. The eigen-functions related to this problem are found using some results from complex analysis.

1. Introduction

The eigenvalue problem for the curl operator has important applications in plasma physics, where the eigen-functions of the curl operator are called free-decay fields. In [1] the free-decay fields have been found as the sum of a poloidal and a toroidal vector fields using a quite particular method. In the theory of fusion plasma, a free-decay field is called Taylor state which is considered the final state that makes the energy a minimum in order to leave the plasma in equilibrium [5]. The free-decay fields play also an important roll to study turbulence in plasma [2]. From a mathematical point of view, we have the studies in [8] and [4]. In [8] spectral properties of curl operator in various function spaces is considered, while in [4] the eigenvalue problem for the curl operator on periodic vector functions is studied. From this same point of view, we study the spectrum of the curl operator on a bounded cylinder

$$G := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2,\ 0 \le x_3 \le l\}.$$

We consider the eigenvalue problem

$$\operatorname{curl} \mathbf{u} = \lambda \mathbf{u} \quad \text{in } G, \tag{1.1}$$

$$u_3\big|_{\partial G} = 0, \quad u_2\big|_{\gamma} = 0, \quad u_1\big|_{p} = 0\,. \tag{1.2}$$

Here $\mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$ is a vector value function of a class $C^1(G) \cap C(\overline{G})$, ∂G is the boundary of G, γ is the circle of radius R in the plane $x_3 = 0$, and p is an arbitrary fixed point taken on γ. In this work $C^\alpha(G)$ denotes the space of α-Hölder continuous functions defined on G and $C^{k,\alpha}(G)$ denotes the space of functions defined on G and possessing there (α-Hölder) continuous derivatives up

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to order k. The subspace of $C^{k,\alpha}(G)$ consisting of all the functions of compact support in $G \setminus \partial G$ will be denoted by $C_o^{k,\alpha}(G)$.

2. The eigenvalues of the curl operator

If $\lambda = 0$, $\operatorname{curl} \mathbf{u} = 0$ on G, and for $w_k \in C_0^2(G)$, $k = 1, 2, \dots$, then

$$\mathbf{u}_k = \nabla w_k(\mathbf{x}),$$
$$\mathbf{u}_k\big|_{\partial G} = 0\,;$$

i.e., we have infinitely many solutions $(0, \nabla w_k)$, $k = 1, 2, \dots$ of the eigenvalue problem (1.1), (1.2).

For the case $\lambda \neq 0$, we apply the divergence and curl operators to (1.1), and obtain

$$\operatorname{div} \mathbf{u} = 0, \quad \text{and} \quad -\Delta u_j = \lambda^2 u_j, \quad \text{for } j = 1, 2, 3.$$

For $j = 3$, let us consider the eigenvalue problem

$$-\Delta u_3 = \lambda^2 u_3 \quad \text{in } G, \tag{2.1}$$
$$u_3 = 0 \quad \text{on } \partial G. \tag{2.2}$$

The solutions of this problem are well known. We shift to cylindrical coordinates (r, θ, z) obtaining, after a straightforward calculation, the eigenvalues λ_κ^2 with

$$\lambda_\kappa = \sqrt{\frac{\rho_{k,j}^2}{R^2} + \frac{m^2\pi^2}{l^2}}, \quad k = 0, 1, \dots;\ j = 1, 2, \dots;\ m = 1, 2, \dots, \tag{2.3}$$

where $\kappa = (k, j, m)$ is multi-index and $\rho_{k,j}$ are the positive roots of the Bessel function $J_k(z)$.

The corresponding eigen-functions are

$$u_3^\kappa = \frac{\sqrt{2}}{\sqrt{l\pi}\, R\, |J_k'(\rho_{kj})|}\, J_k(\rho_{kj}\frac{r}{R}) \exp(ik\theta) \sin(\frac{m\pi}{l} z). \tag{2.4}$$

As $\lambda_\kappa^2 > 0$ the real and imaginary parts of u_3^κ are eigen-functions also and they form the real orthonormal basis in the space $L_2(G)$.

We have the following result.

Theorem 2.1. *The eigenvalue $\lambda = 0$ of problem (1.1)-(1.2) has an infinite multiplicity. In the case $\lambda \neq 0$, we find the eigenvalues $\pm\lambda_\kappa$ for (1.1)-(1.2) through the eigenvalues λ_κ^2 of the problem (2.1)-(2.2)); and λ_κ is given by (2.3); the multiplicities of λ_κ and $-\lambda_\kappa$ are equal and finite. The spectrum for problem (1.1)-(1.2) is a point spectrum and does not have finite limits point.*

3. Determining eigen-functions via complex analysis

In this section, we obtain the eigen-functions of problem (1.1)-(1.2) using some results from complex analysis. We write the eigen-functions as $u^\kappa = (u_1^\kappa, u_2^\kappa, \underline{u}_3^\kappa)$, where $\underline{u}_3^\kappa$ has been already found in the former section and it is the real (or imaginary) part of u_3^κ represented by the expression (2.4). To find u_1^κ and u_2^κ we consider the complex function $\omega = u_2^\kappa + i\, u_1^\kappa$. After this definition and substituting λ by λ_κ and u_3 by $\underline{u}_3^\kappa$, we obtain the following complex form of (1.1):

$$\partial_3 \omega - i\lambda_\kappa \omega = 2i\partial_z \underline{u}_3^\kappa \tag{3.1}$$
$$2 \operatorname{Re} \partial_{\bar z} \omega - \lambda_\kappa \underline{u}_3^\kappa = 0\,, \tag{3.2}$$

where $z = x_1 + i\,x_2$ and
$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2})$$
.

The general solution of the differential equation (3.1) is
$$\omega(\mathbf{x}) = \omega_0^\kappa(\mathbf{x}) + \omega_1(\mathbf{x'})\exp(i\lambda_\kappa x_3), \quad \mathbf{x'} = (x_1, x_2)\,, \tag{3.3}$$
where
$$\omega_0^\kappa(\mathbf{x}) = 2i\int_0^{x_3}\exp(i\lambda_\kappa(x_3 - t))\partial_z\underline{u}_3^\kappa(\mathbf{x'}, t)dt$$
and $\omega_1(\mathbf{x'})$ is a function in $C^{2,\alpha}$ which will be specified later on.

We observe that $\omega_0^\kappa(\mathbf{x}) \in C^{2,\alpha}(G)$ if $u_3^\kappa \in C^{3,\alpha}(G)$.

On the left side of (3.2) we replace ω by the particular solution ω_0^κ of (3.1) and this side will be called V_0. We will need the following lemma.

Lemma 3.1. *If $\underline{u}_3^\kappa \in C^{3,\alpha}(G)$ is a solution of (2.1) (with $\lambda^2 = \lambda_\kappa^2$) on G then V_0 satisfies, on G, the equation*
$$\frac{\partial^2 V_0}{\partial x_3^2} + \lambda_\kappa^2 V_0 = 0. \tag{3.4}$$
Moreover V_0 can be represented in the form
$$V_0(\boldsymbol{x}) = \mathrm{Re}(v_0(\boldsymbol{x'})\exp(i\lambda_\kappa x_3)), \tag{3.5}$$
where
$$v_0(\boldsymbol{x'}) = (V_0(\boldsymbol{x}) - \frac{i}{\lambda_\kappa}\frac{\partial V_0(\boldsymbol{x})}{\partial x_3})\big|_{x_3=0} = i(\partial_3\underline{u}_3^\kappa)\big|_{x_3=0}.$$

Proof. When we apply the matrix differential operator $\begin{pmatrix}\partial_2 & -\partial_1\\ \partial_1 & \partial_2\end{pmatrix}$, where $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$, to the first two equations of system $\operatorname{curl}\underline{u} = \lambda\underline{u}$, where $\lambda = \lambda_\kappa$, $\underline{u} = (\underline{u}_1^\kappa, \underline{u}_2^\kappa, \underline{u}_3^\kappa)$ and $\omega_0^\kappa = \underline{u}_2^\kappa + i\underline{u}_1^\kappa$, and taking into account (2.1), we obtain the equations
$$-\partial_3(\lambda\operatorname{div}\underline{u}) + \lambda^2 V_0 = 0$$
$$\partial_3 V_0 + \lambda\operatorname{div}\underline{u} = 0\,.$$
Because V_0 and $\partial_3 V_0$ belong to the space $C^{1,\alpha}(G)$, we obtain the desired result. □

Substituting the ω given by (3.3) in the left side of (3.2), we obtain
$$2\,\mathrm{Re}\,\partial_{\bar z}(\omega_1(\mathbf{x'})\exp(i\lambda_\kappa x_3)) + V_0 = 0.$$
Using representation (3.5), this last equation can also be written as
$$\mathrm{Re}((2\partial_{\bar z}\omega_1(\mathbf{x'}) + v_0(\mathbf{x'}))\exp(i\lambda_\kappa x_3)) = 0,$$
from which it follows
$$2\partial_{\bar z}\omega_1(\mathbf{x'}) + v_0(\mathbf{x'}) = 0. \tag{3.6}$$
Therefore, the solvability condition for the last equation of system (1.1) is
$$\partial_{\bar z}\omega_1(\mathbf{x'}) = -\frac{1}{2}v_0(\mathbf{x'}) \tag{3.7}$$
It is known [6] that the general solution of the inhomogeneous Cauchy-Riemann system (3.7) is
$$\omega_1(\mathbf{x'}) = \Phi(z) - \frac{1}{2\pi}\int_{\mathcal{D}}\frac{v_0(\xi,\eta)}{z - \zeta}d\xi d\eta, \quad \zeta = \xi + i\eta, \tag{3.8}$$

where $\mathcal{D}$ is the disc $|\mathbf{x}| < R, x_3 = 0$ and $\Phi(z)$ is a function in $C^{2,\alpha}(\bar{\mathcal{D}})$, holomorphic in $\mathcal{D}$ which will be determined soon.

Taking (3.3) and (3.8) into account we have

$$\omega(\mathbf{x}) = \omega_2^\kappa(\mathbf{x}) + \Phi(z)\exp(i\lambda_\kappa x_3) \tag{3.9}$$

where

$$\omega_2^\kappa(\mathbf{x}) = \omega_0^\kappa(\mathbf{x}) + \Big(\frac{1}{2\pi}\int_{\mathcal{D}} \frac{v_0(\xi,\eta)}{\zeta - z} d\xi d\eta\Big)\exp(i\lambda_\kappa x_3)$$

which belongs to $C^{2,\alpha}(G) \cap C^{\alpha}(\bar{G})$ [7]. If $x_3 = 0$ the function $\omega_0^\kappa(\mathbf{x}) = 0$ by definition. Since $u_2^\kappa + iu_1^\kappa = \omega(\mathbf{x})$, according to the solution (3.9), we obtain

$$\operatorname{Re}\Phi(z) = -\operatorname{Re}\omega_2^\kappa(x', 0) \quad \text{on } \gamma \tag{3.10}$$

$$\operatorname{Im}\Phi(z)\Big|_p = -\operatorname{Im}\omega_2^\kappa(p), \quad \text{for } p = (p_1, p_2, 0) \in \gamma. \tag{3.11}$$

Now we use (3.10) and (3.11), and the Schwartz Formula [3] to specify the function $\Phi(z)$:

$$\Phi(z) = -\frac{1}{2\pi i}\int_\gamma \operatorname{Re}\omega_2^\kappa(t,0)\frac{t+z}{t-z}\frac{dt}{t} + i\,C_1.$$

where

$$C_1 = -\operatorname{Im}\Big[\omega_2^\kappa(p',0) + \frac{1}{2\pi i}\int_\gamma \operatorname{Re}\omega_2^\kappa(t,0)\frac{t+z}{t-z}\frac{dt}{t}\Big], \quad \text{for} \quad z = p_1 + ip_2.$$

Therefore, we have

$$u_1^\kappa(\mathbf{x}) = \operatorname{Im}\left[\omega_2^\kappa(\mathbf{x}) + \Phi(z)\exp(i\lambda_\kappa x_3)\right] \tag{3.12}$$

$$u_2^\kappa(\mathbf{x}) = \operatorname{Re}\left[\omega_2^\kappa(\mathbf{x}) + \Phi(z)\exp(i\lambda_\kappa x_3)\right]. \tag{3.13}$$

Then we arrive to the following result.

Theorem 3.2. *The components of the vector value eigen-function $\boldsymbol{u}^\kappa$ of problem* (1.1)-(1.2) *associated to a positive eigen-value λ_κ expressed by* (2.3) *are given by* (3.12), (3.13) *and real (or imaginary) part of* (2.4) *respectively. Moreover, if we replace λ_κ by $-\lambda_\kappa$ in* (3.3),..., (3.12), (3.13), *we obtain the eigen-function $\boldsymbol{u}_-^\kappa$ of problem* (1.1)-(1.2) *associated to a negative eigen-value $-\lambda_\kappa$.*

Thus, we have shown: on one side for any solution $(\lambda_\kappa, \mathbf{u}^\kappa)$ of the problem (1.1)-(1.2) a pair (ν_κ, v^κ) is a solution of the problem (2.1) -(2.2), where $\nu_\kappa = \lambda_\kappa^2$ and $v^\kappa = (\mathbf{u}^\kappa, \mathbf{e}_3) = u_3^\kappa$ is a projection of the vector-function $\mathbf{u}^\kappa$ on the ax of cylinder $\mathbf{e}_3$. Evidently, if the pair $(-\lambda_\kappa, \mathbf{u}_-^\kappa)$ is another solution of the problem (1.1)-(1.2) a pair (ν_κ, v_-^κ) is also a solution of the problem (2.1)- (2.2) (with same $\nu_\kappa = \lambda_\kappa^2$ and $v_-^\kappa = (\mathbf{u}_-^\kappa, \mathbf{e}_3) \neq v^\kappa$ in general case).

On other side, for any solution (ν_κ, v^κ) of the problem (2.1)- (2.2) (with a real function v^κ), we have constructed two solutions $(\lambda_\kappa, \mathbf{u}^\kappa)$ and $(-\lambda_\kappa, \mathbf{u}_-^\kappa)$ of the problem (1.1)-(1.2) such that $\lambda_\kappa = \sqrt{\nu_\kappa}$ and $(u^\kappa, \mathbf{e}_3) = (\mathbf{u}_-^\kappa, \mathbf{e}_3) = v^\kappa$.

Now Theorem 2.1 follows from these relations and properties of eigen-values of the problem (2.1)-(2.2) (see [7], f.e.).

Remark. Later (in 2004) we have calculated the components of eigen-functions $\mathbf{u}^\kappa$ and $\mathbf{u}_-^\kappa$ directly using the series representation of (2.4), which in variables (x_1, x_2, x_3) has the form

$$u_3^\kappa = a_\kappa (x_1 + ix_2)^k \sin(\frac{m\pi}{l}x_3)\sum_{p=0}^{\infty}\frac{(-1)^p \rho_{k,j}^{2p}(x_1^2 + x_2^2)^p}{(2R)^{2p}p!(p+k)!}. \tag{3.14}$$

These results will be published.

A short review of the physical background for the eigenvalue problem of the curl operator in a 3-dimensional bounded domain, with another boundary condition, can be found in [8].

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