2003 Colloquium on Differential Equations and Applications, Maracaibo, Venezuela.
Electronic Journal of Differential Equations, Conference 13, 2005, pp. 65–74.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SPECTRAL DECOMPOSITION THEOREM FOR COMPACT SELF-ADJOINT OPERATORS IN PREHILBERT SPACES

HANZEL LÁREZ

Abstract. In this paper we found a complete orthonormal system for a prehilbert space, in which each element can be expressed as a Fourier series in terms of this system. This result is applied to solve second order differential equations with initial or boundary conditions. In particular, it is applied to Dirichlet problem and to Neumann problem.

1. Introduction

In this work, we show a method for finding a complete orthonormal system in a prehilbert space, such that each element of this space can be developed as a Fourier series in terms of am orthonormal system. This method can be applied to find the solutions of differential equations of second order with initial or boundary conditions: $x''+\lambda x = 0$ $\lambda \in \mathbb{C}$ and $x(0) = x(l) = 0$, $l \in \mathbb{R}$. This Problem is known as Sturm-Liouville Problem and it is part of a more general class, which can be solved in terms of the following result which is known as the Spectral Decomposition Theorem: If T is a linear, bounded, injective, self-adjoint and compact operator; in a prehilbert space $\mathbb{H}$ of finite dimension, then $\mathbb{H}$ possesses a complete orthonormal system $\{e_1, \dots, e_n, \dots\}$ (countable infinite and complete in the sense that $\mathbb{H}$ does not possess another orthonormal system that contains it) such that: $T(e_n) = \lambda_n e_n$ and $\lambda_n \to 0$ if $n \to \infty$, $\lambda_n \neq 0$, for each $n \in \mathbb{N}$. However, it can exist elements of $\mathbb{H}$ that do not admit developments in Fourier series, in terms of this orthonormal system, here we give an example of where this happens. The aim of this work is to show that if we modify the Spectral Decomposition Theorem, and we change the injective hypothesis for the super-injective hypothesis, then all element of $\mathbb{H}$ can be developed in Fourier series in terms of this orthonormal system. Let $L(\mathbb{H})$ be the space of the linear and bounded mappings of $\mathbb{H}$ in itself, $\mathbb{H}$ a prehilbert space, we say that an operator $T \in L(\mathbb{H})$ is super-injective if for each Cauchy sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{H}$ with $T(x_n) \to 0$, if $n \to \infty$, we have that $x_n \to 0$. This last result can be applied to Dirichlet Problems and to Neumann Problems.

2000 *Mathematics Subject Classification.* 34L05, 47B07.
Key words and phrases. Hilbert spaces; prehilbert; self-adjoint; super-injective; compact operator; complete orthonormal systems.

Published May 30, 2005.

2. Statement of the Problem

Many problems of the Mathematical Physics are reduced to second order differential equations in partial derivatives where we can find frequency equations such as

$$\begin{gathered}
\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial x^2} \quad \text{(Heat Equation)} \\
\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \quad \text{(Wave Equation)} \\
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{(Laplace Equation).}
\end{gathered}$$

These three equations are often solved by separation of variables method (Fourier Method), with appropriated condition of boundary, they give ordinary differential equations of the type:

$$x'' + \lambda x = 0 \quad \lambda \in \mathbb{C} x(0) = x(l) = 0 \quad l \in \mathbb{R}. \tag{2.1}$$

This problem is known as The Sturm-Liouville Problem, which belongs to a more general class that can be solved in terms of the Spectral Decomposition Theorem.

Theorem 2.1 (Spectral Decomposition Theorem)**.** *Let $T \in L(\mathbb{H})$ be injective, self-adjoint and compact, and let us suppose that $\dim(\mathbb{H}) = +\infty$. Then $\mathbb{H}$ possesses a countable infinite complete orthonormal system; $\{e_1, \dots, e_n, \dots\} : T(e_n) = \lambda_n e_n$; $|\lambda_1| \geq |\lambda_2| \geq \dots$ (countable infinite and complete in the sense that $\mathbb{H}$ does not possess another orthonormal system that contains it) and $\lambda_n \to 0$ as $n \to \infty$, $\lambda_n \neq 0$, for each $n \in \mathbb{N}$.*

However, it may exist elements of $\mathbb{H}$ that do not admit development in Fourier series, in terms of this orthonormal system. It is the case of the Example 3.1.

The aim of this work is to show that if we change the injectivity hypothesis by the super-injectivity hypothesis, then every element of $\mathbb{H}$ can be developed in Fourier series in terms of this orthonormal system. This last result can be applied to the Dirichlet Problem and the Neumann Problem.

Even more, if we make this change, we obtain a result, which we apply to $(C, \langle \cdot, \cdot \rangle_C)$, to show that:

$$\Big\{\frac{1}{\sqrt{2\pi}}\Big\} \bigcup \Big\{\frac{1}{\sqrt{\pi}} \cos(nt) : n \in \mathbb{N}\Big\} \bigcup \Big\{\frac{1}{\sqrt{\pi}} \sin(nt) : n \in \mathbb{N}\Big\},$$

is an orthonormal basis of C, the set of the continuous and 2π-periodic functions from $\mathbb{R}$ into itself. Also, each element of C can be developed in Fourier series in terms of this basis.

Preliminaries. A prehilbert space is a pair $(\mathbb{V}, \langle \cdot, \cdot \rangle)$, where $\mathbb{V}$ is a vector space and $\langle \cdot, \cdot \rangle$ is an inner product in $\mathbb{V}$.

The following sets of functions with the ordinary operations of addition and multiplication by scalars define vector spaces.

(a) $C_{[a,b]} = \{v : [a, b] \to \mathbb{R}$ such that v is continuous$\}$.
(b) $C^2_{[a,b]} = \{v : [a, b] \to \mathbb{R}$ such that v has second continuous derivative $\}$.

In these spaces, are prehilbert spaces when endowed with inner products defined as follows:

$$\langle f, g\rangle_{C_{[a,b]}} = \int_a^b f(t)g(t)dt, \tag{2.2}$$

$$\langle f, g\rangle_{C^2_{[a,b]}} = \langle f, g\rangle_{C_{[a,b]}} + \langle f', g'\rangle_{C_{[a,b]}} + \langle f'', g''\rangle_{C_{[a,b]}}\,. \tag{2.3}$$

For a prehilbert space $\mathbb{H}$, let $L(\mathbb{H})$ be the set of all linear and bounded operators from $\mathbb{H}$ to $\mathbb{H}$.

Definition 2.2. A linear operator $T \in L(\mathbb{H})$ is said to be super-injective if for each Cauchy sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{H}$ with $T(x_n) \to 0$ (as $n \to \infty$), we have that $x_n \to 0$.

The following results are proved in [5].

Corollary 2.3. *Let $T \in L(\mathbb{H})$ satisfy those hypothesis of Theorem 2.1. If $\mathbb{H}$ is a Hilbert space, then:*

(a) $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ *for each* $x \in \mathbb{H}$.
(b) $\mathrm{Im}(T)$ *is dense in* $\mathbb{H}$.

Corollary 2.4. *Let $(x_n)_{n=1}^{\infty}$ be a continuously differentiable sequence in $C_{[a,b]}$, such that: $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ are bounded in $C_{[a,b]}$. Then $(x_n)_{n=1}^{\infty}$ possesses a subsequence that converges uniformly $[a, b]$.*

Proposition 2.5. *Let $\mathbb{W}$ be a subspace of a vector space $\mathbb{V}$. $\mathbb{W}$ has finite codimension n if and only if there exists epimorphism $\mathcal{T} : \mathbb{V} \to \mathbb{R}^n$ with* $\ker(\mathcal{T}) = \mathbb{W}$.

Proposition 2.6. *Let $\mathbb{W}$ be a subspace of a vector space $\mathbb{V}$ ($\mathbb{W} \subseteq \mathbb{V}$). Then there exists a hyperplane $\mathbb{H} \subseteq \mathbb{V}$ such that $\mathbb{W} \subseteq \mathbb{H}$.*

Proposition 2.7. *Let $\mathbb{V}$ be a normed space and let $\mathbb{H}$ be a hyperplane of $\mathbb{V}$. Then $\mathbb{H}$ is closed or $\mathbb{H}$ is dense.*

Proposition 2.8. *Let $\mathbb{H}$ be a prehilbert space and $(e_n)_{n=1}^{\infty}$ an orthonormal system of $\mathbb{H}$ and $x \in \mathbb{H}$. If we define $x_n = \sum_{i=1}^{n}\langle x, e_i\rangle e_i$, then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.*

Proposition 2.9. *Let $\mathbb{V}$ be an normed space and let $T : \mathbb{V} \to \mathbb{R}$ be a linear operator. Then* $\ker(\mathrm{T})$ *is closed if and only if* T *is continuous.*

3. Main Result

Next, we will build a prehilbert space $\mathbb{H}$ and a linear, bounded, compact, self-adjoint and injective operator $T : \mathbb{H} \to \mathbb{H}$, such that if $(e_n)_{n=1}^{\infty}$ is the orthonormal system of $\mathbb{H}$ given in Theorem 2.1, and we find $x \in \mathbb{H}$ such that

$$x \neq \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n,$$

Example 3.1. Let

$$l_2 = \big\{f : \mathbb{N} \to \mathbb{R}; \sum_{n=1}^{\infty}(f(n))^2 < \infty\big\} \quad \text{and} \quad \langle f, g\rangle_{l_2} = \sum_{n=1}^{\infty} f(n)g(n).$$

Then $(l_2, \langle f, g\rangle_{l_2})$ is a prehilbert space. Let $\mathbb{E} = l_2$. Then the operator

$$T_0 : \mathbb{E} \to \mathbb{E}; \quad T_0(f) = \sum_{i=1}^{\infty} a_i f(i)e_i, \quad (a_i)_{i=1}^{\infty} \in l_2,\ a_i \neq 0$$

is compact, self-adjoint and injective.
• T_0 is well-defined. We can show that $T_0(f) \in \mathbb{E}$ for every $f \in \mathbb{E}$, and that it is equivalent to show that

$$\sum_{i=1}^{\infty} a_i f(i) e_i \in \mathbb{E}, \quad \text{implies} \quad \sum_{i=1}^{\infty} (a_i f(i))^2 < \infty.$$

In fact, as $(a_i)_{i=1}^{\infty} \in \mathbb{E}$, we have that $\sum_{i=1}^{\infty} (a_i)^2 < \infty$, then $(a_i)^2 \to 0$ as $i \to \infty$. Therefore, there exists $i_0 \in \mathbb{N}$ such that $|a_i|^2 < 1$ if $i > i_0$. On the other hand

$$\begin{aligned}
\sum_{i=1}^{\infty} (a_i f(i))^2 &= \sum_{i=1}^{i_0} (a_i f(i))^2 + \sum_{i=1+i_0}^{\infty} (a_i f(i))^2 \\
&\le \sum_{i=1}^{i_0} (a_i f(i))^2 + \sum_{i=1+i_0}^{\infty} (f(i))^2 < \infty,
\end{aligned}$$

because $f \in \mathbb{E}$ and $\sum_{i=1}^{i_0} (a_i f(i))^2$ is finite.
• T_0 is compact. Let $[e_1, \dots, e_n]$ be the space generated by $\{e_1, \dots, e_n\}$ and let $f \in \mathbb{E}$. We define

$$T_{0n}(f) = \sum_{i=1}^{n} a_i f(i) e_i; \quad (a_i)_{i=1}^{\infty} \in l_2,\ a_i \neq 0,\ i \in \mathbb{N}.$$

Then, $\mathrm{Im}(T_{0n}) \subseteq [e_1, \dots, e_n]$.

$$(T_0 - T_{0n})(f) = \sum_{i=n+1}^{\infty} a_i f(i) e_i.$$

Hence,

$$\begin{aligned}
\|(T_0 - T_{0n})(f)\| &= \| \sum_{i=n+1}^{\infty} a_i f(i) e_i \| \\
&\le \sum_{i=n+1}^{\infty} \|a_i f(i) e_i\| \\
&\le \sum_{i=n+1}^{\infty} |a_i||f(i)| \\
&\le (\sum_{i=n+1}^{\infty} |a_i|^2)^{1/2} (\sum_{i=n+1}^{\infty} |f_i|^2)^{1/2} \\
&\le (\sum_{i=n+1}^{\infty} |a_i|^2)^{1/2} \|f\|.
\end{aligned}$$

That implies $\frac{\|(T_0 - T_{0n})(f)\|}{\|f\|} \le (\sum_{i=n+1}^{\infty} |a_i|^2)^{1/2}$. Then

$$\|T_0 - T_{0n}\| \le (\sum_{i=n+1}^{\infty} |a_i|^2)^{1/2}$$

and $T_{0n} \to T_0$, as $n \to \infty$. So the operator T_{0n} is linear and bounded. It has finite range also (because, $\mathrm{Im}(T_{0n}) \subseteq [e_1, \dots, e_n]$). Therefore, T_{0n} is compact and consequently T_0 is compact.

• T_0 is injective. Let $f \in \mathbb{E}$ and assume that $T_0(f) = 0$. Then

$$\sum_{i=1}^{\infty} a_i f(i) e_i = 0, \quad (a_i)_{i=1}^{\infty} \in l_2.$$

with $a_i \neq 0$ for every $i \in \mathbb{N}$. Hence, $f(i) = 0$ for every $i \in \mathbb{N}$. Therefore, $f \equiv 0$.
• T_0 is self-adjoint. Let $f, g \in l_2$ and note that

$$T_0(f) = \sum_{n=1}^{\infty} a_i f(i) e_n = (a_i f(i))_{i=1}^{\infty}.$$

hence

$$\langle T_0(f), g\rangle = \sum_{n=1}^{\infty} a_n f(n) g(n) = \sum_{n=1}^{\infty} a_n g(n) f(n) = \langle f, T_0(g)\rangle.$$

Claim: $\operatorname{Im}(T_0) \subsetneqq \mathbb{E}$.

We will show that $(a_i)_{i=1}^{\infty} \notin \operatorname{Im}(T_0)$. In fact, if $(a_i)_{i=1}^{\infty} \in \operatorname{Im}(T_0)$, then there exists $g \in \mathbb{E}$ such that:

$$T_0(g) = \sum_{i=1}^{\infty} a_i g(i) e_i = \sum_{i=1}^{\infty} a_i e_i.$$

Hence, $g(i) = 1$ for every $i \in \mathbb{N}$. So, $g \notin \mathbb{E}$; therefore, $(a_i)_{i=1}^{\infty} \notin \operatorname{Im}(T_0)$ and $\operatorname{Im}(T_0) \subsetneqq \mathbb{E}$.

Let $\mathbb{F} \subseteq \mathbb{E}$ be a hyperplane in $\mathbb{E}$ which contains $\operatorname{Im}(T_0)$ (see Proposition 2.6). Then $\mathbb{F}$ is dense in $\mathbb{E}$. Therefore, $\operatorname{Im}(T_0) \subsetneqq \mathbb{E}$ is dense in $\mathbb{E}$ (see Corollary 2.3) and there exists a linear operator $\sigma : \mathbb{E} \to \mathbb{R}$ with $\ker(\sigma) = \mathbb{F}$ (see Proposition 2.5), hence σ is discontinuous, because $\ker(\sigma)$ is dense, see (Proposition 2.7 and 2.9).

In $\mathbb{E} \times \mathbb{R}$ we define the inner product

$$\langle (x, \lambda), (y, \alpha)\rangle = \langle x, y\rangle_{\mathbb{E}} + \lambda\alpha.$$

With this product $\mathbb{E} \times \mathbb{R}$ is a Hilbert space. Let

$$\mathbb{H} = \big\{(x, \sigma(x)) \in \mathbb{E} \times \mathbb{R} : x \in \mathbb{E}\big\}.$$

In this way, $\mathbb{H}$ is a subspace of $\mathbb{E} \times \mathbb{R}$. Then $\mathbb{H}$ is dense in $\mathbb{E} \times \mathbb{R}$. Let $\phi : \mathbb{E} \times \mathbb{R} \to \mathbb{R}; \phi(x, t) = \sigma(x) - t$. Then ϕ is discontinuous, because σ is discontinuous. Therefore, $\ker(\phi)$ is dense in $\mathbb{E} \times \mathbb{R}$, and this implies that $\mathbb{H}$ is not a Hilbert space. The operator

$$T : \mathbb{H} \to \mathbb{H}, \quad T((x, \sigma(x))) = (T_0(x), 0)$$

is well-defined, linear, compact, self-adjoint, and injective.
• In order to prove that T is well-defined, let us note that $T_0 \subseteq F = \ker(\sigma)$. In this way $\sigma(T_0(x)) = 0$, that means, $(T_0(x), 0) \in \mathbb{H}$. Hence, T is well-defined.
• T is compact. Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{H}$, this means that $(z_n)_{n=1}^{\infty} = \{((x_n), \phi(x_n))\}_{n=1}^{\infty}$ with $(x_n)_{n=1}^{\infty} \subseteq \mathbb{E}$ and $\{\sigma(x_n)\}_{n=1}^{\infty} \subseteq \mathbb{R}$. Hence $(x_n)_{n=1}^{\infty}$ is bounded (characterization of the product space). T is compact because $T_0$is compact.
• T is self-adjoint. Let $z = (x, \sigma(x))$ and $w = (y, \sigma(y))$ in $\mathbb{H}$. Then

$$\begin{aligned}\langle z, T(w)\rangle &= \langle (x, \sigma(x)), T(y, \sigma(y))\rangle \\ &= \langle (x, \sigma(x)), T_0(y), 0\rangle \\ &= \langle x, T_0(y)\rangle_{\mathbb{E}} + \sigma(x)0 \\ &= \langle x, T_0(y)\rangle_{\mathbb{E}}.\end{aligned}$$

Similarly, it is proved that

$$\langle T(z), w\rangle = \langle T_0(x), y\rangle_{\mathbb{E}}.$$

• T is injective. Let $z = (x, \sigma(x))$ be in $\mathbb{H}$ such that $T(z) = 0$ this means that $T(x, \sigma(x)) = 0$. Hence $T_0(x) = 0$, then $x = 0$, and T_0 is injective. Now $\sigma(x) = 0$ because σ is linear. Therefore $z = 0$. Hence T is injective.

We have obtained that the operator T satisfies the hypotheses of the Spectral Decomposition Theorem, then $\mathbb{H}$ possesses a countable infinite orthonormal basis $\{e_1, \dots, e_n, \dots\}$, such that $T(e_n) = \lambda_n e_n$ and $\lambda_n \to 0$ as $n \to \infty$, $\lambda_n \neq 0$. Hence $e_n = (u_n, \sigma(u_n))$, $u_n \in \mathbb{E}$ and $\sigma(u_n) \in \mathbb{R}$, and by definition

$$T[(u_n, \sigma(u_n))] = (T_0(u_n), 0) = \lambda_n(u_n, \sigma(u_n)).$$

Hence $\lambda_n\sigma(u_n) = 0$, this means that $\sigma(u_n) = 0$ for every $n \in \mathbb{N}$. Therefore, any element of $\mathbb{H}$ of the form $(x, \sigma(x))$, with $\sigma(x) \neq 0$, can be developed in a Fourier series in terms of this orthonormal system.

Now we state the main result in this paper.

Theorem 3.2. *Let $\mathbb{H}$ be a prehilbert space and let $T \in L(\mathbb{H})$ be compact, self-adjoint, and super-injective operator. If $\dim(\mathbb{H}) = \infty$, then $\mathbb{H}$ possesses a countable infinite orthonormal basis $\{e_1, \dots, e_n, \dots\}$, such that $T(e_n) = \lambda_n e_n$; $|\lambda_1| \geq |\lambda_2| \geq \dots$ and $\lambda_n \to 0$ as $n \to \infty$, $\lambda_n \neq 0$. Also, for each $x \in \mathbb{H}$ we have*

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n.$$

In particular, $\operatorname{Im}(T)$ *is dense in* $\mathbb{H}$.

Proof. For $x \in \mathbb{H}$, let

$$x_n = \sum_{i=1}^{n} \langle x, e_i\rangle e_i.$$

According to proposition 2.8, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Then $(T(x_n))_{n=1}^{\infty}$ is a Cauchy sequence. Since T is compact, $(T(x_n))_{n=1}^{\infty}$ possesses a convergent subsequence and thus $(T(x_n))_{n=1}^{\infty}$ is also convergent; say $T(x_n) \to z$.
Claim: $z = T(x)$.

By the properties of the inner product and the linearity of T we have

$$\begin{aligned}
\langle T(x_n), e_j\rangle &= \langle T(\sum_{i=1}^{n} \langle x, e_i\rangle e_i), e_j\rangle \quad \text{for each } j \in \mathbb{N} \\
&= \sum_{i=1}^{n} \langle x, e_i\rangle \langle T(e_i), e_j\rangle \\
&= \sum_{i=1}^{n} \langle x, e_i\rangle \langle \lambda_i e_i, e_j\rangle \\
&= \sum_{i=1}^{n} \langle x, \lambda_i e_i\rangle \langle e_i, e_j\rangle \\
&= \langle x, \lambda_i e_j\rangle \quad \text{if } 1 \leq j \leq n.
\end{aligned}$$

Thus

$$\langle T(x_n), e_j\rangle = \langle x, \lambda_j e_j\rangle = \langle x, T(e_j)\rangle = \langle T(x), e_j\rangle, \quad 1 \leq j \leq n;$$

hence

$$\lim_{n\to\infty}\langle T(x_n), e_j\rangle = \langle \lim_{n\to\infty} T(x_n), e_j\rangle = \langle T(x), e_j\rangle \quad \forall j\in\mathbb{N}.$$

Therefore,

$$\begin{gathered}\langle z, e_j\rangle = \langle T(x), e_j\rangle \quad \forall j\in\mathbb{N},\\ \langle z - T(x), e_j\rangle = 0 \quad \forall j\in\mathbb{N}.\end{gathered}$$

Hence $z - T(x) = 0$ which implies $z = T(x)$.

On the other hand since $T(x_n)\to T(x)$, $T(x_n - x)\to 0$ (because T is linear and bounded). Thus $(x_n - x)\to 0$, (thus $(x_n - x)$ is a Cauchy sequence and T is super-injective) this implies $x_n \to x$.

It just remains to prove that $\mathrm{Im}(T)$ is dense in $\mathbb{H}$. Let $y\in\mathbb{H}$ ($y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$). Put

$$t_n = \sum_{i=1}^{n}\langle y, e_i\rangle e_i.$$

Then $t_n \to y$, as $n\to\infty$. On the other hand

$$t_n = \sum_{i=1}^{n}\langle y, e_i\rangle e_i = \sum_{i=1}^{n}\langle y, e_i\rangle \lambda_i T(e_i) = T[\sum_{i=1}^{n}\frac{1}{\lambda_i}\langle y, e_i\rangle e_i].$$

This implies $t_n\in\mathrm{Im}(T)$, thus $y\in\overline{\mathrm{Im}(T)}$. Therefore, $\mathrm{Im}(T)$ is dense in $\mathbb{H}$. $\square$

Example 3.3. As an application of Theorem 3.2, we show that

$$\{\frac{1}{\sqrt{2\pi}}\}\bigcup\{\frac{1}{\sqrt{\pi}}\cos(nt) : n\in\mathbb{N}\}\bigcup\{\frac{1}{\sqrt{\pi}}\sin(nt) : n\in\mathbb{N}\}, \tag{3.1}$$

is an orthonormal basis for $(C, \langle\cdot,\cdot\rangle_C)$. First we proved that that the mapping

$$L : C^2 \to C, \quad L(x) = x'' - x,$$

is an topologyc isomorphism, where C denotes the set of continuous and $2\pi-$periodic functions from $\mathbb{R}$ into itself and C^2 denotes the set of the 2π-periodic functions from $\mathbb{R}$ into itself, which possess continuous second derivative.

Proposition 3.4. *The composition $T = (I\circ L^{-1}) : C\to C(L : C^2\to C)$, $L(x) = x'' - x$, $I : C^2\to C$, $I(x) = x)$ is compact, self-adjoint, and injective. In addition, $\mu\in\mathbb{R}$ is an eigenvalue of T if and only if there is a not trivial $\mu\in C^2$ such that $\mu u'' = (1+\mu)u$.*

In addition, we can observe that the operator T, given in the previous proposition is super-injective. We prove this using some results of Lebesgue Theory, which are presented after some definitions.

A property which is certain, except for a set of measure zero, it is said to be valid *almost everywhere* (a.e.).

The space of all functions f for which $|f|^p$ is Lebesgue integrable on $[a, b]$; in other words $\int_a^b |f(x)|^p dx < \infty$ with $p\geq 1$, is denoted by $L^p_{[a,b]}$ (or L^p, if no confusion arises). Every space L^p is complete.

Corollary 3.5. *Let f_n be a sequence of functions that possesses continuous first derivative, and such that:*

1. *f_n converges uniformly to 0 on $[a, b]$*
2. *If f_n' converges to h in L^2.*

Then $h = 0$ *a.e.*

With this result, we will prove that the operator T is super-injective. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in C such that

$$T(x_n) \to 0 \quad \text{on } C \text{ as } n \to \infty. \tag{3.2}$$

Put

$$y_n = T(x_n) = (I \circ L^{-1})(x_n) = L^{-1}(x_n).$$

Hence, $x_n = L(y_n)$. Since is L an isomorphism, there exists $k > 0$, such that

$$\|L(y_n)\|_C \geq k\|(y_n)\|_{C^2}.$$

Then

$$\|L(y_n) - L(y_m)\|_C = \|L(y_n - y_m)\|_C \geq k\|y_n - y_m\|_{C^2}.$$

This implies

$$\|x_n - x_m\|_C \geq k\|y_n - y_m\|_{C^2}.$$

Hence, $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in C^2. Thus, $(y_n')_{n=1}^{\infty}$ and $(y_n'')_{n=1}^{\infty}$ are Cauchy sequence in C.

On the other hand, because L is continuous, there exists $M > 0$, such that

$$\|L(y_n)\|_C \leq M\|y_n\|_{C^2}.$$

this implies

$$\|x_n\|_C \leq M\|y_n\|_{C^2}.$$

Now we just must to prove that $y_n \to 0$ in C^2, and thus $x_n \to 0$ en C. In order to do it, we prove that $y_n' \to 0$, and $y_n'' \to 0$ in C.

First, we prove that $y_n' \to 0$ in C. In fact, as $(y_n)_{n=1}^{\infty}$ and $(y_n')_{n=1}^{\infty}$ are Cauchy sequences in C, then $(y_n)_{n=1}^{\infty}$ and $(y_n')_{n=1}^{\infty}$ are bounded in C. In virtue of Corollary 2.4, there exists w in C and a subsequence $(y_{n_k})_{n_{k=1}}^{\infty}$ of $(y_n)_{n=1}^{\infty}$, such that $(y_{n_k})_{n_{k=1}}^{\infty}$ converges uniformly to w in $[0, 2\pi]$. Consequently, $(y_{n_k})_{n_{k=1}}^{\infty}$ converges uniformly to w in C. As $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in C, then $(y_n)_{n=1}^{\infty}$ converges also in C and it converges to w.

Claim: $(y_n)_{n=1}^{\infty}$ converges uniformly to w.

Assume that there exists $\epsilon_0 > 0$ and a subsequence $(v_n)_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ such that

$$\|v_n - w\|_{\infty} > \epsilon_0 \quad \text{for all } k \in \mathbb{N}. \tag{3.3}$$

Since $(v_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are bounded in C, then according to Corollary 2.4, there exists h in C and a subsequence $(w_k)_{k=1}^{\infty}$ de $(v_n)_{n=1}^{\infty}$ which converges uniformly to h on $[0, 2\pi]$. According to the argument above, we have $v_k \to h$ in C. Hence, $(y_n)_{n=1}^{\infty}$ converges also to h. Therefore, $h \equiv w$. By (3.3),

$$\|h - w\|_{\infty} > \epsilon_0$$

which is a contradiction. This proves the Claim and by (3.2) that $w = 0$.

Since $(y_n')_{n=1}^{\infty}$ is a Cauchy sequence in C and L^2 is complete, then there exists $z \in L^2$, such that $y_n' \to z$ in L^2, in virtue of Corollary 3.5, $z \equiv 0$, a.e., that means, $y_n' \to 0$ in C.

To work with $(y_n')_{n=1}^{\infty}$, instead of $(y_n)_{n=1}^{\infty}$ we have that $(y_n'')_{n=1}^{\infty}$ is a Cauchy sequence in C. By applying the Corollary 3.5, we obtain that $y_n'' \to 0$. Thus, we have proved that $(x_n)_{n=1}^{\infty}$ converges to 0 in C. Therefore the operator T is super-injective. Then according to the Theorem 3.2, each element of C can be developed in Fourier series, in terms of an orthonormal basis of C, and it will be shown that it is given by (3.1).

On the other hand, let μ be an eigenvalue of T, then there exists $u \in C$, not trivial, such that $T(u) = \mu u$. But

$$T(u) = (I \circ L)^{-1}(u) = L^{-1}(u).$$

Hence $u = \mu(u'' - u)$. This implies

$$\mu u'' = (1+\mu)u.$$

Now, suppose that there exists a non trivial function $u \in C^2$ such that

$$\mu u'' = (1+\mu)u \quad \text{for some } \mu \in \mathbb{R}.$$

Hence, it is obtained that $T(u) = \mu u$.

The previous proposition says that to determine the eigenvalues of T, we must find those values $\alpha \in \mathbb{R}$, for which there exists a non trivial $u \in C^2$, such that $u'' = \alpha u$.

Remarks. 1. Let $\alpha = k^2$. Then

$$u = c_1 e^{kt} + c_2 e^{-kt}$$
$$u' = k(c_1 e^{kt} - c_2 e^{-kt})$$

Because $u \in C^2$, we have $u(0) = u(\varpi)$ and $u'(0) = u'(\varpi)$; therefore,

$$c_1 + c_2 = c_1 e^{k2\varpi} + c_1 2e^{-k2\varpi},$$
$$c_1 - c_2 = c_1 e^{k2\varpi} - c_1 2e^{-k2\varpi}$$

which is a homogenous system having only trivial solutions for c_1, c_2.

2. If $u'' = 0$ and $u \in C^2$, then u is constant. In fact, u has the form $u(t) = A + Bt$ for some constants A, $B \in \mathbb{R}$. Since u is $2\pi-$periodic, we obtain that $B = 0$.

3. If $\alpha = -k^2 < 0$ and there exists $u \in C^2$ such that $u'' = -k^2 u$, then k is an integer and $u(t) = A\sin(kt) + B\cos(kt)$ for certain constants A, $B \in \mathbb{R}$. Put $u_1(t) = \sin(kt)$ $u_2(t) = \cos(kt)$, then $u_i'' = k^2 u_i$ for $i = 1, 2$.

Let $u : \mathbb{R} \to \mathbb{R}$ be of class C^2 such that $u'' = -k^2 u$. Define

$$\varphi(t) = k^{-1}[u'(0)\sin(kt) + u(0)]\cos(kt).$$

Then

$$\varphi'' = -k^2\varphi, \quad \varphi(0) = u(0), \quad \varphi'(0) = u'(0).$$

To reason as we did above, let

$$u(t) = A\sin(kt) + B\cos(kt),$$

where A, $B \in \mathbb{R}$ are constants. An expression of the type $A\sin(kt) + B\cos(kt)$, is 2π-periodic if and only if k is an integer.

To conclude we have that the problem: $x'' = \alpha x$, with $x \in C^2$ has a non trivial solution u if and only if $\alpha = -k^2$, where $k \geq 0$ is an integer. Consequently, the eigenvalues of T are $\mu = -1/(1+k^2)$, with k a positive integer. In addition,

$$N(\mu_k) = \{A\sin(kt) + B\cos(kt) : A,\ B \in \mathbb{R}\}$$

for $k \geq 0$, and

$$N(\mu_0) = \big\{u \in C^2_{[0,2\pi]} : u \text{ is constant}\big\}.$$

In particular, $\dim[N(\mu_0)] = 1$ and $\dim[N(\mu_k)] = 2$ if $k \geq 1$. On the other hand, the constant $1/\sqrt{2\pi}$ is an orthonormal basis of $N(\mu_0)$, while

$$\big\{[1/\sqrt{\pi}]cos(kt),\ [1/\sqrt{\pi}]\sin(kt)\big\}$$

is an orthonormal basis of $N(\mu_k)$ for $k \geq 1$. Thus, from the proof of Theorem 3.2, we obtain that

$$\{\frac{1}{\sqrt{2\pi}}\}\bigcup\{\frac{1}{\sqrt{\pi}}\cos(nt) : n \in \mathbb{N}\}\bigcup\{\frac{1}{\sqrt{\pi}}\sin(nt) : n \in \mathbb{N}\},$$

is an orthonormal basis of C.

References

[1] A. N. Tijonov and A. A. Samarsky, *Ecuaciones de la Física Matemática*, Editorial Mir, Moscú 1980.

[2] J. Dieudonne, *Foundations of Moderno Analysis*, Academic Pres, New York-London 1960.

[3] S.R. Murray, *Teoría y Problemas de Variables Reales. Medida e Integración de Lebesgue*, Resselear Politechnic Intitute, Mexico 1976.

[4] G. Bachman and L. Narici, *Funtional Analysis*. Academic Press IMC., 1966.

[5] Hanzel Lárez. *Teorema de Descomposición Espectral para Operadores Compactos y Autoadjuntos en Espacios Prehilbert.* ULA, Mérida, Venezuela 1994.

[6] John B. Conway, *A Coursse in Functinal Analysis.* Springer-Verlag, Bioomington, Indiana 1989.

Departamento de Física y Matemáticas, Núcleo Universitario "Rafael Rangel" Universidad de Los Andes, Trujillo, Trujillo, Venezuela

E-mail address: `larez@ula.ve`