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A SYSTEM OF SEMILINEAR EVOLUTION EQUATIONS WITH HOMOGENEOUS BOUNDARY CONDITIONS FOR THIN PLATES COUPLED WITH MEMBRANES

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Abstract. In this work we consider a semilinear initial boundary-value problem modelling an elastic thin plate (in the context of the so-called Kirchhoff-Love theory) coupled with an elastic membrane, regarding homogeneous boundary conditions. By means of the theory of strongly continuous semigroups of linear operators applied to abstract semilinear initial valued problems [16], we obtain existence and uniqueness of a weak solution which is defined in a suitable way.

1. Introduction

In this work we consider a semilinear evolution problem which we pose as follows: Let Ω and Ω_m be two open bounded connected subsets of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$ and $\partial\Omega_m$ so that $\Omega_m \subset\subset \Omega$. Let $\Omega_p := \Omega\backslash\overline{\Omega}_m$ and $\Gamma_1 := \partial\Omega_m$. We decompose $\partial\Omega$ in two connected parts Γ_2 and Γ_3 with $\Gamma_2 \cap \Gamma_3 = \emptyset$, $\sigma_1(\Gamma_2) \neq 0$ and $\sigma_1(\Gamma_3) \neq 0$, where σ_1 is the surface measure on $\partial\Omega$, induced by the Lebesgue measure on $\mathbb{R}$ (see figure 1). Then we consider the system of partial differential equations

$$\begin{aligned}
&\rho_p h \frac{\partial^2 u_p}{\partial t^2}(t,x) + \frac{h^3}{12} \sum_{\alpha,\beta\gamma,\theta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\big(A_{\alpha\beta\gamma\theta}(x) \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}(t,x)\big) \\
&= f_p(t,x,u_p(t,x)) \quad \text{in }]0,T] \times \Omega_p
\end{aligned} \tag{1.1}$$

$$\rho_m \frac{\partial^2 u_m}{\partial t^2}(t,x) - C\Delta u_m(t,x) = f_m(t,x,u_m(t,x)) \quad \text{in }]0,T] \times \Omega_m\,, \tag{1.2}$$

$$\begin{aligned}
&\frac{h^3}{12} \sum_{\alpha,\beta,\gamma,\theta=1}^{2} \nu_\alpha \frac{\partial}{\partial x_\beta}\big(A_{\alpha\beta\gamma\theta} \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}\big) + \frac{h^3}{12} \frac{\partial}{\partial \vec{\tau}}\Big(\sum_{\alpha,\beta,\gamma,\theta=1}^{2} \nu_\alpha \tau_\beta A_{\alpha\beta\gamma\theta} \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}\Big) \\
&= 0 \quad \text{on }]0,T] \times \Gamma_2\,,
\end{aligned} \tag{1.3}$$

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$$\begin{aligned} &\frac{h^3}{12}\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\frac{\partial}{\partial x_\beta}\big(A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}\big)+\frac{h^3}{12}\frac{\partial}{\partial\vec\tau}\Big(\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\tau_\beta A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}\Big)\\ &+C\frac{\partial u_m}{\partial\vec\nu}=0\quad\text{on }]0,T]\times\Gamma_1\,, \end{aligned}\tag{1.4}$$

$$\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\nu_\beta A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}=0\quad\text{on }]0,T]\times(\partial\Omega_p\setminus\Gamma_3),\tag{1.5}$$

$$u_p=\frac{\partial u_p}{\partial\vec\nu}=0\quad\text{on }]0,T]\times\Gamma_3,\tag{1.6}$$

$$u_p=u_m\quad\text{on }]0,T]\times\Gamma_1\,,\tag{1.7}$$

with the initial conditions

$$u_p(0,\cdot)=g_p^0\quad\text{in }\Omega_p,\tag{1.8}$$

$$u_m(0,\cdot)=g_m^0\quad\text{in }\Omega_m,\tag{1.9}$$

$$\frac{\partial u_p}{\partial t}(0,\cdot)=g_p^1\quad\text{in }\Omega_p,\tag{1.10}$$

$$\frac{\partial u_m}{\partial t}(0,\cdot)=g_m^1\quad\text{in }\Omega_m.\tag{1.11}$$

Equations (1.1)-(1.11) describe the vibrations of a structure which consists of a thin elastic anisotropic plate (in the context of the so called Kirchhoff-Love theory) with its middle surface occupying the domain Ω_p, coupled with a membrane occupying the domain Ω_m (see figure 1).

It is supposed that ρ_p and ρ_m are positive constants, where ρ_p (resp. ρ_m) is the density of the middle surface of the plate (resp. the membrane) and h is the thickness of the plate. The coefficients $A_{\alpha\beta\gamma\theta}$ depend on the elastic modulus of the plate and are assumed as C^∞ functions on $\overline{\Omega}_p$; they satisfy the symmetry assumption

$$A_{\alpha\beta\gamma\theta}=A_{\beta\alpha\gamma\theta},\quad A_{\alpha\beta\gamma\theta}=A_{\alpha\beta\theta\gamma},\quad A_{\alpha\beta\gamma\theta}=A_{\gamma\theta\alpha\beta}\tag{1.12}$$

and the coercivity hypothesis

$$\sum_{\alpha,\beta,\gamma,\theta=1}^{2}A_{\alpha\beta\gamma\theta}(x)\xi_{\gamma\theta}\xi_{\alpha\beta}\ge\rho\sum_{\alpha,\beta=1}^{2}\xi_{\alpha\beta}^2\tag{1.13}$$

for all $x\in\Omega_p$ and for all real matrices $(\xi_{\alpha\beta})_{2\times2}$ with $\xi_{\alpha\beta}=\xi_{\beta\alpha}$ for $\alpha,\beta\in\{1,2\}$, where $\rho>0$ is a constant. Moreover it is supposed that the plate is clamped on Γ_3 (equation (1.6)) and is free on Γ_2 (see figure 1).

The vector $\vec\nu=(\nu_1,\nu_2)$ is the unitary outward normal to $\partial\Omega_p$ and $\tau=(\tau_1,\tau_2)=(-\nu_2,\nu_1)$ is the positive oriented unitary tangent vector. C is a positive constant depending on the material forming the membrane. f_p (resp. f_m) is the pressure supported by the plate (resp. the membrane) and depend on the transverse displacement u_p (resp. u_m) of the plate (resp. the membrane). The initial conditions g_p^0 and g_p^1 (resp. g_m^0 and g_m^1) are real functions defined on Ω_p (resp. Ω_m). The equations (1.4) and (1.7) are the boundary conditions expressing the coupling between the plate and the membrane.

We give the definition of weak solution for our semilinear problem (1.1)-(1.11) and with help of the theory of C^0-semigroups of linear operators we obtain a result of existence and uniqueness for this type of solution. For other works in the area of

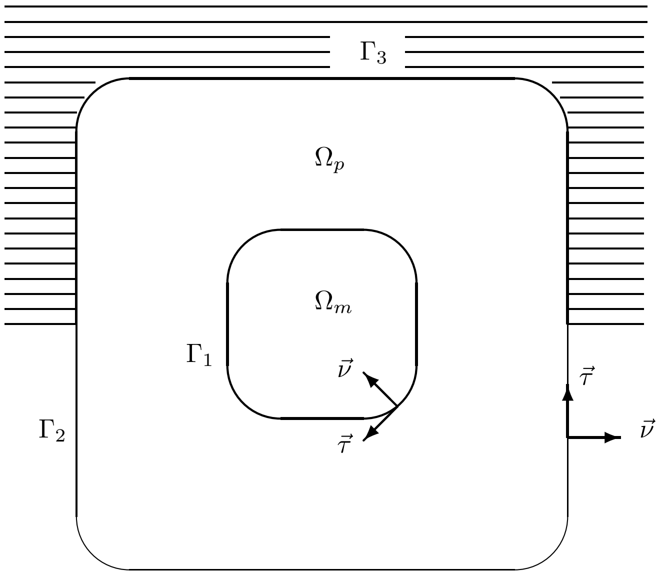

FIGURE 1. $\overline{\Omega}_m$ (resp. $\overline{\Omega}_p$) is occupied by the membrane (resp. the middle surface of the Plate). The Plate is clamped on Γ_3.

transmission problems and networks we refer the reader to [2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 15].

2. Notation and mathematical preliminaries

In this section we shall present the concepts and abstract framework that we need for the treatment of our problem (1.1)-(1.11). We shall consider only real valued functions. Let n a positive integer. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_2)$ (i.e. $\alpha \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of all nonnegative integers), we write

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \text{where} \quad |\alpha| := \alpha_1 + \dots + \alpha_n.$$

Sometimes we write ∂_i for $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. For the rest of this section, let Ω be an open bounded connected set in $\mathbb{R}^n$ with sufficiently smooth boundary.

For any nonnegative integer k let $C^k(\Omega)$ be the vector space consisting of all functions ϕ which, together with all their partial derivatives $\partial^\alpha \phi$ of orders $|\alpha| \le k$, are continuous in Ω. $C^\infty(\Omega)$ is the vector space consisting of all functions ϕ, such that $\phi \in C^k(\Omega)$ for all nonnegative integer k.

We write $C^k(\overline{\Omega})$ for the Banach space consisting of all functions $\phi \in C^k(\Omega)$ for which $\partial^\alpha \phi$ is bounded and uniformly continuous on Ω for $|\alpha| \le k$, with norm given by

$$\|\phi\|_{C^k(\overline{\Omega})} := \max_{|\alpha| \le k} \sup_{x \in \Omega} |\partial^\alpha \phi(x)|.$$

For a nonnegative integer k and $1 \leq p \leq \infty$ let $W^{k,p}(\Omega)$ be the usual Sobolev space defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega); \partial^\alpha u \in L^p(\Omega) \text{forall} \alpha \in \mathbb{N}_0^n, |\alpha| \leq k\}, \tag{2.1}$$

where $\partial^\alpha u$ is understood in distributional (or weak) sense, with the usual norm

$$\|u\|_{k,p,\Omega} := \Big\{ \sum_{|\alpha|\leq k} \int_\Omega |\partial^\alpha u(x)|^p dx \Big\}^{1/p} \quad \text{if} 1 \leq p < \infty, \tag{2.2}$$

$$\|u\|_{k,\infty,\Omega} := \max_{|\alpha|\leq k} \operatorname*{ess\,sup}_{x\in\Omega} |\partial^\alpha u(x)|. \tag{2.3}$$

As usual we shall write $H^k(\Omega) := W^{k,2}(\Omega)$.

Lemma 2.1. *The set $\mathcal{D}(\overline{\Omega})$ of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ (i.e. the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support) is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.*

For the proof of the above lemma, see Adams [1, theorem 3.18,].

Lemma 2.2. *If $kp = n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \leq q < \infty$.*

For the proof of the above lemma, see Adams [1, lemma 5.14].

Lemma 2.3. *If $kp > n$, then $W^{k,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.*

The proof of the above lemma can be found in Evans [9, sec. 5.6, Theorem 6] and in Adams [1, lemma 5.17].

Lemma 2.4. *Let $1 \leq p < \infty$. Then there exists a linear operator*

$$\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{2.4}$$

such that

(i) $\gamma_0 u = u|_{\partial\Omega}$ *if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.*
(ii) $\|\gamma_0 u\|_{L^p(\partial\Omega)} \leq c(p,\Omega)\|u\|_{1,p,\Omega}$ *for each $u \in W^{1,p}(\Omega)$, where $c(p,\Omega)$ is a constant depending only on p and Ω.*

For the proof of the above lemma, see Evans [9, theorem 5.5.1].

Remark 2.5. We call $\gamma_0 u$ the trace of order zero of u on $\partial\Omega$.

Definition 2.6. Let $j, k \in \mathbb{N}$, $k > 1$, $1 \leq j \leq k-1$ and $u \in W^{k,p}(\Omega)$. We define the trace of order j of u on $\partial\Omega$ by

$$\gamma_j u := \sum_{|\alpha|=j} \frac{j!}{\alpha_1! \cdots \alpha_n!} \gamma_0(\partial^\alpha u) \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}, \tag{2.5}$$

where $\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the unit outward normal along $\partial\Omega$.

Remark 2.7. $\gamma_j : W^{k,p}(\Omega) \to L^p(\partial\Omega)$ is a linear operator with

(i) $\gamma_j u = \frac{\partial^j u}{\partial \vec{\nu}^j}\Big|_{\partial\Omega} := \sum_{|\alpha|=j} \frac{j!}{\alpha_1!\cdots\alpha_n!} \partial^\alpha u|_{\partial\Omega} \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$ for $j = 1, \dots, k-1$ if $u \in W^{k,p}(\Omega) \cap C^{k-1}(\overline{\Omega})$.
(ii) $\|\gamma_j u\|_{L^p(\partial\Omega)} \leq c(k,p,\Omega)\|u\|_{k,p,\Omega}$ for each $u \in W^{k,p}(\Omega)$ and for all $j = 1, \dots, k-1$.

Now for $j, k \in \mathbb{N}_0$, $0 \le j \le k$, and $1 \le p < \infty$ we define the the functional given by

$$|u|_{j,p,\Omega} := \Big\{ \sum_{|\alpha|=j} \int_\Omega |\partial^\alpha u(x)|^p dx \Big\}^{1/p}, \quad u \in W^{k,p}(\Omega). \tag{2.6}$$

Clearly, $|u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \|u\|_{L^p(\Omega)}$. We have the following statement.

Lemma 2.8. *The functional*

$$((u))_{k,p,\Omega} = \Big\{ |u|^p_{k,p,\Omega} + |u|^p_{0,p,\Omega} \Big\}^{1/p}$$

is a norm on $W^{k,p}(\Omega)$, equivalent to the usual norm $\|\cdot\|_{k,p,\Omega}$.

The proof of the above lemma can be found in Adams [1, corollary 4.16].

We need some crucial results of the theory of semigroups of linear operators in Banach spaces. We refer to Pazy [16] or Dautray-Lions [8], chapter XVII, with respect to this theory.

Let V (resp. H) be a real separable Hilbert space with scalar product $(\cdot|\cdot)_V$ (resp. $(\cdot|\cdot)_H$) and norm $\|\cdot\|_V$ (resp. $\|\cdot\|_H$). We assume $V \hookrightarrow H$ and V dense in H.
Let $a(\cdot|\cdot) : V \times V \to \mathbb{R}$ be a continuous bilinear form, V-coercive with respect to H i.e., there exists $\lambda_0 \in \mathbb{R}$ and $c_0 > 0$ such that

$$a(v|v) + \lambda_0 \|v\|_H^2 \ge c_0 \|v\|_V^2, \quad \forall v \in V. \tag{2.7}$$

We put

$$D(\mathcal{A}) := \{u \in V; V \ni v \mapsto a(u|v) \text{is continuous for the topology of} H\}. \tag{2.8}$$

Theorem 2.9. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be the operator given by $(\mathcal{A}u|v)_H = a(u|v)$ $\forall u \in D(\mathcal{A})$ and $\forall v \in V$. Then $-\mathcal{A}$ is the infinitesimal generator of a C^0- semigroup $\{T(t)\}_{t\ge 0}$ in H which satisfies*

$$\|T(t)\|_{\mathcal{L}(H)} \le e^{\lambda_0 t} \quad \forall t \ge 0\,.$$

For a proof of the above theorem, see Dautray-Lions [8, theorem XVII.3.3].

Now we assume furthermore that $a(\cdot|\cdot)$ is symmetrical ($a(u|v) = a(v|u)$ $\forall u, v \in V$). Let $\mathcal{H} := V \times H$. $\mathcal{H}$ equipped with the scalar product defined by $(u|v)_{\mathcal{H}} := a(u_1|v_1) + (u_2|v_2)_H$ for $u = (u_1, u_2)^t, v = (v_1, v_2)^t \in \mathcal{H}$ (we write the elements of $\mathcal{H}$ as columns) is a Hilbert space (cf. Dautray-Lions [8], Section VII.3.4., p. 331).

Let $D(\mathbb{A}) := D(\mathcal{A}) \times V$. We define the operator $\mathbb{A}$ over $D(\mathbb{A})$ by

$$\mathbb{A}u := \begin{pmatrix} 0 & -id \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_2 \\ \mathcal{A}u_1 \end{pmatrix}, \quad \forall u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D(\mathbb{A}). \tag{2.9}$$

It follows that $D(\mathbb{A})$ is dense in $\mathcal{H}$ and $\mathbb{A}$ is a closed operator.

Theorem 2.10. *$-\mathbb{A}$ is the infinitesimal generator of a C^0-semigroup in $\mathcal{H}$.*

For the proof of the above theroem, see Dautray-Lions [8, theorem XVII.3.4].

Theorem 2.11. *Let $-A$ be the infinitesimal generator of a C^0-semigroup of linear operators on a Banach space X and $u_0 \in D(A)$. If $f : [t_0, T] \times X \to X$*

is continuously differentiable with bounded partial derivatives then there exists a unique classical solution $u \in C^1([t_0,T];X)$ *of the initial value problem*

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = f(t,u(t)) \quad \text{in } X, \text{ on }]t_0,T] \\ u(t_0) = u_0\,. \end{gathered} \tag{2.10}$$

The proof of this lemma ca be found in Pazy [16, theorem 6.1.5].

3. Function spaces and bilinear forms for the semilinear problem plate-membrane

We define the vector space (with the usual vectorial sum and multiplication by scalars)

$$V := \big\{(u_p,u_m) \in H^2(\Omega_p)\times H^1(\Omega_m); u_p|_{\Gamma_3} = \gamma_1 u_p|_{\Gamma_3} = 0, u_p|_{\Gamma_1} = \gamma_0 u_m|_{\Gamma_1}\big\} \tag{3.1}$$

(In this work we only consider real vector spaces). The vector space V, endowed with the inner product

$$((u_p,u_m)|(v_p,v_m))_V := (u_p|v_p)_{H^2(\Omega_p)} + (u_m|v_m)_{H^1(\Omega_m)}, \tag{3.2}$$

is a separable Hilbert space. The norm in V is given by

$$\|(u_p,u_m)\|_V := \big(\|u_p\|^2_{2,2,\Omega_p} + \|u_m\|^2_{1,2,\Omega_m}\big)^{1/2}. \tag{3.3}$$

We consider also

$$H := L^2(\Omega_p)\times L^2(\Omega_m) \tag{3.4}$$

with inner product and norm given by

$$((u_p,u_m)|(v_p,v_m))_H := (u_p|v_p)_{L^2(\Omega_p)} + (u_m|v_m)_{L^2(\Omega_m)} \tag{3.5}$$

and

$$\|(u_p,u_m)\|_H := \Big(\|u_p\|^2_{0,2,\Omega_p} + \|u_m\|^2_{0,2,\Omega_m}\Big)^{1/2}. \tag{3.6}$$

Also we consider

$$\tilde V := \big\{(\tilde u_p,\tilde u_m) \in H^2(\Omega_p)\times H^1(\Omega_m); \big(\frac{1}{\sqrt{\rho_p h}}\tilde u_p, \frac{1}{\sqrt{\rho_m}}\tilde u_m\big) \in V\big\}, \tag{3.7}$$

endowed with the norm

$$\|(\tilde u_p,\tilde u_m)\|_{\tilde V} := \big(\frac{1}{\rho_p h}\|\tilde u_p\|^2_{2,2,\Omega_p} + \frac{1}{\rho_m}\|\tilde u_m\|^2_{1,2,\Omega_m}\big)^{1/2}. \tag{3.8}$$

We have the imbedding $\tilde V \hookrightarrow H$ with $\tilde V$ dense in H. Identifying H with its dual H' we obtain $\tilde V \overset{i}{\hookrightarrow} H = H' \overset{i'}{\hookrightarrow} \tilde V'$, where $i : \tilde V \to H$ is the identity operator and $i' : H \to \tilde V'$ is the dual operator of $i : V \to H$. Since $i : \tilde V \to H$ is injective and its range is dense in H, the same holds for $i' : H \to \tilde V'$. Furthermore we identify $i'f$ with f for $f \in H$. Therefore we regard H as subspace of $\tilde V'$.

We consider the symmetric bilinear form

$$\begin{aligned} &a((u_p,u_m)|(v_p,v_m)) \\ &:= \frac{h^3}{12}\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\int_{\Omega_p} A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}\frac{\partial^2 v_p}{\partial x_\alpha \partial x_\beta}dx + C\int_{\Omega_m}\nabla u_m\cdot\nabla v_m dx \end{aligned} \tag{3.9}$$

for $(u_p, u_m), (v_p, v_m) \in V$ (The symmetry is a consequence of the assumption (1.12)). For technical reasons it is convenient to consider also

$$\tilde{a}\left((\tilde{u}_p, \tilde{u}_m)|(\tilde{v}_p, \tilde{v}_m)\right) := a\Big(\big(\frac{1}{\sqrt{\rho_p h}}\tilde{u}_p, \frac{1}{\sqrt{\rho_m}}\tilde{u}_m\big)\big|\big(\frac{1}{\sqrt{\rho_p h}}\tilde{v}_p, \frac{1}{\sqrt{\rho_m}}\tilde{v}_m\big)\Big) \tag{3.10}$$

for $(\tilde{u}_p, \tilde{u}_m), (\tilde{v}_p, \tilde{v}_m) \in \tilde{V}$.

Lemma 3.1. *Under the assumptions introduced for the coefficients $A_{\alpha\beta\gamma\theta}$, the bilinear form (3.9) (resp. (3.10)) is continuous and V-coercive (resp. $\tilde{V}-$coercive) with respect to H.*

Proof. From the Schwarz inequality we have the continuity of the bilinear forms (3.9) and (3.10). Now let $u = (u_p, u_m) \in V$. From Lemma 2.8 we have that there exists $c_p > 0$ such that

$$((u_p))_{2,2,\Omega_p} \ge c_p \|u_p\|_{2,2,\Omega_p}.$$

Then

$$\begin{aligned}
a(u|u) &= \frac{h^3}{12} \sum_{\alpha,\beta,\gamma,\theta=1}^{2} \int_{\Omega_p} A_{\alpha,\beta,\gamma,\theta} \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta} \frac{\partial^2 u_p}{\partial x_\alpha \partial x_\beta} dx + C \int_{\Omega_m} |\nabla u_m|^2 dx \\
&\ge \frac{h^3}{12}\rho \sum_{\alpha,\beta=1}^{2} \int_{\Omega_p} \Big|\frac{\partial^2 u_p}{\partial x_\alpha \partial x_\beta}\Big|^2 dx + C|u_m|^2_{1,2,\Omega_m} \\
&= \frac{h^3}{12}\rho |u_p|^2_{2,2,\Omega_p} + C|u_m|^2_{1,2,\Omega_m} \\
&\ge \frac{h^3}{12}\rho c_p \|u_p\|^2_{2,2,\Omega_p} - \frac{h^3}{12}\rho |u_p|^2_{0,2,\Omega_p} + C\|u_m\|^2_{1,2,\Omega_m} - C|u_m|^2_{0,2,\Omega_m}.
\end{aligned}$$

With $\lambda_0 := \max\{\frac{h^3}{12}\rho, C\}$ and $c_0 := \min\{\frac{h^3}{12}\rho c_p, C\}$ we obtain the V-coerciveness of $a(\cdot|\cdot)$ with respect to H. From this follows immediately the $\tilde{V}$-coerciveness of $\tilde{a}(\cdot|\cdot)$ with respect to H. □

Let$D(\tilde{\mathcal{A}}) := \tilde{A}^{-1}(H)$ and $\tilde{\mathcal{A}} := \tilde{A}|_{D(\tilde{\mathcal{A}})}$, where $\tilde{A} : \tilde{V} \to \tilde{V}'$ is given by $\langle \tilde{A}\tilde{u}|\tilde{v}\rangle = \tilde{a}(\tilde{u}|\tilde{v})$, for all $\tilde{u}, \tilde{v} \in \tilde{V}$. We have that $-\tilde{\mathcal{A}}$ is the infinitesimal generator of a C^0-semigroup in H (see [11, p. 54].

4. Weak solution

For the function

$$(t, x, u) \mapsto f_p(t, x, u) : [0, T] \times \Omega_p \times \mathbb{R} \to \mathbb{R} \tag{4.1}$$

we assume the following:

(i) For all $t \in [0, T]$, $x \mapsto f_p(t, x, u(x)) : \Omega_p \to \mathbb{R}$ is measurable, if $u : \Omega_p \to \mathbb{R}$ is measurable.

(ii) $|f_p(t, x, u)| \le q_p(t, x) + k_p|u|$ for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$, where $q_p(t, \cdot) \in L^2(\Omega_p)$ for all $t \in [0, T]$ and $k_p > 0$ is a constant.

(iii) $\frac{\partial f_p}{\partial t}(t, x, u)$ exists for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$. It is bounded and Lipschitz continuous on $[0, T] \times \Omega_p \times \mathbb{R}$.

(iv) $\frac{\partial f_p}{\partial u}(t, x, u)$ exists for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$. It is bounded and Lipschitz continuous on $[0, T] \times \Omega_p \times \mathbb{R}$.

For the function

$$(t,x,u)\mapsto f_m(t,x,u):[0,T]\times\Omega_m\times\mathbb{R}\to\mathbb{R} \tag{4.2}$$

we assume the following:

(i) For all $t\in[0,T]$, $x\mapsto f_m(t,x,u(x)):\Omega_m\to\mathbb{R}$ is measurable, if $u:\Omega_m\to\mathbb{R}$ is measurable.
(ii) $|f_m(t,x,u)|\le q_m(t,x)+k_m|u|$, for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$, where $q_m(t,\cdot)\in L^2(\Omega_m)$ for all $t\in[0,T]$ and $k_m>0$ a constant.
(iii) $\frac{\partial f_m}{\partial t}(t,x,u)$ exists for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$. It is bounded and Lipschitz continuous on $[0,T]\times\Omega_m\times\mathbb{R}$.
(iv) $\frac{\partial f_m}{\partial u}(t,x,u)$ exists for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$. It is bounded and Lipschitz continuous on $[0,T]\times\Omega_m\times\mathbb{R}$.

Let $\mathbf{f}_p:[0,T]\times L^2(\Omega_p)\to L^2(\Omega_p)$ and $\mathbf{f}_m:[0,T]\times L^2(\Omega_m)\to L^2(\Omega_m)$ be defined by

$$[\mathbf{f}_p(t,u_p)](x):=f_p(t,x,u_p(x))\quad\text{for } (t,x)\in[0,T]\times\Omega_p\ u_p\in L^2(\Omega_p)\,, \tag{4.3}$$

$$[\mathbf{f}_m(t,u_m)](x):=f_m(t,x,u_m(x))\quad\text{for } (t,x)\in[0,T]\times\Omega_m\ u_m\in L^2(\Omega_m). \tag{4.4}$$

From assumptions on (4.1) and (4.2), we see that $\mathbf{f}_p(t,u_p)\in L^2(\Omega_p)$ and $\mathbf{f}_m(t,u_m)\in L^2(\Omega_m)$, for $u_p\in L^2(\Omega_p)$ and $u_m\in L^2(\Omega_m)$.

For technical reasons we introduce the following functions:

$$\tilde{\mathbf{f}}_p(t,u_p):=\frac{1}{\sqrt{\rho_p h}}\mathbf{f}_p\big(t,\frac{1}{\sqrt{\rho_p h}}u_p\big)\quad\text{for } t\in[0,T]\ u_p\in L^2(\Omega_p)\,, \tag{4.5}$$

$$\tilde{\mathbf{f}}_m(t,u_m):=\frac{1}{\sqrt{\rho_m}}\mathbf{f}_m\big(t,\frac{1}{\sqrt{\rho_m}}u_m\big)\quad\text{for } t\in[0,T]\ u_m\in L^2(\Omega_m)\,. \tag{4.6}$$

Let us suppose that $u_p:[0,T]\times\overline{\Omega}_p\to\mathbb{R}$ and $u_m:[0,T]\times\overline{\Omega}_m\to\mathbb{R}$ are smooth enough in such a way that the system (1.1) - (1.11) for (u_p,u_m) holds; i.e., we suppose that (u_p,u_m) is a classical solution of the semilinear problem (1.1)-(1.11). Furthermore we assume that $(\tilde u_p(t,.),\tilde u_m(t,.))\in D(\tilde{\mathcal{A}})$ for $t\in]0,T]$, where $(\tilde u_p,\tilde u_m):=\big(\sqrt{\rho_p h}u_p,\sqrt{\rho_m}u_m\big)$. If we multiply (1.1) (resp. (1.2)) with $\frac{1}{\sqrt{\rho_p h}}\tilde v_p$ (resp. $\frac{1}{\sqrt{\rho_m}}\tilde v_m$), where $(\tilde v_p,\tilde v_m)\in\tilde V$, by use of integration by parts, (1.3)-(1.7) and the fact that $\tilde V$ is dense in H we obtain

$$\big(\frac{\partial^2\tilde u_p}{\partial t^2}(t,\cdot),\frac{\partial^2\tilde u_m}{\partial t^2}(t,\cdot)\big)+\tilde{\mathcal{A}}(\tilde u_p(t,\cdot),\tilde u_m(t,\cdot))=\big(\tilde{\mathbf{f}}_p(t,\tilde u_p(t,\cdot)),\tilde{\mathbf{f}}_m(t,\tilde u_m(t,\cdot))\big) \tag{4.7}$$

in H, for $t\in]0,T]$. On the other hand we have

$$\tilde u_p(0,\cdot)=\tilde g_p^0,\quad \tilde u_m(0,\cdot)=\tilde g_m^0,\quad \frac{\partial\tilde u_p}{\partial t}(0,\cdot)=\tilde g_p^1,\quad \frac{\partial\tilde u_p}{\partial t}(0,\cdot)=\tilde g_m^1, \tag{4.8}$$

where $\tilde g_p^0:=\sqrt{\rho_p h}g_p^0$, $\tilde g_m^0:=\sqrt{\rho_m}g_m^0$, $\tilde g_p^1:=\sqrt{\rho_p h}g_p^1$ and $\tilde g_m^1:=\sqrt{\rho_m}g_m^1$.
We suppose

$$(i)\ (g_p^0,g_m^0)\in A^{-1}(H),\quad (ii)\ (g_p^1,g_m^1)\in V \tag{4.9}$$

where $A:V\to V'$ is given by $\langle Au|v\rangle=a(u|v)$, for all $u,v\in V$.

Equations (4.7) and (4.8) motivate the following definition: Consider the Hilbert space $\mathcal{H} := \tilde{V} \times H$ endowed with the inner product

$$\Big(\begin{pmatrix} (\tilde{u}_p^1, \tilde{u}_m^1) \\ (\tilde{u}_p^2, \tilde{u}_m^2) \end{pmatrix} \Big| \begin{pmatrix} (\tilde{v}_p^1, \tilde{v}_m^1) \\ (\tilde{v}_p^2, \tilde{v}_m^2) \end{pmatrix} \Big)_{\mathcal{H}} := a((\tilde{u}_p^1, \tilde{u}_m^1)|(\tilde{v}_p^1, \tilde{v}_m^1)) + ((\tilde{u}_p^2, \tilde{u}_m^2)|(\tilde{v}_p^2, \tilde{v}_m^2))_H. \tag{4.10}$$

Moreover let $D(\tilde{\mathbb{A}}) := D(\tilde{\mathcal{A}}) \times \tilde{V}$ and $\tilde{\mathbb{A}} := \begin{pmatrix} 0 & -id \\ \tilde{\mathcal{A}} & 0 \end{pmatrix}$. It follows from theorem 2.10 that $-\tilde{\mathbb{A}}$ is the infinitesimal generator of a C^0-semigroup of contractions in $\mathcal{H}$. We put

$$\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) := \begin{pmatrix} 0 \\ \big(\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p^1), \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m^1)\big) \end{pmatrix} \quad \text{for } \tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix} \in \mathcal{H}, \tag{4.11}$$

$$\tilde{\mathbb{G}} := \begin{pmatrix} (\tilde{g}_p^0, \tilde{g}_m^0) \\ (\tilde{g}_p^1, \tilde{g}_m^1) \end{pmatrix}. \tag{4.12}$$

Next we define weak solution for our semilinear problem.

Definition 4.1. Assume that (1.12), (1.13), (4.1), (4.2) and (4.9) are satisfied. We say that a function $(\mathbf{u}_p, \mathbf{u}_m) \in C^1([0,T]; V) \cap C^2([0,T]; H)$ is a weak solution of the semilinear problem (1.1)-(1.11) if the function

$$(\tilde{\mathbf{u}}_p, \tilde{\mathbf{u}}_m) := \big(\sqrt{\rho_p h}\mathbf{u}_p, \sqrt{\rho_m}\mathbf{u}_m\big) \in C^1([0,T]; \tilde{V}) \cap C^2([0,T]; H)$$

has the following properties:

$$\begin{aligned} &(i)\Big(\frac{d^2\tilde{\mathbf{u}}_p(t)}{dt^2}, \frac{d^2\tilde{\mathbf{u}}_m(t)}{dt^2}\Big) + \tilde{\mathcal{A}}(\tilde{\mathbf{u}}_p(t), \tilde{\mathbf{u}}_m(t)) = \big(\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p(t)), \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m(t))\big) \\ &\quad \text{in } H, \text{ on }]0,T] \\ &(ii)(\tilde{\mathbf{u}}_p(0), \tilde{\mathbf{u}}_m(0)) = (\tilde{g}_p^0, \tilde{g}_m^0). \\ &(iii)\Big(\frac{d\tilde{\mathbf{u}}_\mathbf{p}}{dt}(0), \frac{d\tilde{\mathbf{u}}_\mathbf{m}}{dt}(0)\Big) = (\tilde{g}_p^1, \tilde{g}_m^1). \end{aligned} \tag{4.13}$$

Lemma 4.2. *Assume* (1.12)*,* (1.13)*,* (4.1) *and* (4.2)*. Then the function* $(t, \mathbb{U}) \mapsto \mathbb{F}(t, \mathbb{U}) : [0,T] \times \mathcal{H} \to \mathcal{H}$ *which is defined by* (4.11)*, is continuously differentiable with bounded partial derivatives.*

Proof. **1.** The assumptions (4.1)(i),(ii) and (4.2)(i),(ii) lead to

$$\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p^1) \in L^2(\Omega_p) \quad \text{and} \quad \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m^1) \in L^2(\Omega_m)$$

for $\tilde{\mathbf{u}}_p^1 \in L^2(\Omega_p)$ and $\tilde{\mathbf{u}}_m^1 \in L^2(\Omega_m)$ and for all $t \in [0,T]$ (cf. [5, theorem 2.1]). Then we have $\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) \in \mathcal{H}$ for $(t, \tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$.
2. It follows from (4.1)(iii) that

$$\frac{\partial f_p}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big) \in L^2(\Omega_p) \quad \forall t \in [0,T]\, \forall \tilde{\mathbf{u}}_p^1 \in L^2(\Omega_p).$$

Let $t \in [0,T]$. For $\tau \in \mathbb{R}$ with $-t \le \tau \le T-t$ we have

$$\begin{aligned}
&\Big\| \frac{\tilde{\mathbf{f}}_p(t+\tau,\tilde{\mathbf{u}}_p^1) - \tilde{\mathbf{f}}_p(t,\tilde{\mathbf{u}}_p^1)}{\tau} - \frac{1}{\sqrt{\rho_p h}}\frac{\partial f_p}{\partial t}\big(t,\cdot,\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big)\Big\|^2_{L^2(\Omega_p)} \\
&= \int_{\Omega_p} \frac{1}{\rho_p h}\Big| \int_0^1 \big[\frac{\partial f_p}{\partial t}\big(t+\xi\tau, x, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(x)\big) - \frac{\partial f_p}{\partial t}\big(t, x, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(x)\big)\big] d\xi\Big|^2 dx \\
&\le \int_{\Omega_p} \frac{1}{\rho_p h}\Big[\int_0^1 \big|\frac{\partial f_p}{\partial t}\big(t+\xi\tau, x, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(x)\big) - \frac{\partial f_p}{\partial t}\big(t, x, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(x)\big)\big| d\xi\Big]^2 dx \\
&\le \frac{1}{\rho_p h}\text{const.}\mu_{_p}(\Omega_p)\tau^2 \xrightarrow[\tau\to 0]{} 0
\end{aligned} \tag{4.14}$$

The above inequality because the Lipschitz continuity of $\frac{\partial f_p}{\partial t}$.

3. It follows from (4.2)(iii) that

$$\frac{\partial f_m}{\partial t}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big) \in L^2(\Omega_m) \quad \forall t\in[0,T]\ \forall \tilde{\mathbf{u}}_m^1 \in L^2(\Omega_m).$$

Let $t \in [0,T]$. For $\tau \in \mathbb{R}$ with $-t \le \tau \le T-t$ we have as above

$$\Big\| \frac{\tilde{\mathbf{f}}_m(t+\tau,\tilde{\mathbf{u}}_m^1) - \tilde{\mathbf{f}}_m(t,\tilde{\mathbf{u}}_m^1)}{\tau} - \frac{1}{\sqrt{\rho_m}}\frac{\partial f_m}{\partial t}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big)\Big\|^2_{L^2(\Omega_m)} \tag{4.15}$$

approaches zero as $\tau \to 0$.

4. Let $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$ with $\tilde{\mathbb{U}} := \begin{pmatrix}(\tilde{\mathbf{u}}_p^1,\tilde{\mathbf{u}}_m^1)\\(\tilde{\mathbf{u}}_p^2,\tilde{\mathbf{u}}_m^2)\end{pmatrix}$. We consider the operator $D_1\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) \in \mathcal{L}(\mathbb{R};\mathcal{H})$ which is defined by

$$D_1\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tau := \begin{pmatrix} 0 \\ \big(\frac{1}{\sqrt{\rho_p h}}\frac{\partial f_p}{\partial t}\big(t,\cdot,\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big)\tau, \frac{1}{\sqrt{\rho_m}}\frac{\partial f_m}{\partial t}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big)\tau\big)\end{pmatrix} \tag{4.16}$$

For $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$ and from (4.14) and (4.15) we have that

$$\frac{\|\tilde{\mathbb{F}}(t+\tau,\tilde{\mathbb{U}}) - \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) - D_1\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tau\|_{\mathcal{H}}}{|\tau|} \xrightarrow[-t\le\tau\le T-t,\ \tau\ne 0,\ \tau\to 0]{0} . \tag{4.17}$$

Then there exists the partial derivative of $\tilde{\mathbb{F}}$ with respect to t for all $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$ and it is equal to $D_1\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})$. By the Lipschitz continuity of $\frac{\partial f_p}{\partial t}$ and $\frac{\partial f_m}{\partial t}$ it can be showed that

$$\|D_1\tilde{\mathbb{F}}(t_1,\tilde{\mathbb{U}}_1) - D_1\tilde{\mathbb{F}}(t_2,\tilde{\mathbb{U}}_2)\|_{\mathcal{L}(\mathbb{R};\mathcal{H})} \le \text{const.}\big(|t_1-t_2| + \|\tilde{\mathbb{U}}_1 - \tilde{\mathbb{U}}_2\|_{\mathcal{H}}\big). \tag{4.18}$$

Then the maping

$$(t,\tilde{\mathbb{U}}) \mapsto D_1\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) : [0,T]\times\mathcal{H} \to \mathcal{L}(\mathbb{R};\mathcal{H})$$

is continuous. The boundedness of $\frac{\partial f_p}{\partial t}$ and $\frac{\partial f_m}{\partial t}$ implied by the boundedness of $D_1\tilde{\mathbb{F}}$.

5. From (4.1)(iv) and (4.2)(iv) we have

$$\frac{\partial f_p}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big)\tilde{\mathbf{v}}_p^1 \in L^2(\Omega_p)$$

and

$$\frac{\partial f_m}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big)\tilde{\mathbf{v}}_m^1 \in L^2(\Omega_m)$$

for all $t \in [0,T]$ and all $(\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1), (\tilde{\mathbf{v}}_p^1, \tilde{\mathbf{v}}_m^1) \in H$. For $t \in [0,T]$, $\tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix} \in \mathcal{H}$ and $\tilde{\mathbb{V}} := \begin{pmatrix} (\tilde{\mathbf{v}}_p^1, \tilde{\mathbf{v}}_m^1) \\ (\tilde{\mathbf{v}}_p^2, \tilde{\mathbf{v}}_m^2) \end{pmatrix} \in \mathcal{H}$ we put

$$D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}} := \begin{pmatrix} 0 \\ \big(\frac{1}{\rho_p h}\frac{\partial f_p}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big)\tilde{\mathbf{v}}_p^1, \frac{1}{\rho_m}\frac{\partial f_m}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big)\tilde{\mathbf{v}}_m^1\big) \end{pmatrix} \tag{4.19}$$

Since $\frac{\partial f_p}{\partial u}$ (resp. $\frac{\partial f_m}{\partial u}$) is bounded on $[0,T] \times \Omega_p \times \mathbb{R}$ (resp. $[0,T] \times \Omega_m \times \mathbb{R}$), we see that $D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) \in \mathcal{L}(\mathcal{H})$ for all $(t,\tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$.

For $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$ and $\tilde{\mathbb{V}} \in \mathcal{H}$ with $\|\tilde{\mathbb{V}}\|_{\mathcal{H}} \neq 0$ we have (with "const" denoting different constants)

$$\begin{aligned}
&\frac{\|\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}+\tilde{\mathbb{V}}) - \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) - D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2} \\
&\le \frac{\text{const}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\Big\{\int_{\Omega_p}\Big[\int_0^1\big|\frac{\partial f_p}{\partial u}\big(t,x,\frac{1}{\sqrt{\rho_p h}}(\tilde{\mathbf{u}}_p^1(x)+\xi\tilde{\mathbf{v}}_p^1(x))\big) \\
&\quad - \frac{\partial f_p}{\partial u}\big(t,x,\frac{\tilde{\mathbf{u}}_p^1(x)}{\sqrt{\rho_p h}}\big)\big|d\xi\Big]^2\frac{|\tilde{\mathbf{v}}_p^1(x)|^2}{\rho_p h}dx \\
&\quad + \int_{\Omega_m}\Big[\int_0^1\big|\frac{\partial f_m}{\partial u}\big(t,x,\frac{1}{\sqrt{\rho_m}}(\tilde{\mathbf{u}}_m^1(x)+\xi\tilde{\mathbf{v}}_m^1(x))\big) \\
&\quad - \frac{\partial f_m}{\partial u}\big(t,x,\frac{\tilde{\mathbf{u}}_m^1(x)}{\sqrt{\rho_m}}\big)\big|d\xi\Big]^2\frac{|\tilde{\mathbf{v}}_m^1(x)|^2}{\rho_m}dx\Big\} \\
&\le \frac{\text{const}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\Big\{\frac{1}{\rho_p^2 h^2}\int_{\Omega_p}|\tilde{\mathbf{v}}_p^1(x)|^4dx + \frac{1}{\rho_m^2}\int_{\Omega_m}|\tilde{\mathbf{v}}_m^1(x)|^4dx\Big\}.
\end{aligned} \tag{4.20}$$

The above holds because of the Lipschitz continuity of $\frac{\partial f_p}{\partial u}$ and $\frac{\partial f_m}{\partial u}$. Since

$$\tilde{\mathbf{v}}_p^1 \in H^2(\Omega_p) \hookrightarrow C^0(\overline{\Omega}_p) \hookrightarrow L^4(\Omega_p) \quad\text{and}\quad \tilde{\mathbf{v}}_m^1 \in H^1(\Omega_m) \hookrightarrow L^4(\Omega_m)$$

(see lemmas 2.2 and 2.3), from (4.20), we have

$$\begin{aligned}
&\frac{\|\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}+\tilde{\mathbb{V}}) - \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) - D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2} \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\big(\frac{1}{\rho_p^2h^2}\|\tilde{\mathbf{v}}_p^1\|_{H^2(\Omega_p)}^4 + \frac{1}{\rho_m^2}\|\tilde{\mathbf{v}}_m^1\|_{H^1(\Omega_m)}^4\big) \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\|(\tilde{\mathbf{v}}_p^1,\tilde{\mathbf{v}}_m^1)\|_{\tilde{V}}^4 \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^4 = \text{const.}\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.21}$$

It follows that the partial derivative of $\tilde{\mathbb{F}}$ with respect to the second variable $\tilde{\mathbb{U}}$ exists and it is equal to $D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})$ for all $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$. We can show similarly that the Lipschitz continuity (resp. the boundedness) of $\frac{\partial f_p}{\partial u}$ and $\frac{\partial f_m}{\partial u}$ leads to the continuity (resp. the boundedness) of

$$(t,\tilde{\mathbb{U}}) \mapsto D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) : [0,T]\times\mathcal{H} \to \mathcal{L}(\mathcal{H}).$$

So the proof is complete. $\square$

Lemma 4.3. *Let* $\tilde{\mathbb{F}} : [0,T] \times \mathcal{H} \to \mathcal{H}$ *(resp.* $\tilde{\mathbb{G}}$*) be defined by* (4.11) *(resp. (4.12)). Under assumptions* (1.12), (1.13), (4.9), (4.1) *and* (4.2)*, there exists a unique function* $\tilde{\mathbb{U}} : [0,T] \to \mathcal{H}$ *with the following properties:*

$$
\begin{aligned}
&(i)\tilde{\mathbb{U}} \in C^1([0,T];\mathcal{H}).\\
&(ii)\frac{d\tilde{\mathbb{U}}(t)}{dt} + \tilde{\mathbb{A}}\tilde{\mathbb{U}}(t) = \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}(t)) \quad \text{in } \mathcal{H} \quad \text{on}]0,T].\\
&(iii)\tilde{\mathbb{U}}(0) = \tilde{\mathbb{G}}.
\end{aligned}
\tag{4.22}
$$

Proof. **1.** It follows from theorem 2.10 that $-\tilde{\mathbb{A}}$ is the infinitesimal generator of a C^0-semigroup of linear operators in $\mathcal{H}$.
2. From lemma 4.2 we have that $\tilde{\mathbb{F}} : [0,T] \times \mathcal{H} \to \mathcal{H}$ is continuously differentiable with bounded partial derivatives.
3. It can be seen that $\tilde{\mathbb{G}}$ belongs to $D(\tilde{\mathbb{A}})$.
4. From theorem 2.11 we have the desired result. $\square$

Theorem 4.4. *Under assumptions* (1.12), (1.13), (4.9), (4.1) *and* (4.2)*, there exists a unique weak solution of the semilinear problem* (1.1)-(1.11).

Proof. Let

$$
\tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix} : [0,T] \to \mathcal{H}
$$

be the unique function satisfying (4.22) (Lemma 4.3). It can be showed that $(\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1)$ belongs to $C^1([0,T];\tilde{V}) \cap C^2([0,T];H)$ and that it satisfies (4.13). Then $(\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1, \frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1)$ is the desired weak solution. The uniqueness follows from the uniqueness of $\tilde{\mathbb{U}}$. $\square$

Remark 4.5. For sufficiently smooth solutions in the sense of definition 4.1 we can obtain as usual a classical pointwise solution of system (1.1)-(1.11). See [12].

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