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CRITICAL POINTS OF THE STEADY STATE OF A FOKKER-PLANCK EQUATION

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Abstract. In this paper we consider a set of vector fields over the torus for which we can associate a positive function v_ϵ which define for some of them in a solution of the Fokker-Planck equation with ϵ diffusion:

$$\epsilon\Delta v_\epsilon - \operatorname{div}(v_\epsilon X) = 0\,.$$

Within this class of vector fields we prove that X is a gradient vector field if and only if at least one of the critical points of v_ϵ is a stationary point of X, for an $\epsilon > 0$. In particular we show a vector field which is stable in the sense of Zeeman but structurally unstable in the Andronov-Pontriaguin sense. A generalization of some results to other kind of compact manifolds is made.

1. Vector fields in covering spaces

Let $\pi : \widetilde{M} \to M$ be a covering space of a Riemannian and oriented manifold M. In $\widetilde{M}$ there exists one and only one Riemannian structure such that

$$d\pi_y : T_y(\widetilde{M}) \to T_{\pi(y)}(M)$$

is an isometry for all $y \in \widetilde{M}$. Then it is able to associate to every C^r vector field X in M, another C^r vector field $\widetilde{X}$ in $\widetilde{M}$ in the following way:

$$\widetilde{X}(y) = (d(\pi)(y))^{-1}(X(\pi(y))).$$

It is easy to verify the following theorem:

Theorem 1.1.

$$(\widetilde{X+Y}) = \widetilde{X} + \widetilde{Y}m \tag{1.1}$$

$$\widetilde{\nabla f} = \nabla(\pi \circ f) \tag{1.2}$$

$$\operatorname{div}(\widetilde{X}(y) = \operatorname{div}(X)(\pi(y)) \tag{1.3}$$

Definition A vector field X is called almost gradient respect to the projection π, if and only if $\widetilde{X}$ is a gradient in M. This set will be denoted by $V_{ag}(\pi)$. Particular we can write

$$\operatorname{grad}(M) = V_{ag}(1_M)$$

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There are non trivial projections for which is true the preceding statement, so we have the following theorem.

Theorem 1.2. *If $\pi : \widetilde{M} \to M$ is a finite covering and M is compact, then V_{ag} is the set of gradient vector fields in M.*

Proof. Let X be a vector field in M and let
$$X = \nabla f + W$$
be its Hodge's decomposition. If $\widetilde{X}$ is gradient, then there exists a C^∞ function g such that:
$$\nabla g = \widetilde{X} = \nabla(f \circ \pi) + \widetilde{W}.$$
Then by Theorem 1.1 it follows that $\widetilde{W}$ is a gradient vector field in $\widetilde{M}$ and we can write $\widetilde{W} = \nabla h$. Finally from Theorem 1.1 we get, $\operatorname{div}(\widetilde{W}) = \operatorname{div}(W) \circ \pi = 0$. Thus $\nabla h = 0$ and by compactness of $\widetilde{M}$ it follows h to be a constant. So $W = 0$ and $W = 0$. □

2. Vector Fields in T_n

Let $\pi : \mathbb{R}^n \to T_n = \mathbb{R}^n/\mathbb{Z}^n$ be the universal covering space of the torus T_n. So there exists a Riemannian structure in T_n such that $\widetilde{T_n} = \mathbb{R}^n$, where $\mathbb{R}^n$ is considered with the usual Riemannian structure. It is easy to realize that $V_{ag}(\pi)$ is different from $\operatorname{grad}(M)$. More precisely we have the following statement.

Theorem 2.1. *$X \in V_{ag}(\pi)$ if and only if X is in the form $X = \nabla f + \lambda$, where $\lambda \in \mathbb{R}^n$.*

Proof. Let $X = \nabla f + W$ be the Hodge's decomposition of X. Then $\widetilde{X} \in V_{ag}(\pi)$ implies
$$\widetilde{X} = \nabla g,$$
but
$$\widetilde{X} = \widetilde{\nabla f} + \widetilde{W} = \nabla(f \circ \pi) + \widetilde{W}$$
and $\widetilde{W} = \nabla h$ with $h = g - f \circ \pi$. By the periodicity of $\widetilde{W}$ it follows that $\|\widetilde{W}\| \leq K$. Then
$$|h(x)| \leq K\|x\| \quad \forall x \in \mathbb{R}^n. \tag{2.1}$$
Because W has divergence zero so it does $\widetilde{W}$, and h is an harmonic function in whole $\mathbb{R}^n$. From the estimate (4) h is a linear function; i.e.,
$$h(x) = a + \lambda \cdot x \tag{2.2}$$
with $\lambda \in \mathbb{R}^n$ so $\nabla h = \widetilde{W} = \lambda$, consequently $W = \lambda$. □

3. The function v_ϵ

For the rest of this article, we consider X in T_n to be of the form $X = \nabla f + \lambda$. Let us consider
$$v_\epsilon(x) = \int_{T_n} \exp\big(\frac{h(x,z)}{\epsilon}\big) dz \tag{3.1}$$
where
$$g(x) = f(x) + \lambda \cdot x\,, \tag{3.2}$$
$$h(x,z) = g(x) - g(x+z) \tag{3.3}$$

Lemma 3.1. *X is a gradient if $\lambda = 0$ and we have*

$$v_\epsilon = L(\epsilon)\exp(f/\epsilon)$$

Proof. If $\lambda = 0$, we have

$$v_\epsilon = \exp(\frac{f}{\epsilon})\int_{T_n}\exp(\frac{-f(x+z)}{\epsilon})dz = \exp(\frac{f}{\epsilon})\int_{T_n}\exp(\frac{-f(z)}{\epsilon})dz\,.$$

□

Definition. X will be called without coupling, if

$$\big((i\neq j) \text{ and } (\frac{\partial X_i}{\partial x_j}\neq 0)\big) \Rightarrow \lambda_i = 0.$$

Theorem 3.2. *Let X be a vector field without coupling. Then v_ϵ is a solution of the Fokker-Planck equation*

$$\epsilon\Delta v - \operatorname{div}(vX) = 0. \tag{3.4}$$

Proof. Let I be the set of indices for which $\lambda_i \neq 0$. Then the i-component of the vector field X is

$$X_i = f_i'(x_i) + \lambda_i$$

with f_i a function in the variable x_i. Therefore, $X = \nabla f + \lambda$ with

$$f = \sum(f_i(x_i)) + p(x)$$

where $p(x)$ is a periodic function and

$$\frac{\partial}{\partial x_i}(p(x)) = 0, \quad i \in I.$$

Then

$$h(x,z) = \sum_{i\epsilon I} h_i(x_i,z_i) + p(x) - p(x+z).$$

Because X is without coupling, applying Lemma 3.1 to ∇p,

$$v_\epsilon = \int_{T_n}\exp\big(\frac{h(x,z)}{\epsilon}\big)dz = K\big(\prod_{i\epsilon I} v_\epsilon^i(x_i)\big)\exp\big(\frac{p(x)}{\epsilon}\big)\,, \tag{3.5}$$

where

$$v_\epsilon^i(x_i) = \int_0^1 \exp\big(\frac{h_i(x_i,z_i)}{\epsilon}\big)dz_i$$

is associated with the vector field $\nabla f_i + \lambda_i$. If $i \in I$ it follows that

$$\epsilon(\nabla v_\epsilon)_i = (X_i(x_i)v_\epsilon^i - \epsilon R_i)\prod_{k\epsilon(I-\{i\})} v_\epsilon^k \exp\big(\frac{p(x)}{\epsilon}\big)$$

where

$$R_i = \int_0^1 \frac{X_i(x_i+z_i)}{\epsilon}\exp\big(\frac{f_i(x_i) - f(x_i+z_i) - \lambda_i z_i}{\epsilon}\big)dz_i - \exp\big(-\frac{-\lambda_i}{\epsilon}\big) + 1.$$

For $i \notin I$,

$$(\epsilon\nabla v_\epsilon)_i = (\nabla p(x))_i v_\epsilon = X_i v_\epsilon$$

thus v_ϵ is solution of (3.4). □

4. Dynamics and Steady State

We begin this section with some definitions:

Definition Let X be a vector field in T_n and let $u_\epsilon = \sum_0^\infty \frac{F_i}{\epsilon^i}$ be a series with a positive ratio of convergence. Suppose that u, is a solution of (3.4). We will denote:

$$C_\epsilon = \{x \in M : \nabla u_\epsilon = 0\} \tag{4.1}$$

$$E(X) = \{x \in M : X(x) = 0\} \tag{4.2}$$

$$D(X) = \{x \in M : \det\big(\frac{\partial X_i}{\partial x_j}\big) = 0\} \tag{4.3}$$

In [1] and [2], we have such series on T_n and S_n.

Theorem 4.1. *Consider $X \in V_{ag}(T_n)$ such that*

(i) *There exists a convergent series $u_\epsilon = \sum_{i=0}^\infty \frac{F_i}{\epsilon^i}$ solving (3.4) for $\frac{1}{\epsilon} \le r, r > 0$*
(ii) *There exists an infinity set $S \subset [r_1, r]$, $r1 > 0$ and a point x in T_n such that*

$$x \in C_\epsilon \cap E(X) \quad \forall \frac{1}{\epsilon} \in S \tag{4.4}$$

Then X is a gradient vector field.

Proof. Because $X \in V_{ag}(T_n)$, by Theorem 2.1 we can write

$$X = \nabla f + \lambda\,, \tag{4.5}$$

$$\nabla u_\epsilon = 0 = \sum_{i=0}^\infty \nabla F_i(x)\big(\frac{1}{\epsilon}\big)^i \quad \forall \epsilon \in S. \tag{4.6}$$

Then $\nabla F_i(x) = 0$, for every i. In particular $\nabla F_1(x) = \nabla f(x) = 0$ and by (15) $\lambda = 0$. □

Theorem 4.2. *Let X be a vector field without coupling. Then the following statements are equivalent.*

(i) *There exists ϵ such that $C_\epsilon \cap E(X) \neq \emptyset$*
(ii) *For all ϵ, $C_\epsilon \cap E(X) \neq \emptyset$*
(iii) *X is gradient.*

Proof. For $x \in C_\epsilon \cap E(X)$ and $i \in I$ we have a contradiction:

$$0 = (v_\epsilon^i)'(x) = e^{\frac{\lambda_i}{\epsilon}} - 1 + \frac{\lambda_i}{\epsilon} v_\epsilon(x)\,.$$

wich completes the proof. □

Remark. The main idea here is that for non-gradient cases critical points of a steady state are different from stationary points of the vector field. This fact enable us to find a vector field X with an associated u_ϵ which has not generated critical points, even when X has degenerated stationary points.

Lemma 4.3. *Let's suppose that $X = \nabla f + \lambda$ is without coupling and let I_+ be the set of index such that $\lambda_i > 0$ and let I_- be the set of index such that $\lambda_i < 0$. Then*

$$C_\epsilon \subset \cap_{i\epsilon I_+} X_i^{-1}((0, +\infty)) \cap_{i\epsilon I_-} X_i^{-1}((-\infty, 0))$$

Proof. For a such f we can write
$$f = \sum (f_i(x_i)) + p(x)$$
where $p(x)$ is not depending of x_i for $i \in I = I_- \cup I_+$ and
$$v_\epsilon = K\big(\prod_{i\epsilon I} v_\epsilon^i(x_i)\big)\exp\big(\frac{p(x)}{\epsilon}\big) \tag{4.7}$$
where
$$v_\epsilon^i = \int_0^1 \exp\big(\frac{f_i(x_i) - f(x_i+z_i) - \lambda_i z_i}{\epsilon}\big) dz_i \tag{4.8}$$
So for every $i \in I$, we have
$$\frac{\partial v_\epsilon}{\partial x_i} = \big(\frac{X_i(x_i)v_\epsilon^i}{\epsilon} - R_i\big) \prod_{k\epsilon(I-\{i\})} v_\epsilon^k \exp\big(\frac{p(x)}{\epsilon}\big)\,, \tag{4.9}$$
where $R_i = -\exp(-\lambda_i/\epsilon) + 1$. So if $x \in C_\epsilon$,
$$\frac{X_i(x_i)}{\epsilon} v_\epsilon^i = -\exp\big(-\frac{\lambda_i}{\epsilon}\big) + 1, \quad i \in I \tag{4.10}$$
Then for $i \in I_+$ we have $X_i(x_i) > 0$ and for $i \in I_-$ we have $X_i(x_i) < 0$. □

Lemma 4.4. *Under the hypothesis of Lemma 4.3, the set of degenerated critical points of u_ϵ is a subset of*
$$D_1(X) = \cup_{i\epsilon I_+}[D(X_i) \cap (X_i^{-1}(0,+\infty))] \cup_{i\epsilon I_-} [D(X_i) \cap (X_i^{-1}(-\infty,0))] \cup D(\nabla_p)$$
Here $x \in D(X_i)$, means $X'(x_i) = 0$ and $x \in D(\nabla_p)$ means $\det(\frac{\partial^2 p}{\partial x_i x_j}) = 0$.

Proof. With the notation of Lemma 4.3 and by (4.9) and (4.10), for every $x \in C_\epsilon$,
$$\frac{\partial^2 u_\epsilon}{\partial x_i^2}(x) = X'(x_i)u_\epsilon, \quad i \in I \tag{4.11}$$
$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x) = 0, \quad i,j \in I, i \neq j \tag{4.12}$$
$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x) = \frac{\partial^2 p}{\partial x_i \partial x_j}(x)u_\epsilon, \quad i,j \in I' \tag{4.13}$$
where $I' = \{1,2,\dots,n\} - I$. So
$$\det\big(\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(x)\big) = \big(\prod_{i\epsilon I} X_i'(x_i)\big)\big(\det\big(\frac{\partial^2 p}{\partial x_i \partial x_j}(x)\big)_{j,i\epsilon I'}\big)(u_\epsilon(x))^n \tag{4.14}$$
If x is a degenerated critical point of u_ϵ, by Lemma 4.3, we get $x \in D_1(X)$. □

Theorem 4.5. *Let $X = \nabla f + \lambda$ be a vector field without coupling and suppose*
$$I_+ = \{i : \lambda_i > 0\}, \quad I_- = \{i : \lambda_i < 0\}, \quad I = I_+ \cup I_-, \quad k = \mathrm{card}(I)$$
Let also suppose:

(i) *For every $i \in I_+$ the set $D_1(X_i) = D(X_i) \cap X_i^{-1}(0,+\infty)$ is finite and for $x_i \in D_1(X_i)$ there exits $z_i \in (0,1)$ such that $f(x_i) - f(x_i+z_i) - \lambda_i z_i > 0$.*

(ii) *For $i \in I_-$ the set $D_1(X_i) = D(X_i) \cap X_i^{-1}(-\infty,0)$ is finite and for every $z_i \in (0,1)$ we have $f(x_i) - f(x_i+z_i) - \lambda_i z_i \leq 0$.*

(iii) *Considering p as a function in T_{n-k} do not has critical points which are degenerated.*

Then there exists $\epsilon_0 > 0$ such that u_ϵ does not have degenerated critical points for $0 < \epsilon < \epsilon_0$.

Proof. Suppose there exists a sequence of values ϵ_n with $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence of point x_n in such way that x_n is a critical degenerated point of u_{ϵ_n}. Then by the proceeding Lemma and under conditions (i), (ii) and (iii) we can find a sequence of (ϵ_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = x$ with $x_i \in D_1(X_{i)}$ for some index $i \in I$. Clearly $(x_{n_k})_i = x_i$ for $k > k_0$ because $D_1(X_i)$ is finite set. Then for that index i, it follows:

$$X_i(x_i) u^i_{\epsilon_{n_k}}(x_i) = \epsilon_{n_k}\big(- \exp\big(\frac{\lambda_i}{\epsilon_{n_k}}\big) + 1\big) \tag{4.15}$$

then for (i) or (ii) we have a contradiction when $k \to \infty$. □

Example. Consider the vector field

$$X(x) = \begin{cases} \alpha \exp\big(-\frac{1}{\sin(2\pi x)}\big) & 0 \le x \le 1/2, \\ -\beta \exp\big(-\frac{1}{\sin(2\pi x)}\big) & 1/2 \le x \le 1 \end{cases}$$

It is a C^∞ vector field on T_1. We put

$$\begin{gathered} H = \int_0^{1/2} \exp\big(-\frac{1}{sen 2\pi x}\big), \\ X = \nabla f + \lambda, \end{gathered}$$

$$\begin{aligned} h(x,z) &= f(x_i) - f(x+z) - \lambda z \\ &= \int_0^x X(t)dt - \int_0^{x+z} X(t)dt \end{aligned}$$

Then we have

$$h(1/4, 3/4) = (\beta - \frac{\alpha}{2})H, \quad \lambda = (\alpha - \beta)H$$

Then if $\alpha > \beta > \frac{\alpha}{2}$, $\lambda = (\alpha - \beta) > 0$, $D_1(X) = \{1/4\}$ and by theorem 4.5, we have $\epsilon_0 > 0$ such that u_ϵ does not have degenerated critical points. In this case, v_ϵ is a Morse function for $\epsilon < \epsilon_0$ and X is Zeeman Stable vector field [3].

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