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# EXACT CONTROLLABILITY OF A NON-LINEAR GENERALIZED DAMPED WAVE EQUATION: APPLICATION TO THE SINE-GORDON EQUATION

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ABSTRACT. In this paper, we give a sufficient conditions for the exact controllability of the non-linear generalized damped wave equation

$$\ddot{w} + \eta\dot{w} + \gamma A^{\beta} w = u(t) + f(t, w, u(t)),$$

on a Hilbert space. The distributed control $u \in L^2$ and the operator $A$ is positive definite self-adjoint unbounded with compact resolvent. The non-linear term $f$ is a continuous function on $t$ and globally Lipschitz in the other variables. We prove that the linear system and the non-linear system are both exactly controllable; that is to say, the controllability of the linear system is preserved under the non-linear perturbation $f$. As an application of this result one can prove the exact controllability of the Sine-Gordon equation.

## 1. INTRODUCTION

In this paper, we give sufficient conditions for the exact controllability of the following non-linear generalized damped wave equation on a Hilbert space $X$,

$$\ddot{w} + \eta\dot{w} + \gamma A^{\beta} w = u(t) + f(t, w, u(t)), \quad t \geq 0, \tag{1.1}$$

where $\gamma > 0$, $\eta > 0$, $\beta \geq 0$, the distributed control $u$ is in $L^2(0, t_1; X)$, and $A : D(A) \subset X \to X$ is a positive definite self-adjoint unbounded linear operator in $X$ with compact resolvent. This implies the following spectral decomposition of the operator $A$:

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k} \rangle \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x, \quad x \in D(A).$$

The non-linear term $f : [0, t_1] \times X \times X \to X$ is a continuous function on $t$ and globally Lipschitz in the other variables. i.e., there exists a constant $l > 0$ such that for all $x_1, x_2, u_1, u_2 \in X$ we have

$$\|f(t, x_2, u_2) - f(t, x_1, u_1)\| \leq l \left\{ \|x_2 - x_1\| + \|u_2 - u_1\| \right\}, \quad t \in [0, t_1]. \tag{1.2}$$

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We consider the operator
$$\mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^{\beta} & -\eta I \end{bmatrix} \tag{1.3}$$
which corresponds to the equation $\ddot{w} + \eta \dot{w} + \gamma A^{\beta} w = 0$ written as a first order system in the space $D(A^{\beta/2}) \times X$. Then we prove the following statements:

- (I) $\mathcal{A}$ generates a strongly continuous group $\{T(t)\}_{t\in\mathbb{R}}$ on $D(A^{\beta/2}) \times X$ such that $\|T(t)\| \leq M(\eta,\gamma)e^{-\frac{\eta}{2}t}, \quad t \geq 0$.
- (II) The linear system (1.4) ($f = 0$) is exactly controllable on $[0, t_1]$.
- (III) The non-linear system (1.1) is also exactly controllable on $[0, t_1]$.

Moreover, each of the following statements are equivalent to the exact controllability of the linear system
$$\ddot{w} + \eta \dot{w} + \gamma A^{\beta} w = u(t) \quad t \geq 0, \tag{1.4}$$

- (a) Each of the following finite dimensional systems is controllable on $[0, t_1]$,
$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \dots, \infty. \tag{1.5}$$
- (b) $B^* P_j^* e^{A_j^* t} y = 0$, for all $t \in [0, t_1]$, implies $y = 0$
- (c) Rank $\begin{bmatrix} P_jB & A_jP_jB & A_j^2P_jB & \cdots & A_j^{2\gamma_j-1}P_jB \end{bmatrix} = 2\gamma_j$
- (d) The operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ given by
$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} B B^* e^{-A_j^* s} ds, \tag{1.6}$$
is invertible, where $\lambda_j$ are the eigenvalues of $A$, $\{P_j\}$ are the projections on the corresponding eigenspace,
$$B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad A_j = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_j^{\beta} & -\eta \end{bmatrix} P_j\,, \quad j \geq 1.$$

The operator, $W_j(t_1)$, allows us to compute explicitly the control $u \in L^2(0, t_1; X)$ steering an initial state $z_0$ to a final state $z_1$ in time $t_1 > 0$ for the linear system (1.4). This control is given by the formula
$$u(t) = B^* T^*(-t) \sum_{j=1}^{\infty} W_j^{-1}(t_1) P_j (T(-t_1)z_1 - z_0). \tag{1.7}$$
We use this formula to construct a sequence of controls $u_n$ that converges to a control $u$ that steers an initial state $z_0$ to a final state $z_1$ for the non-linear system (1.1). That is to say, we proved the exact controllability of this system.

As an application of this result we can prove the exact controllability of The Sine-Gordon Equation
$$\begin{gathered} w_{tt} + cw_t - dw_{xx} + k \sin w = p(t,x), \quad 0 < x < 1, \ t \in \mathbb{R}, \\ w(t,0) = w(t,1) = 0, \quad t \in \mathbb{R} \end{gathered} \tag{1.8}$$
where $d > 0$, $c > 0$, $k > 0$ and $p : \mathbb{R} \times [0,1] \to \mathbb{R}$ is continuous and bounded function acting as an external force.

The existence of an attractor for the Sine-Gordon equation is proved in [9] where we can find a study of this equation, and the existence of bounded solutions for this model (1.8) and others similar one has been carried out recently in [5], [6] and [3]. To our knowledge, the exact controllability of this model under non-linear action of the control has not been studied before. So, in this paper we give a sufficient conditions for the exact controllability of the system (1.1) that can be applied to the following controlled Sine-Gordon equation

$$\begin{gathered} w_{tt} + cw_t - dw_{xx} + k\sin w = p(t,x) + u(t,x) + g(t,w,u(t,x)), \quad 0 < x < 1 \\ w(t,0) = w(t,1) = 0, \quad t \in \mathbb{R} \end{gathered} \tag{1.9}$$

where $g : [0,t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function on $t$ and globally Lipschitz in the other variables. i.e., there exists a constant $m > 0$ such that for all $x_1, x_2, u_1, u_2 \in \mathbb{R}$ we have

$$\|g(t,x_2,u_2) - g(t,x_1,u_1)\| \le m\left\{\|x_2 - x_1\| + \|u_2 - u_1\|\right\}, \quad t \in [0,t_1]. \tag{1.10}$$

This system can be written in the form of system (1.1) if we choose $X = L^2[0,1]$, $A\phi = -\phi_{xx}$, with domain $D(A) = H^2 \cap H_0^1$ and $f(t,w,u) = -k\sin w + p(t,\cdot) + g(t,w,u)$. Moreover, the exact controllability of (1.9) does not depend on the bounded function $p(t,\cdot)$.

Also, in [4] the authors study the exact *null* controllability of the second order linear equation

$$\ddot{w} + \rho A^r \dot{w} + Aw = u(t), \quad \rho > 0,\ \frac{1}{2} \le r \le 1,\ t \ge 0, \tag{1.11}$$

where the distributed control $u \in L^2(0,t_1;X)$ and $A : D(A) \subset X \to X$ is a positive definite self-adjoint unbounded linear operator in $X$ with compact resolvent. They prove that if $\frac{1}{2} \le r < 1$, then the system (1.11) is exactly *null* controllable on $[0,t_1]$. However, if $\alpha = 1$, the system (1.11) is not exactly *null* controllable. In [2, Example 3.27] it is shown that exact *null* controllability of an infinite dimensional system does not imply exact controllability of the system.

## 2. Notation and Preliminaries

The fact that $A : D(A) \subset X \to X$ is a positive definite self-adjoint unbounded linear operator in $X$ with compact resolvent implies the following:

(a) The spectrum of $A$ consists of only eigenvalues
$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$
Each $\lambda_j$ has finite multiplicity, $\gamma_n$, equal to the dimension of the corresponding eigenspace.

(b) There exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of $A$.

(c) For all $x \in D(A)$ we have
$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k}\rangle \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{2.1}$$
where $\langle\cdot,\cdot\rangle$ is the inner product in $X$ and
$$E_n x = \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k}\rangle \phi_{n,k}. \tag{2.2}$$
So, $\{E_n\}$ is a family of complete orthogonal projections in $X$ and $x = \sum_{n=1}^{\infty} E_n x$, $x \in X$.

(d) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by
$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{2.3}$$

(e) The fractional powered spaces $X^r$ are given by

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \Big\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \Big\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{2.4}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space endow with the norm

$$\Big\| \begin{bmatrix} w \\ v \end{bmatrix} \Big\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2.$$

Using the change of variables $w' = v$, the second order equation (1.1) can be written as a first order system of ordinary differential equations in the Hilbert space $Z_{\beta/2} = D(A^{\beta/2}) \times X = X^{\beta/2} \times X$ as

$$z' = \mathcal{A}z + Bu + F(t, z, u(t)) \quad z \in Z_{\beta/2},\ t \geq 0, \tag{2.5}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^\beta & -\eta I_X \end{bmatrix}. \tag{2.6}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A^\beta) \times X$ and

$$F(t, z, u) = \begin{bmatrix} 0 \\ f(t, w, u) \end{bmatrix}, \tag{2.7}$$

is a function $F : [0, t_1] \times Z_{\beta/2} \times X \to Z$. Since $X^{\beta/2}$ is continuously included in $X$ we obtain for all $z_1, z_2 \in Z_{\beta/2}$ and $u_1, u_2 \in X$ that

$$\|F(t, z_2, u_2) - F(t, z_1, u_1)\|_{Z_{\beta/2}} \leq L\left\{\|z_2 - z_1\| + \|u_2 - u_1\|\right\}, \quad t \in [0, t_1]. \tag{2.8}$$

In this paper, without lose of generality we shall assume the following condition

$$\eta^2 < 4\gamma\lambda_1^\beta.$$

## 3. The Uncontrolled Linear Equation

In this section we shall study the well-posedness of the abstract linear Cauchy initial-value problem

$$\begin{gathered} z' = \mathcal{A}z, \quad (t \in \mathbb{R}) \\ z(0) = z_0 \in D(\mathcal{A}), \end{gathered} \tag{3.1}$$

which is equivalent to prove that the operator $\mathcal{A}$ generates a strongly continuous group. To this end, we shall use the following Lema from [7].

**Lemma 3.1.** *Let $Z$ be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in $Z$ with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_n P_n = P_n A_n, \quad n = 1, 2, 3, \ldots \tag{3.2}$$

*Define the family of linear operators*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad t \geq 0. \tag{3.3}$$

*Then*

(a) $T(t)$ *is a linear bounded operator if*

$$\|e^{A_n t}\| \leq g(t), \quad n = 1, 2, 3, \dots \tag{3.4}$$

*for some continuous real-valued function* $g(t)$.

(b) *Under the condition* (3.4) $\{T(t)\}_{t\geq 0}$ *is a* $C_0$*-semigroup in the Hilbert space* $Z$ *whose infinitesimal generator* $\mathcal{A}$ *is given by*

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \quad z \in D(\mathcal{A}) \tag{3.5}$$

*with* $D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\}$

(c) *the spectrum* $\sigma(\mathcal{A})$ *of* $\mathcal{A}$ *is given by*

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{3.6}$$

*where* $\bar{A}_n = A_n P_n$.

**Theorem 3.2.** *The operator* $\mathcal{A}$ *given by* (2.6), *is the infinitesimal generator of a strongly continuous group* $\{T(t)\}_{t\mathbb{R}}$ *given by*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z_{\beta/2},\ t \geq 0 \tag{3.7}$$

*where* $\{P_n\}_{n\geq 0}$ *is a complete family of orthogonal projections in the Hilbert space* $Z_{\beta/2}$: $P_n = diag[E_n, E_n]$, $n \geq 1$, *and*

$$A_n = B_n P_n, \quad B_n = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_n^{\beta} & -\eta \end{bmatrix},\ n \geq 1. \tag{3.8}$$

*This group decays exponentially to zero. In fact, we have the estimate* $\|T(t)\| \leq M(\eta, \gamma)e^{-\frac{\eta}{2}t}$, $t \geq 0$, *where*

$$\frac{M(\eta,\gamma)}{2\sqrt{2}} = \sup_{n\geq 1}\Big\{2\Big|\frac{\eta \pm \sqrt{4\gamma\lambda_n^{\beta} - \eta^2}}{\sqrt{\eta^2 - 4\gamma\lambda_n^{\beta}}}\Big|, \Big|(2+\gamma)\sqrt{\frac{\lambda_n^{\beta}}{4\gamma\lambda_n^{\beta} - \eta^2}}\Big|\Big\}.$$

*Proof.* Computing $\mathcal{A}z$ yields,

$$\begin{aligned}
\mathcal{A}z &= \begin{bmatrix} 0 & I \\ -\gamma A^{\beta} & -\eta \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\
&= \begin{bmatrix} v \\ -\gamma A^{\beta} w - \eta v \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=1}^{\infty} E_n v \\ -\gamma \sum_{n=1}^{\infty} \lambda_n^{\beta} E_n w - \eta \sum_{n=1}^{\infty} E_n v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} \begin{bmatrix} E_n v \\ -\gamma\lambda_n^{\beta} E_n w - \eta E_n v \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_n^{\beta} & -\eta \end{bmatrix} \begin{bmatrix} E_n & 0 \\ 0 & E_n \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} A_n P_n z.
\end{aligned}
$$

It is clear that $A_nP_n = P_nA_n$. Now, we need to check condition (3.4) from Lemma 3.1. To this end, compute the spectrum of the matrix $B_n$. The characteristic equation of $B_n$ is given by

$$\lambda^2 + \eta\lambda + \gamma\lambda_n^{\beta} = 0,$$

and the eigenvalues $\sigma_1(n)$, $\sigma_2(n)$ of the matrix $B_n$ are given by

$$\sigma_1(n) = -c + il_n, \quad \sigma_2(n) = -c - il_n,$$

where,

$$c = \frac{\eta}{2} \quad \text{and} \quad l_n = \frac{1}{2}\sqrt{4\gamma\lambda_n^{\beta} - \eta^2}.$$

Therefore,

$$
\begin{aligned}
e^{B_nt} &= e^{-ct}\big\{ \cos l_ntI + \frac{1}{l_n}(B_n + cI)\big\} \\
&= e^{-ct} \begin{bmatrix} \cos l_nt + \frac{\eta}{2l_n}\sin l_nt & \frac{\sin l_nt}{l_n} \\ -\gamma S(n)\lambda_n^{\beta/2}\sin l_nt & \cos l_nt - \frac{\eta}{2l_n}\sin l_nt \end{bmatrix},
\end{aligned}
$$

From the above formulas, we obtain

$$e^{B_nt} = e^{-ct}\begin{bmatrix} a(n) & \frac{b(n)}{l_n} \\ -\gamma S(n)\lambda_n^{\beta/2}c(n) & d(n) \end{bmatrix}$$

where

$$a(n) = \cos l_nt + \frac{\eta}{2l_n}\sin l_nt, \quad b(n) = \sin l_nt,$$

$$c(n) = \sin l_nt, \quad d(n) = \cos l_nt - \frac{\eta}{2l_n}\sin l_nt, \quad S(n) = \sqrt{\frac{\lambda_n^{\beta}}{4\gamma\lambda_n^{\beta} - \eta^2}}.$$

Now, consider $z = (z_1, z_2)^T \in Z_{\beta/2}$ such that $\|z\|_{Z_{\beta/2}} = 1$. Then

$$\|z_1\|^2_{\beta/2} = \sum_{j=1}^{\infty}\lambda_j^{\beta}\|E_jz_1\|^2 \le 1 \quad \text{and} \quad \|z_2\|^2_X = \sum_{j=1}^{\infty}\|E_jz_2\|^2 \le 1.$$

Therefore, $\lambda_j^{\beta/2}\|E_jz_1\| \le 1$, $\|E_jz_2\| \le 1$, $j = 1, 2, \dots$. and so,

$$
\begin{aligned}
\|e^{A_nt}z\|^2_{Z_{\beta/2}} &= e^{-2ct}\Big\| \begin{bmatrix} a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2 \\ -\gamma S(n)c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1 + d(n)E_nz_2 \end{bmatrix} \Big\|^2_{Z_{\beta/2}} \\
&= e^{-2ct}\|a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2\|^2_{\frac{\beta}{2}} + e^{-2ct}\| \\
&\quad - \gamma S(n)c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1 + d(n)E_nz_2\|^2_X \\
&= e^{-2ct}\sum_{j=1}^{\infty}\lambda_j^{\beta}\|E_j\big(a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2\big)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{-2ct}\sum_{j=1}^{\infty}\|E_j\big(-\gamma S(n)c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\big)\|^2\\
&= e^{-2ct}\lambda_n^{\beta}\|a(n)E_nz_1+\frac{b(n)}{l_n}E_nz_2\|^2+e^{-2ct}\|\\
&\quad -\gamma S(n)c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\|^2\\
&\le e^{-2ct}(|a(n)|+|\frac{\lambda_n^{\frac{\beta}{2}}}{\lambda_n^{\alpha}}b(n)|)^2+e^{-2ct}(|\gamma S(n)c(n)|+|d(n)|)^2,
\end{aligned}
$$

where

$$
|\frac{\lambda_n^{\frac{\beta}{2}}}{l_n}b(n)|=\Big|\sqrt{\frac{\lambda_n^{\beta}}{\eta^2-4\gamma\lambda_n^{\beta}}}\Big|.
$$

If we set,

$$
\frac{M(\eta,\gamma)}{2\sqrt{2}}=\sup_{n\ge 1}\Big\{2\Big|\frac{\eta\pm\sqrt{4\gamma\lambda_n^{\beta}-\eta^2}}{\sqrt{\eta^2-4\gamma\lambda_n^{\beta}}}\Big|,\Big|(2+\gamma)\sqrt{\frac{\lambda_n^{\beta}}{4\gamma\lambda_n^{\beta}-\eta^2}}\Big|\Big\},
$$

we have,

$$
\|e^{A_nt}\|\le M(\eta,\gamma)e^{-ct},\quad t\ge 0,\ n=1,2,\dots .
$$

Hence, applying Lemma 3.1 we obtain that $\mathcal{A}$ generates a strongly continuous group given by (3.7). Next, we will prove this group decays exponentially to zero. In fact,

$$
\begin{aligned}
\|T(t)z\|^2 &\le \sum_{n=1}^{\infty}\|e^{A_nt}P_nz\|^2\\
&\le \sum_{n=1}^{\infty}\|e^{A_nt}\|^2\|P_nz\|^2\\
&\le M^2(\eta,\gamma)e^{-2ct}\sum_{n=1}^{\infty}\|P_nz\|^2\\
&= M^2(\eta,\gamma)e^{-2ct}\|z\|^2.
\end{aligned}
$$

Therefore, $\|T(t)\|\le M(\eta,\gamma)e^{-ct}$, $t\ge 0$. □

## 4. Exact Controllability of the Linear System

Now, we shall give the definition of controllability in terms of the linear systems

$$
z'=\mathcal{A}z+Bu\quad z\in Z_{\beta/2},\ t\ge 0, \tag{4.1}
$$

where

$$
z=\begin{bmatrix} w\\ v\end{bmatrix},\quad B=\begin{bmatrix}0\\ I_X\end{bmatrix},\quad \mathcal{A}=\begin{bmatrix}0 & I_X\\ -\gamma A^{\beta} & -\eta I_X\end{bmatrix}. \tag{4.2}
$$

For all $z_0\in Z_{\beta/2}$ equation (4.1) has a unique mild solution given by

$$
z(t)=T(t)z_0+\int_0^t T(t-s)Bu(s)ds,\quad 0\le t\le t_1. \tag{4.3}
$$

The following definition of exact controllability can be found in [2].

**Definition 4.1.** We say that system (4.1) is exactly controllable on $[0, t_1]$, $t_1 > 0$, if for all $z_0, z_1 \in Z_{\beta/2}$ there exists a control $u \in L^2(0, t_1; X)$ such that the solution $z(t)$ of (4.3) corresponding to $u$, satisfies $z(t_1) = z_1$.

Consider the bounded linear operator

$$G : L^2(0, t_1; U) \to Z_{\beta/2}, \quad Gu = \int_0^{t_1} T(-s)B(s)u(s)ds. \tag{4.4}$$

Then the following proposition is a characterization of the exact controllability of system (4.1).

**Proposition 4.2.** *The system (4.1) is exactly controllable on* $[0, t_1]$ *if and only if, the operator* $G$ *is surjective, that is to say*

$$GL^2(0, t_1; X) = \mathrm{Range}(G) = Z_{\beta/2}.$$

Now, consider the family of finite dimensional systems

$$y' = A_j P_j y + P_j Bu, \quad y \in \mathcal{R}(P_j);\ j = 1, 2, \dots, \infty. \tag{4.5}$$

Then the following proposition can be shown as in [8, Lemma 1].

**Proposition 4.3.** *The following statements are equivalent:*

(a) *System* (4.5) *is controllable on* $[0, t_1]$
(b) $B^* P_j^* e^{A_j^* t} y = 0$*, for all* $t \in [0, t_1]$*, implies* $y = 0$
(c) Rank $\begin{bmatrix} P_j B & A_j P_j B & A_j^2 P_j B & \cdots & A_j^{2\gamma_j - 1} P_j B \end{bmatrix} = 2\gamma_j$
(d) *The operator* $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ *given by*

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} BB^* e^{-A_j^* s} ds, \tag{4.6}$$

*is invertible.*

Now, we are ready to formulate the main result on exact controllability of the linear system (4.1).

**Theorem 4.4.** *The system* (4.1) *is exactly controllable on* $[0, t_1]$. *Moreover, the control* $u \in L^2(0, t_1; X)$ *that steers an initial state* $z_0$ *to a final state* $z_1$ *at time* $t_1 > 0$ *is given by the formula*

$$u(t) = B^* T^*(-t) \sum_{j=1}^{\infty} W_j^{-1}(t_1) P_j (T(-t_1) z_1 - z_0). \tag{4.7}$$

*Proof.* Since $\{T(t)\}_{t \geq 0}$ is a group, the operator $G$ in (5) can be replaced by

$$G : L^2(0, t_1; X) \to Z_{\beta/2}, \quad Gu = \int_0^{t_1} T(-s)B(s)u(s)ds. \tag{4.8}$$

Then system (4.1) is exactly controllable on $[0, t_1]$ if and only if, the operator $G$ is surjective, that is to say

$$GL^2(0, t_1; X) = \mathrm{Range}(G) = Z_{\beta/2}.$$

First, we shall prove that each of the following finite dimensional systems is controllable on $[0, t_1]$

$$y' = A_j P_j y + P_j Bu, \quad y \in \mathcal{R}(P_j);\ j = 1, 2, \dots, \infty. \tag{4.9}$$

In fact, we can check the condition for controllability of this systems,

$$B^*P_j^*e^{A_j^*t}y=0,\quad \forall t\in[0,t_1],\quad \Rightarrow y=0.$$

In this case the operators $A_j=B_jP_j$ and $\mathcal{A}$ are given by

$$B_j=\begin{bmatrix}0 & 1\\ -\gamma\lambda_j^\beta & -\eta\end{bmatrix},\quad \mathcal{A}=\begin{bmatrix}0 & I_X\\ -\gamma A^\beta & -\eta I\end{bmatrix},$$

and the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$ of the matrix $B_j$ are given by $\sigma_1(j)=-c+il_j$ and $\sigma_2(j)=-c-il_j$, where

$$c=\frac{\eta}{2}\quad\text{and}\quad l_j=\frac{1}{2}\sqrt{4\gamma\lambda_j^\beta-\eta^2}.$$

Therefore, $A_j^*=B_j^*P_j$ with $B_j^*=\begin{bmatrix}0 & -1\\ \gamma\lambda_j^\beta & -\eta\end{bmatrix}$ and

$$\begin{aligned}
e^{B_jt}&=e^{-ct}\big\{\cos l_jtI+\frac{1}{l_j}(B_j+cI)\big\}\\
&=e^{-ct}\begin{bmatrix}\cos l_jt+\frac{\eta}{2l_j}\sin l_jt & \frac{\sin l_jt}{l_j}\\ -\gamma S(j)\lambda_j^{\beta/2}\sin l_jt & \cos l_jt-\frac{\eta}{2l_j}\sin l_jt\end{bmatrix},\\
e^{B_j^*t}&=e^{-ct}\big\{\cos l_jtI+\frac{1}{l_j}(B_j^*+cI)\big\}\\
&=e^{-ct}\begin{bmatrix}\cos l_jt+\frac{\eta}{2l_j}\sin l_jt & -\frac{\sin l_jt}{l_j}\\ \gamma S(j)\lambda_j^{\beta/2}\sin l_jt & \cos l_jt-\frac{\eta}{2l_j}\sin l_jt\end{bmatrix},\\
B&=\begin{bmatrix}0\\ I_X\end{bmatrix},\quad B^*=[0,I_X]\quad\text{and}\quad BB^*=\begin{bmatrix}0 & 0\\ 0 & I_X\end{bmatrix}.
\end{aligned}$$

Now, let $y=(y_1,y_2)^T$ be in $\mathcal{R}(P_j)$ such that $B^*P_j^*e^{A_j^*t}y=0$ for all $t\in[0,t_1]$. Then

$$e^{-ct}\Big[\gamma S(j)\lambda_j^{\beta/2}\sin l_jty_1+\big(\cos l_jt-\frac{\eta}{2l_j}\sin l_jt\big)y_2\Big]=0,\quad \forall t\in[0,t_1],$$

which implies $y=0$. From Proposition 4.3 the operator $W_j(t_1):\mathcal{R}(P_j)\to\mathcal{R}(P_j)$ given by

$$W_j(t_1)=\int_0^{t_1}e^{-A_js}BB^*e^{-A_j^*s}ds=P_j\int_0^{t_1}e^{-B_js}BB^*e^{-B_j^*s}dsP_j=P_j\overline{W}_j(t_1)P_j$$

is invertible. Since

$$\begin{gathered}
\|e^{-A_jt}\|\le M(\eta,\gamma)e^{ct},\quad \|e^{-A_j^*t}\|\le M(\eta,\gamma)e^{ct},\\
\|e^{-A_jt}BB^*e^{-A_j^*t}\|\le M^2(\eta,\gamma)\|BB^*\|e^{2ct},
\end{gathered}$$

we have

$$\|W_j(t_1)\|\le M^2(\eta,\gamma)\|BB^*\|e^{2ct_1}\le L(\eta,\gamma),\quad j=1,2,\dots.$$

Now, we shall prove that the family of linear operators,

$$W_j^{-1}(t_1)=\overline{W}_j^{-1}(t_1)P_j:Z_{\beta/2}\to Z_{\beta/2}$$

is bounded and $\|W_j^{-1}(t_1)\|$ is uniformly bounded. To this end, we shall compute explicitly the matrix $\overline{W}_j^{-1}(t_1)$. From the above formulas we obtain that

$$e^{B_j t} = e^{-ct}\begin{bmatrix} a(j) & b(j) \\ -a(j) & c(j) \end{bmatrix}, \quad e^{B_j^* t} = e^{-ct}\begin{bmatrix} a(j) & -b(j) \\ d(j) & c(j) \end{bmatrix},$$

where

$$a(j) = \cos l_j t + \frac{\eta}{2l_j}\sin l_j t, \quad b(j) = \frac{\sin l_j t}{l_j},$$

$$c(j) = \gamma S(j)\lambda_j^{\beta/2}\sin l_j t, \quad d(j) = \cos l_j t - \frac{\eta}{2l_j}\sin l_j t, \quad S(j) = \sqrt{\frac{\lambda_j^\beta}{4\gamma\lambda_j^\beta - \eta^2}}.$$

Then

$$e^{-B_j s}BB^* e^{-B_j^* s} = \begin{bmatrix} b(j)c(j)\lambda_j^{\beta/2} I & -b(j)d(j)I \\ -d(j)c(j)\lambda_j^{\beta/2} I & d^2(j)I \end{bmatrix}.$$

Therefore,

$$\overline{W}_j(t_1) = \begin{bmatrix} \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}k_{11}(j) & \frac{1}{l_j}k_{12}(j) \\ -\gamma S(j)\lambda_j^{\beta/2}k_{21}(j) & k_{22}(j) \end{bmatrix},$$

where

$$\begin{aligned}
k_{11}(j) &= \int_0^{t_1} e^{2cs}\sin^2 l_j s ds \\
k_{12}(j) &= -\int_0^{t_1} e^{2cs}\big[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\big]ds \\
k_{21}(j) &= \int_0^{t_1} e^{2cs}\big[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\big]ds \\
k_{22}(j) &= \int_0^{t_1} e^{2cs}\big[\cos l_j s - \frac{\eta\sin l_j s}{2l_j}\big]^2 ds.
\end{aligned}$$

The determinant $\Delta(j)$ of the matrix $\overline{W}_j(t_1)$ is

$$\begin{aligned}
\Delta(j) &= \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}\left[k_{11}(j)k_{22}(j) - k_{12}(j)k_{21}(j)\right] \\
&= \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}\big\{\big(\int_0^{t_1} e^{2cs}\sin^2 l_j s ds\big)\big(\int_0^{t_1} e^{2cs}\big[\cos l_j s - \frac{\eta\sin l_j s}{2l_j}\big]^2 ds\big) \\
&\quad - \big(\int_0^{t_1} e^{2cs}\big[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\big]ds\big)^2\big\}.
\end{aligned}$$

Passing to the limit as $j$ approaches $\infty$, we obtain

$$\lim_{j\to\infty}\Delta(j) = \frac{(e^{2ct_1} - 1)(1 - 2e^{ct_1} + e^{2ct_1})}{2^4 c^3}.$$

Therefore, there exist constants $R_1, R_2 > 0$ such that $0 < R_1 < |\Delta(j)| < R_2$, $j = 1, 2, 3, \dots$. Hence,

$$\overline{W}^{-1}(j) = \frac{1}{\Delta(j)}\begin{bmatrix} k_{22}(j) & -\frac{1}{l_j}k_{12}(j) \\ \gamma S(j)\lambda_j^{\beta/2}k_{21}(j) & \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}k_{11}(j) \end{bmatrix} = \begin{bmatrix} b_{11}(j) & b_{12}(j) \\ b_{21}(j)\lambda_j^{\beta/2} & b_{22}(j) \end{bmatrix},$$

where $b_{n,m}(j)$ are bounded for $n = 1,2$; $m = 1,2$; $j = 1,2,\dots$. Using the same computation as in Theorem 3.2 we can prove the existence of a constant $L_2(\eta,\gamma)$ such that

$$\|W_j^{-1}(t_1)\|_{Z_{\beta/2}} \le L_2(\eta,\gamma), \quad j = 1,2,\dots.$$

Now, we define the linear bounded operators $W(t_1) : Z_{\beta/2} \to Z_{\beta/2}$, $W^{-1}(t_1) : Z_{\beta/2} \to Z_{\beta/2}$, by

$$W(t_1)z = \sum_{j=1}^{\infty} W_j(t_1)P_j z, \quad W^{-1}(t_1)z = \sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j z.$$

Using these definitions we see that $W(t_1)W^{-1}(t_1)z = z$ and

$$W(t_1)z = \int_0^{t_1} T(-s)BB^*T^*(-s)z ds.$$

Finally, we show that given $z \in Z_{\beta/2}$ there exists a control $u \in L^2(0,t_1;X)$ such that $Gu = z$. In fact, let $u$ be the control

$$u(t) = B^*T^*(-t)W^{-1}(t_1)z, \quad t \in [0,t_1].$$

Then

$$\begin{aligned}
Gu &= \int_0^{t_1} T(-s)Bu(s)ds \\
&= \int_0^{t_1} T(-s)BB^*T^*(-s)W^{-1}(t_1)zds \\
&= \Big(\int_0^{t_1} T(-s)BB^*T^*(-s)ds\Big)W^{-1}(t_1)z \\
&= W(t_1)W^{-1}(t_1)z = z.
\end{aligned}$$

Then the control steering an initial state $z_0$ to a final state $z_1$ in time $t_1 > 0$ is given by

$$\begin{aligned}
u(t) &= B^*T^*(-t)W^{-1}(t_1)(T(-t_1)z_1 - z_0) \\
&= B^*T^*(-t)\sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j(T(-t_1)z_1 - z_0).
\end{aligned}$$

□

## 5. Exact Controllability of the Non-Linear System

Now, we give the definition of controllability in terms of the non-linear systems

$$\begin{gathered}
z' = \mathcal{A}z + Bu + F(t,z,u(t)) \quad z \in Z_{\beta/2},\ t > 0, \\
z(0) = z_0.
\end{gathered} \tag{5.1}$$

For all $z_0 \in Z_{\beta/2}$, equation (5.1) has a unique mild solution

$$z(t) = T(t)z_0 + \int_0^t T(t)T(-s)[Bu(s) + F(s,z(s),u(s))]ds. \tag{5.2}$$

**Definition 5.1.** We say that system (5.1) is exactly controllable on $[0,t_1]$, $t_1 > 0$, if for all $z_0, z_1 \in Z_{\beta/2}$ there exists a control $u \in L^2(0,t_1;X)$ such that the solution $z(t)$ of (5.2) corresponding to $u$, verifies: $z(t_1) = z_1$.

Consider the non-linear operator $G_F : L^2(0,t_1;U) \to Z_{\beta/2}$, given by

$$G_F u = \int_0^{t_1} T(-s)B(s)u(s)ds + \int_0^{t_1} T(-s)F(s,z(s),u(s))ds, \tag{5.3}$$

where $z(t) = z(t;z_0,u)$ is the corresponding solution of (5.2). Then the following proposition is a characterization of the exact controllability of the non-linear system (5.1).

**Proposition 5.2.** *The system* (5.1) *is exactly controllable on* $[0,t_1]$ *if and only if, the operator* $G_F$ *is surjective, that is to say*

$$G_F L^2(0,t_1;X) = \mathrm{Range}(G_F) = Z_{\beta/2}.$$

**Lemma 5.3.** *Let* $u_1,u_2 \in L^2(0,t_1;X)$, $z_0 \in Z_{\beta/2}$ *and* $z_1(t;z_0,u_1)$, $z_2(t;z_0,u_2)$ *the corresponding solutions of (5.2). Then*

$$\|z_1(t) - z_2(t)\|_{Z_{\beta/2}} \le M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_1 - u_2\|_{L^2(0,t_1;X)}, \tag{5.4}$$

*where* $0 \le t \le t_1$ *and*

$$M = \sup_{0\le s\le t\le t_1}\{\|T(t)\|\|T(-s)\|\}. \tag{5.5}$$

*Proof.* Let $z_1,z_2$ be solutions of (5.2) corresponding to $u_1,u_2$ respectively. Then

$$\begin{aligned}
\|z_1(t) - z_2(t)\| &\le \int_0^t \|T(t)\|\|T(-s)\|\|B\|\|u_1(s) - u_2(s)\| \\
&\quad + \int_0^t \|T(t)\|\|T(-s)\|\|F(s,z_1(s),u_1(s)) - F(s,z_2(s),u_2(s))\|ds \\
&\le M[\|B\| + L]\int_0^t \|u_1(s) - u_2(s)\| + ML\int_0^t \|z_1(s) - z_2(s)\|ds \\
&\le M[\|B\| + L]\sqrt{t_1}\|u_1 - u_2\| + ML\int_0^{t_1} \|z_1(s) - z_2(s)\|ds.
\end{aligned}$$

Using Gronwall's inequality, we obtain

$$\|z_1(t) - z_2(t)\|_{Z_{\beta/2}} \le M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_1 - u_2\|_{L^2(0,t_1;X)},$$

for $0 \le t \le t_1$. □

Now, we are ready to formulate and prove the main Theorem of this section.

**Theorem 5.4.** *If in addition of condition* (2.8),

$$\|B\|ML\|W^{-1}(t_1)\|K(t_1)t_1 < 1, \tag{5.6}$$

*where* $K(t_1) = M[\|B\| + L]e^{MLt_1}t_1 + 1$, *then the non-linear system* (5.1) *is exactly controllable on* $[0,t_1]$.

*Proof.* Given the initial state $z_0$ and the final state $z_1$, and $u_1 \in L^2(0,t_1;X)$, there exists $u_2 \in L^2(0,t_1;X)$ such that

$$0 = z_1 - \int_0^{t_1} T(-s)F(s,z_1(s),u_1(s))ds - \int_0^{t_1} T(-s)Bu_2(s)ds,$$

where $z_1(t) = z(t;z_0,u_1)$ is the corresponding solution of (5.2). Moreover, $u_2$ can be chosen as

$$u_2(t) = B^*T^*(-t)W^{-1}(t_1)\Big(z_1 - \int_0^{t_1} T(-s)F(s,z_1(s),u_1(s))ds\Big).$$

For such $u_2$ there exists $u_3 \in L^2(0, t_1; X)$ such that

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z_2(s), u_2(s))ds - \int_0^{t_1} T(-s)Bu_3(s)ds,$$

where $z_2(t) = z(t; z_0, u_2)$ is the corresponding solution of (5.2), and $u_3$ can be taken as follows:

$$u_3(t) = B^*T^*(-t)W^{-1}(t_1)\Big(z_1 - \int_0^{t_1} T(-s)F(s, z_2(s), u_2(s))ds\Big).$$

Following this process we obtain two sequences

$$\{u_n\} \subset L^2(0, t_1; X), \quad \{z_n\} \subset L^2(0, t_1; Z_{\beta/2}), \quad (z_n(t) = z(t; z_0, u_n))\ n = 1, 2, \dots,$$

such that

$$u_{n+1}(t) = B^*T^*(-t)W^{-1}(t_1)\Big(z_1 - \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds\Big) \tag{5.7}$$

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds - \int_0^{t_1} T(-s)Bu_{n+1}(s)ds. \tag{5.8}$$

Now, we shall prove that $\{z_n\}$ is a Cauchy sequence in $L^2(0, t_1; Z_{\beta/2})$. In fact, from formula (5.7) we obtain that

$$\begin{aligned}
&u_{n+1}(t) - u_n(t)\\
&= B^*T^*(-t)W^{-1}(t_1)\Big(\int_0^{t_1} T(-s)(F(s, z_{n-1}(s), u_{n-1}(s)) - F(s, z_n(s), u_n(s)))ds\Big).
\end{aligned}$$

Hence, using lemma 5.3 we obtain

$$\begin{aligned}
&\|u_{n+1}(t) - u_n(t)\|\\
&\le \|B\|ML\|W^{-1}(t_1)\| \int_0^{t_1} (\|z_n(s) - z_{n-1}(s)\| + \|u_n(s) - u_{n-1}(s)\|)\, ds\\
&\le \|B\|ML\|W^{-1}(t_1)\| \int_0^{t_1} M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_n(s) - u_{n-1}(s)\|ds\\
&\quad + \|B\|ML\|W^{-1}(t_1) \int_0^{t_1} \|u_n(s) - u_{n-1}(s)\|ds.
\end{aligned}$$

Using Hóder's inequality, we obtain

$$\|u_{n+1} - u_n\|_{L^2(0,t_1;X)} \le \|B\|ML\|W^{-1}(t_1)\|K(t_1)t_1\|u_{n+1} - u_n\|_{L^2(0,t_1;X)}. \tag{5.9}$$

Since $\|B\|ML\|W^{-1}(t_1)\|K(t_1)t_1 < 1$, it follows that $\{u_n\}$ is a Cauchy sequence in $L^2(0, t_1; X)$. Therefore, there exists $u \in L^2(0, t_1; X)$ such that $\lim_{n\to\infty} u_n = u$ in $L^2(0, t_1; X)$.

Let $z(t) = z(t; z_0, u)$ the corresponding solution of (5.2). Then we shall prove that

$$\lim_{n\to\infty} \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds = \int_0^{t_1} T(-s)F(s, z(s), u(s))ds.$$

In fact, using lemma 5.3 we obtain that

$$\begin{aligned}
&\Big\| \int_0^{t_1} T(-s)[F(s, z_n(s), u_n(s)) - F(s, z(s), u(s))]ds\Big\|\\
&\le \int_0^{t_1} ML[\|z_n(s) - z(s)\| + \|u_n(s) - u(s)\|]ds
\end{aligned}$$

$$\leq \int_0^{t_1} ML[M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_n - u\|_{L^2(0,t_1;X)} + \|u_n(s) - u(s)\|]ds$$
$$\leq MLK(t_1)\sqrt{t_1}\|u_n - u\|_{L^2(0,t_1;X)}.$$

From here we obtain the result. Finally, passing to the limit in (5.8) as $n$ approaches $\infty$, we obtain

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z(s), u(s))ds - \int_0^{t_1} T(-s)Bu(s)ds.$$

i.e., $G_F u = z_1$. □

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