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FRIEDRICHS MODEL OPERATORS OF ABSOLUTE TYPE WITH ONE SINGULAR POINT

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Abstract. Problems of existence of the singular spectrum on the continuous spectrum emerges in some mathematical aspects of quantum scattering theory and quantum solid physics. In the latter field, this phenomenon results from physical effects such as the Anderson transitions in dielectrics. In the study of this problem, selfadjoint Friedrichs model operators play an important part and constitute quite an apt model of real quantum Hamiltonians. The Friedrichs model and the Schrödinger operator are related via the integral Fourier transformation. Similarly, the relationship between the Friedrichs model and the one dimensional discrete Schrödinger operator on $\mathbb{Z}$ is established with the help of the Fourier series. We consider a family of selfadjoint operators of the Friedrichs model. These absolute type operators have one singular point $t = 0$ of positive order. We find conditions that guarantee the absence of point spectrum and the singular continuous spectrum for such operators near the origin. These conditions are actually necessary and sufficient. They depend on the finiteness of the rank of a perturbation operator and on the order of singularity. The sharpness of these conditions is confirmed by counterexamples.

1. Introduction

Problem of existence of the singular spectrum on the continuous spectrum emerges in some mathematical aspects of quantum scattering theory and quantum solid physics. In the latter field this phenomenon results from physical effects such as the Anderson transitions in dielectrics. Note that we understand the singular spectrum as the union of the point spectrum and the singular continuous spectrum. In the study of this problem an important part is played by the selfadjoint Friedrichs model operator $S_1 := t \cdot +V$ acting in $L_2(\mathbb{R})$ (where $t\cdot$ stands for the operator of multiplication by the independent variable $t \in \mathbb{R}$, and V is an integral operator with a continuous Hermitian kernel). This operator constitutes quite an apt model of real quantum Hamiltonians. In [2] it was shown how the Friedrichs model can be used for the study of the spectral properties of the Schrödinger operator $(-\Delta + q)$. These operators are related via the integral Fourier transformation. A large body of literature is devoted to this model; we mention the papers by Faddeev, Pavlov,

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Naboko, Iakovlev and others [1, 2, 10, 4, 5, 6, 7, 13, 14]. For the first time the fact that here the singular spectrum may arise indeed was established by Pavlov and Petras (1970) in [10]. Radically new conditions on V guaranteeing the finiteness of the point spectrum of S_1 (the singular continuous spectrum is missing) have been found in the papers [1, 7]. Since, actually, these conditions are necessary and sufficient in the context of the selfadjoint Friedrichs model the problem in question was solved completely in [1].

Further elaboration of this topic seems to be of value. Namely, it is of interest to investigate the singular spectrum of perturbations of the operators of multiplication by a function $f(t)$ of the independent variable (for example, $f(t)$ is equal to $\cos t$ or t^2). Such operators naturally appear when various models of the Schrödinger operator are considered in a momentum representation. For example, the operator of multiplication by t^2 is obtained if we write the Schrödinger operator in a momentum representation. Similarly, the relationship between the Friedrichs model and the one dimensional discrete Schrödinger operator S on $\mathbb{Z}$ is established with the help of the Fourier series. The operator S is equal to $(U + U^*) + q$ and is defined on the space $l_2(\mathbb{Z})$ of square summable complex sequences $u = \{u_n\}_{n=-\infty}^{+\infty}$; here U is the operator of right shift, U^* is its adjoint, and $q = \{q_n\}_{-\infty}^{\infty}$, so that $(Uu)_n = u_{n-1}$, $(U^*u)_n = u_{n+1}$, and $(q\,u)_n = q_n \cdot u_n$ [8, 9]. Under the isomorphism between $l_2(\mathbb{Z})$ and $L_2(-\pi,\pi)$ given by the map

$$\Phi^{-1} : u \to \tilde{u}(t) = \sum_{n=-\infty}^{+\infty} u_n \cdot e^{\imath nt}, \tag{1.1}$$

the operator S turns into $\tilde{S}$ acting by the formula

$$\tilde{S}\tilde{u}(t) = 2\cos(t) \cdot \tilde{u}(t) + \int_{-\pi}^{\pi} \tilde{q}(t-x) \cdot \tilde{u}(x)\,dx, \tag{1.2}$$

where $\tilde{q}(t) = \sum_n q_n \cdot e^{\imath nt}$, $\tilde{u} \in L_2(-\pi,\pi)$. Indeed,

$$\begin{aligned}(\tilde{S}\tilde{u}) =& \Phi^{-1}\left[(U+U^*) + q\right] u = \sum_{n=-\infty}^{+\infty} (u_{n-1} + u_{n+1} + q_n u_n) e^{\imath nt} \\ =& \sum\nolimits_n u_n e^{\imath(n+1)t} + \sum\nolimits_n u_n e^{\imath(n-1)t} + \sum\nolimits_n q_n u_n e^{\imath nt} \\ =& 2\cos(t) \sum\nolimits_n u_n e^{\imath nt} + \int_{-\pi}^{\pi} \tilde{q}(t-x) \cdot \tilde{u}(x)\,dx\,. \end{aligned} \tag{1.3}$$

Obviously, that the change of variables $2\cos t = x$ would reduce the study of $\sigma_{sing}(\tilde{S})$, the singular spectrum of the operator $\tilde{S}$, to that of $\sigma_{sing}(S_1)$. However, since $(\cos t)' = -\sin t|_{\pm\pi} = 0$, this substitution is not smooth (that is, not diffeomorphism) near the points $\pm\pi$ and, therefore, may lead to a loss of subtle information concerning the structure of $\sigma_{sing}(\tilde{S})$. It is clear that, having an idea of the structure of the set $\sigma_{sing}(S_1)$, we can deduce some information about the set $\sigma_{sing}(\tilde{S})$ (also near the singular points $t = \pm\pi$) with the help of the change of variables, but the results obtained in this way will be expressed in inconvenient terms, and their sharpness near zero will be less than satisfactory.

Thus, as a model in the theory of continuous spectrum perturbations it seems reasonable to consider the perturbations not of the operator of multiplication by the independent variable t, but of the operator of multiplication by a function of t.

In this case the main attention must be paid to the singular spectrum in a neighborhood of so called singular points next to which it is impossible to introduce a smooth (locally) change of variables reducing our problem to the standard Friedrichs model. It will be shown that in a neighborhood of such points the behavior of the singular spectrum acquires a quite different character.

The following two functions $f_1(t) = |t|^m$ and $f_2(t) = \operatorname{sgn} t \cdot |t|^m$ have one zero of order $m > 0$ at the point $t = 0$. And near the origin f_1 and f_2 have a different behavior. To these functions there correspond the selfadjoint Friedrichs model operators A_m, $m > 0$, with one singular point $t = 0$

$$A_m = |t|^m \cdot +V \qquad \text{(the absolute type operators)}\ , \tag{1.4}$$

and the operators S_m

$$S_m = \operatorname{sgn} t \cdot |t|^m \cdot +V \qquad \text{(the symmetric type operators)} \tag{1.5}$$

also with one singular point $t = 0$ for $m \neq 1$. The operator $S_1 = t \cdot +V$ is the main operator of the Friedrichs model. It has no singular points. In this paper we study the case of the operators A_m. The operators $S_m\,, m \neq 1$, were partially considered in [3].

2. Statement of the problem and main result.

In $L_2(\mathbb{R})$ we consider a family of selfadjoint operators $A_m\,, m > 0$, given by

$$A_m = |t|^m \cdot +V\,. \tag{2.1}$$

Here $|t|^m\cdot$ is the operator of multiplication by the function $|t|^m$ of the independent variable $t \in \mathbb{R}$,and V (perturbation) is an integral operator with a continuous Hermitian kernel $v(t,x)$. Thus, the action of the operator A_m can be written as follows

$$\big(A_m u\big)(t) = |t|^m \cdot u(t) + \int_{\mathbb{R}} v(t,x)u(x)\,dx\,. \tag{2.2}$$

We assume that V is non-negative and belongs to the trace class σ_1 :

$$V \geq 0\,, \quad V \in \sigma_1\,. \tag{2.3}$$

Consequently, the operator A_m is defined on the domain of functions $u(t) \in L_2(\mathbb{R})$ such that $|t|^m u(t) \in L_2(\mathbb{R})$. The kernel $v(t,x)$ is assumed to satisfy the following smoothness condition

$$v(t+h,t+h) + v(t,t) - v(t+h,t) - v(t,t+h) \leq \omega^2(|h|), \quad |h| \leq 1\,, \tag{2.4}$$

with the function $\omega(t)$ (the modulus of continuity of V) monotone and satisfying a Dini condition:

$$\omega(t) \downarrow 0 \quad \text{as } t \downarrow 0\,, \quad \text{and} \quad \int_0^1 \frac{\omega(t)}{t}\,dt < \infty\,. \tag{2.5}$$

Inequality (2.4) may be regarded as a smoothness condition for the kernel $v_{1/2}(t,x)$ of the integral operator $V^{1/2}$, because, as shown in [6], the expression on the left in (2.4) can be written as the integral $\int_{\mathbb{R}} |v_{1/2}(t+h,x) - v_{1/2}(t,x)|^2\,dx$ (and, therefore, is nonnegative). Together with (2.4) the fact that V is of class σ_1 means that the kernel $v(t,x)$satisfies a certain condition of decrease at infinity. The requirement that the operator V be of trace class σ_1 is sufficient for the absolutely continuous spectrum of A_m to coincide with the real semi-axis $\mathbb{R}_+ = [0\,,+\infty)$ (see [11]).

Near the singular point $t = 0$ we study the dependence of the behavior of the point and singular continuous spectrum on the smoothness of the kernel $v(t, x)$. As noted above the structure of the singular spectrum $\sigma_{sing}(S_1)$ of the operator $S_1 = t \cdot +V$ (the usual Friedrichs model operator) is pretty well studied [1, 2, 4, 5, 6, 7, 10, 13, 14]. In particular, in the papers [7, 1] it was shown that for this operator there exists a sharp condition of finiteness of the singular spectrum. Namely, if $\omega(t) = O(\sqrt{t})$ as $t \to 0$, the singular spectrum of S_1 consists of at most a finite number of eigenvalues of finite multiplicity (the singular continuous spectrum is absent). On the other hand, if $\overline{\lim}_{t\to 0}\omega(t)/\sqrt{t} = +\infty$, then examples are constructed showing that even in the case when V is a rank 1 perturbation the eigenvalues of S_1 may have cluster points. By using the simple change of variables $|t|^m = x$, we can show that outside any neighborhood of the origin on the interval $[0, +\infty)$ the structure of the spectrum $\sigma_{sing}(A_m)$ is locally identical with that of the operator S_1. This result is explained by the smoothness of the above change of variables outside any neighborhood of the origin, and also by the local character of the main results of [1, 2, 10, 4, 5, 6, 7, 13, 14] relating to the structure of $\sigma_{sing}(S_1)$. Here by locality we mean the following. Suppose that conditions (2.4), (2.5) are fulfilled only in some interval $(c, d) \subset \mathbb{R}$, then the main results in [1, 2, 10, 4, 5, 6, 7, 13, 14] about the structure of $\sigma_{sing}(S_1)$ remain true in any closed subinterval $\Delta \subset (c, d)$. However, as shown in this paper, in a neighborhood of the origin the behavior of $\sigma_{sing}(A_m)$ is quite different. Here, near zero, we can still use the change of variables $|t|^m = x$ mentioned above, but, since, e.g., $(|t|^m)'_{|0} = 0$ for $m > 1$, this change is not smooth (that is, not a diffeomorphism) near zero. In this sense the zero point is a singular point of the operators $A_m\,, m > 0$, so it needs a special inspection. Observe that the origin is also a boundary point of the continuous spectrum of A_m, which coincides with the interval $[0, +\infty)$.

Naturally, there appears a problem of finding sharp, in a sense, conditions on the kernel $v(t, x)$ that guarantee that the singular spectrum is absent near the origin. In this paper it is shown that such sufficient conditions are given in terms of asymptotic behavior of the modulus of continuity $\omega(t)$ as t tends to zero. It appears that for $m \in (1\,, 3]$ these conditions also depend on a rank of the perturbation operator V. Namely, if $\operatorname{rank} V < \infty$, then provided that $\omega(t) = O(t^{(m-1)/2})$, $t \to 0$, the spectrum near zero is purely absolutely continuous. But if $\operatorname{rank} V = +\infty$, then the structure of $\sigma_{sing}(A_m)$ depends on the value of a constant C in the condition $\omega(t) = Ct^{(m-1)/2}$. The sharpness of these conditions is confirmed by counterexamples. For $m \leq 1$ the spectrum is always purely absolutely continuous in some neighborhood of the zero point on the interval $[0, +\infty)$. At the same time for $m > 3$ the singular spectrum may appear near zero for any modulus of continuity $\omega(t)$. Hence, for $m > 3$ near zero there is no condition of the singular spectrum absence in terms of $\omega(t)$ as for $m \in (1\,, 3]$.

In Sections 3, 4 the main results of the paper are formulated. In Section 3 sufficient conditions on the perturbation V are given ensuring the singular spectrum absence near the origin. Counterexamples constructed in Section 4 show that these conditions are sharp. Note that some results of this paper (for the case $m \in \mathbb{N}$) were announced in [15], and the case of $m = 2$ has been in detail considered in [16].

3. Sufficient conditions for absolute continuity of the spectrum on $[0,+\infty)$ near zero

For $z \in \mathbb{C} \setminus [0,+\infty)$ we define an analytic operator–valued function $T_m(z) : E \to E$, where $E := \overline{R(V)}$ is the closure of the range of V, as follows:

$$T_m(z) := -\sqrt{V}\,(|t|^m - z)^{-1}\sqrt{V}\,. \tag{3.1}$$

Here $(|t|^m - z)^{-1}$ denotes the operator of multiplication by the corresponding function in $L_2(\mathbb{R})$. Obviously, that $\operatorname{Im} T_m(z) \ge 0$ if $\operatorname{Im} z > 0$, and $T_m(z) \in \sigma_1$.

proposition 3.1. *If V satisfies conditions* (2.3)–(2.5)*, then in the complex plane cut along $[0,+\infty)$ the analytic operator–valued function $T_m(z)$ admits a σ_1–norm continuous extension to the upper and the lower parts of the cut on the interval $(\,0\,;+\infty)$.*

Let $T_m(\lambda) := T_m(\lambda + i0)$, $\lambda > 0$, denote the corresponding boundary values of $T_m(z)$. The set $N_m := \{\lambda > 0 : \exists g \in l_2, g \ne 0, T_m(\lambda)g = g\} \equiv \{\lambda > 0 : \ker(I - T_m(\lambda)) \ne \emptyset\}$ is called a set of roots of the operator–function T_m. The vector g is called a root vector corresponding to the root λ.

proposition 3.2. *If V satisfies conditions* (2.3)– (2.5)*, then $\sigma_{sing}(A_m)$, the singular spectrum of the operator $A_m = |t|^m \cdot +V\,, m > 0$, embeds into the set N_m supplemented by the origin, i.e.,*

$$\sigma_{sing}(A_m) = \overline{\sigma_p(A_m)} \cup \sigma_{s.c.}(A_m) \subseteq N_m \cup \{0\}\,, \tag{3.2}$$

where $\sigma_p(A_m)$ is the point spectrum, and $\sigma_{s.c.}(A_m)$ is the singular continuous spectrum of the selfadjoint operator A_m.

From the Fredholm analytic alternative (see [12, §8]) it follows that the set $N_m \cup \{0\} \subset \mathbb{R}$ is a closed set of Lebesgue measure zero. Also [16, Theorem 3] says that under the condition $V \ge 0$ the point 0 is not an eigenvalue of the operator $A_m = |t|^m \cdot +V$. Below some conditions on the modulus of continuity $\omega(t)$ of the perturbation operator V are given guaranteeing the absolute continuity of the spectrum of the operator A_m on the interval $[0,+\infty)$ near zero. For $m \in (1,3]$ these conditions depend on a rank of the operator V.

Theorem 3.3. *Suppose that conditions* (2.3)–(2.5) *are fulfilled. Then for $m \in (0;1]$ the roots set N_m is empty in some neighborhood of the origin. And, hence, the spectrum of the operator $A_m = |t|^m \cdot +V$, defined by* (2.2)*, is purely absolutely continuous in some neighborhood of the origin on the interval $[0,+\infty)$.*

Theorem 3.4. *Suppose that the perturbation V satisfies conditions* (2.3)– (2.5) *with the function $\omega(t) = C_\omega t^\alpha$, where $\alpha = (m-1)/2$, and $m \in (1;3]$. If*

$$C_\omega < C_m := (2\int_0^1 \frac{(1-x)^{m-1}}{1-x^m}dx)^{-1/2}\,, \tag{3.3}$$

then the roots set N_m is empty in some neighborhood of the origin. Consequently, the spectrum of the operator $A_m = |t|^m \cdot +V$, defined by (2.2)*, is purely absolutely continuous in some neighborhood of the origin on the interval $[0,+\infty)$.*

Note: Clearly that for the modulus of continuity $\omega(t) = C_\omega t^\alpha$ the greatest possible value of α is 1. The value $\alpha = 1$ exactly corresponds to $m = 3$.

Observation: It is not difficult to obtain for the constant C_m a two–sided estimate. Indeed, since for $m > 1$ and $x \in [0, 1]$

$$1 - x \le 1 - x^m \le m(1 - x)\,, \tag{3.4}$$

we have

$$\frac{1}{m}\int_0^1 \frac{(1-x)^{m-1}}{1-x}\,dx \le \int_0^1 \frac{(1-x)^{m-1}}{1-x^m}\,dx \le \int_0^1 \frac{(1-x)^{m-1}}{1-x}\,dx\,. \tag{3.5}$$

Whence

$$\Big(\frac{m-1}{2}\Big)^{1/2} \le C_m \le \Big(\frac{m(m-1)}{2}\Big)^{1/2}. \tag{3.6}$$

At the same time for $m = 2$ and $m = 3$ the integral $\int_0^1 (1-t)^{m-1}/(1-t^m)\,dt$ is evaluated exactly. For $m = 2$ we obtain $C_2 = (1/\ln 4)^{1/2} = 0,849\dots$ that coincides with the value of this constant from the paper [16, Theorem 1]. Likewise, we find that $C_3 = (\pi/\sqrt{3} - \ln 3)^{-1/2} = 1,182\dots$. Note also that from (3.6) it follows immediately that $C_m \to +0$ as $m \to 1^+$ (compare that with the assertion of Theorem 3.3).

If the perturbation V is a finite rank operator, the result of Theorem 3.4 can be improved.

Theorem 3.5. *Suppose that the perturbation operator V satisfies conditions* (2.3)–(2.5) *and* $\operatorname{rank} V < \infty$. *If $m \in (1;3]$ and $\omega(t) = O(t^{(m-1)/2})$ as $t \to 0^+$, then the origin is not a cluster point of the set of roots N_m of the operator-valued function T_m. Consequently, the spectrum of the operator $A_m = |t|^m \cdot +V$, defined by* (2.2), *is purely absolutely continuous in some neighborhood of the origin on the interval* $[0, +\infty)$.

4. Sharpness of the absence conditions for the singular spectrum: Counterexamples

The following theorem states that for an infinite rank perturbation ($\operatorname{rank} V = \infty$) the absence condition for the singular spectrum of A_m, $m \in (1;3]$, near the origin is indeed related to the constant C_ω by $\omega(t) = C_\omega \cdot t^{(m-1)/2}$ as $t \to 0$, (see Theorem 3.4). In particular, this means that the result of Theorem 3.5 cannot be extended to perturbations of infinite rank. Namely, Theorem 3.4 is sharp in the following sense.

Theorem 4.1. *Let $m \in (1;3]$. For any value of $C_\omega \ge \tilde{C}_m := 2^{(m-1)/2}m$ there exists an operator V with* $\operatorname{rank} V = \infty$, *such that V satisfies conditions* (2.3)–(2.5) *with $\omega(t) = C_\omega t^{(m-1)/2}$, and the origin is a cluster point of the set of eigenvalues of the operator $A_m = |t|^m \cdot +V$, defined by* (2.2).

Note: It is easy to verify that $\tilde{C}_m = 2^{(m-1)/2}m \ge (m(m-1)/2)^{1/2} \ge C_m$ for $m > 1$.

If $\operatorname{rank} V = 1$, then $V = (\cdot, \varphi)\varphi$ with $\varphi \in L_2(\mathbb{R})$. In this case the smoothness condition (2.4) is written in the form

$$|\varphi(t+h) - \varphi(t)| \le \omega\big(|h|\big), \quad |h| \le 1\,, \tag{4.1}$$

with the function $\omega(t)$ satisfying condition (2.5). Note that for any function $\varphi(t)$ its actual modulus of continuity $\widetilde{\omega}(h) := \sup\{|\varphi(x) - \varphi(y)| : |x - y| < h\}$ always satisfies the additional constraint of semiadditivity: $\widetilde{\omega}(t_1 + t_2) \le \widetilde{\omega}(t_1) + \widetilde{\omega}(t_2)$ for all $t_1, t_2 \ge 0$.

Theorem 3.5 involves the condition $\omega(t) = O(t^{(m-1)/2})$ as $t \to 0$ ensuring for the finite rank perturbation operator V the emptiness of the roots set N_m near the origin. This condition appears to be sharp in the class of semiadditive functions $\omega(t)$.

Theorem 4.2. *Let $m > 1$. Suppose that $\omega(t)$, $t \geq 0$, is a monotone nondecreasing function satisfying the condition $\omega(0^+) = \omega(0) = 0$ as well as the natural additional condition of semiadditivity: $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ for all $t_1, t_2 \geq 0$. If $\limsup_{t\to 0} \omega(t)/t^{(m-1)/2} = +\infty$, then a compactly supported function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying condition* (4.1) *is constructed and such that the operator $A_m = |t|^m \cdot +(\cdot, \varphi)\varphi$ has a sequence of positive eigenvalues converging to zero.*

Corollary 4.3. *It is not hard to show (see* [3, Lemma 2.2]*) that if $\omega(t)$ is a nonnegative semiadditive function and $\omega(t) \downarrow 0$ as $t \downarrow 0$, then for any $a > 0$ there exists a constant $C > 0$ such that $Ct \leq \omega(t)$ for $t \in [0, a]$. Hence,*

$$\limsup_{t\to 0} \omega(t)/t^{(m-1)/2} = +\infty$$

for all $m > 3$. Therefore, it follows from Theorem 4.2 that for every $m > 3$ and for each monotone and semiadditive function $\omega(t)$, $t \geq 0$, satisfying the condition $\omega(0^+) = \omega(0) = 0$ (and thus nonnegative) a real–valued compactly supported function φ is constructed satisfying the smoothness condition (4.1) *and such that the operator $A_m = |t|^m \cdot +(\cdot, \varphi)\varphi$ has a sequence of positive eigenvalues converging to zero. This means, in particular, that for $m > 3$ there is no condition guaranteeing the absence of the singular spectrum of the operator $A_m = |t|^m \cdot +V$ near the origin in terms of the modulus of continuity $\omega(t)$ of the perturbation V.*

Corollary 4.4. *If $m \in (1, 3]$, then, according to Theorem 4.2, the sufficient condition $\omega(t) = O(t^{(m-1)/2})$ as $t \to 0$ guaranteeing the absence of the singular spectrum of the operator A_m near the origin for the finite rank perturbation operator,* $\operatorname{rank} V < \infty$*, (see Theorem 3.5) is sharp. If this condition is not fulfilled, that is, $\limsup_{t\to 0} \omega(t)/t^{(m-1)/2} = +\infty$, then even in the case when V is a rank 1 perturbation there can exist nontrivial singular spectrum near zero, and, in particular, the operator A_m can have a sequence of positive eigenvalues converging to zero.*

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References

[1] E. M. Dinkin, S. N. Naboko, and S. I. Yakovlev; *The boundary of finiteness of the singular spectrum in the selfadjoint Friedrichs model*, Algebra i Anal. **3** (1991), no. 2, 77–90.

[2] L. D. Faddeev; *On a model of Friedrichs in the theory of perturbations of the continuous spectrum*, Trudy Mat. Inst. Steklov **73** (1964), 292–313.

[3] S. I. Iakovlev; *Examples of Friedrichs model operators with a cluster point of eigenvalues.* International Journal of Mathematics and Mathematical Sciences. . Vol. 2003, pp. 625–638.

[4] Ya. V .Mikityuk; *The singular spectrum of selfadjoint operators*, Dokl. Akad. Nauk SSSR **303** (1988), no.1, 33–36.

[5] S. N. Naboko; *Uniqueness theorems for operator–functions with positive imaginary part and the singular spectrum in the selfadjoint Friedrichs model*, Dokl. Akad. Nauk SSSR **275** (1984), no.6, 1310–1313.

[6] S. N. Naboko; *Uniqueness theorems for operator-valued functions with positive imaginary part, and the singular spectrum in the selfadjoint Friedrichs model*, Arkiv för matematik **25** (1987), no.1, 115–140.

[7] S. N. Naboko and S. I. Yakovlev; *Conditions for the finiteness of the singular spectrum in a selfadjoint Friedrichs model*, Functional Anal. i Pril. **24** (1990), no.4, 88–89.
[8] S. N. Naboko, S. I. Yakovlev; *On point spectrum of the discrete Schrödinger operator*, Funktsional. anal. i Prilozhen. **26** (1992), no. 2, 85–88.
[9] S. N. Naboko, S. I. Yakovlev: *Discrete Schrödinger operator: The point spectrum on the continuous one,* Algebra i Anal. **4** (1992), no. 3, 183–195.
[10] B. S. Pavlov and S. V. Petras; *The singular spectrum of a weakly perturbed multiplication operator*, Functional Anal. i Pril. **4** (1970), no.2, 54–61.
[11] M. Reed and B. Simon; *Methods of modern mathematical physics 4. Analysis of operators.* Academic press, New York–London,1978.
[12] D. R. Yafaev; *Mathematical scattering theory*, Saint-Petersburg University press, 1994.
[13] S. I. Yakovlev; *Perturbations of a singular spectrum in a selfadjoint Friedrichs model*, Vestnik Leningrad Univ. Mat. Mekh. Astronom. (1990), no.1, 116–117.
[14] S. I. Yakovlev; *On the structure of the singular spectrum in selfadjoint Friedrichs model*, L., (1991) (Manuscript, Depon. VINITI, no. 2050–B, 17.05.91).
[15] S. I. Yakovlev; *On the singular spectrum of the Friedrichs model operators in a neighborhood of a singular point*, Functional Anal. i Pril. **32** (1998), no.3, 91–94.
[16] S. I. Yakovlev; *The finiteness bound for the singular spectrum of Friedrichs model operators near a singular point*, Algebra i analis, **10** (1998), no.4, 210–237.

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