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THE RELATIVISTIC ENSKOG EQUATION NEAR THE VACUUM

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Abstract. We prove an existence and uniqueness theorem for the solution with data near the vacuum in the Hard sphere.

1. Introduction

The relativistic Boltzmann equation is written as

$$V \cdot \nabla_x F = -C(F, F),$$

where the dot represents the Lorentz inner product $(+ - --)$ of 4-vectors $v = (v_1, v_2, v_3)$, $V = (v_0, v_1, v_2, v_3)$, $X = (x_0, x_1, x_2, x_3)$, $x = (x_1, x_2, x_3)$, $x_0 = -t$ and $C(F, F)$ is the collision integral. Normalizing the speed of light $c = 1$ and the particle mass $m = 1$, we have $V \cdot V = 1$ or $v_0 = \sqrt{1 + |v|^2}$.

For convenience, we separate the time and space variables, and then divide by v_0 the relativistic Bolttzman equation to obtain,

$$\partial_t F + \hat{v} \cdot \nabla_x F = Q(F, F) \tag{1.1}$$

where

$$Q(F, F) = v_0^{-1} C(F, F) \text{ y } \hat{v} = \frac{v}{v_0} = \frac{v}{\sqrt{1 + |v|^2}}\,,$$

$$\begin{aligned} Q(F, F)(v) =& \frac{1}{2v_0} \int\int\int \delta(U^2 - 1)\delta(U'^2 - 1)\delta(V'^2 - 1) s\sigma(s, \theta)\delta^4 \\ & \times (U + V - U' - V')[F(u')F(v') - F(u)F(v)] d^4U d^4U' d^4V \end{aligned}$$

where $U^2 = U \cdot U = u_0^2 - |u|^2$, $|u|^2 = u_1^2 + u_2^2 + u_3^2$, δ is the delta function in one variable, δ^4 is the delta function in four variables, and all of the F are evaluated at the same space-time point (t, x). Furthermore $\sigma(s, \theta)$ is called the *differential cross section or the scattering kernel*; it is a function of variables s and θ which will be defined below. The delta functions express the conservation of momentum and

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energy:

$$u' + v' = u + v\,.$$

$$\sqrt{1+|u'|^2} + \sqrt{1+|v'|^2} = \sqrt{1+|u|^2} + \sqrt{1+|v|^2}$$

Let us begin by defining the remaining variables in the collision integral. We define

$$\begin{aligned} S = (U+V)^2 &= (u_0+v_0)^2 - |u+v|^2 \\ &= 2u_0v_0 - 2u\cdot v + u_0^2 - |u|^2 + v_0^2 - |v|^2 \\ &= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - 2u\cdot v + 1)\,. \end{aligned}$$

Now

$$\begin{aligned} 4g^2 &= -(U-V)^2 \\ &= -(u_0-v_0)^2 + |u-v|^2 \\ &= 2u_0v_0 - 2u\cdot v - u_0^2 + |u|^2 - v_0^2 + |v|^2 \\ &= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - u\cdot v + 1) \\ &= s - 4 \end{aligned}$$

and

$$\cos\theta = \frac{(V-U)\cdot(V'-U')}{(V-U)^2}.$$

Furthermore, we define the Moller velocity as the scalar v_M given by

$$v_M^2 = |\hat{v} - \hat{u}|^2 - |\hat{v}\times\hat{u}|^2 = \frac{s(s-4)}{4v_0^2u_0^2}$$

or

$$v_M = \frac{2g\sqrt{1+g^2}}{v_0u_0}\,.$$

The two expressions for v_M^2 are equal because

$$\begin{aligned} \frac{1}{4}s(s-4) = sg^2 &= (u_0v_0 - u\cdot v + 1)(u_0v_0 - u\cdot v - 1) \\ &= |u|^2 + |v|^2 + |u|^2|v|^2 - 2u_0v_0u\cdot v + (u\cdot v)^2 \\ &= u_0^2|v|^2 + v_0^2|u|^2 - 2u_0v_0u\cdot v - (u\times v)^2 \\ &= u_0^2v_0^2\Big[\frac{|v|^2}{v_0^2} + \frac{|u|^2}{u_0^2} - 2\frac{u}{u_0}\cdot\frac{v}{v_0} - \big|\frac{u}{u_0}\times\frac{v}{v_0}\big|^2\Big]\,. \end{aligned}$$

The relativistic equation resulting is

$$\partial_t F + \hat{v}\cdot\nabla_x F = \int_{\mathbb{R}^3}\int_{S^2} v_M\sigma(s,\theta)[F(u')F(v') - F(u)F(v)]d\Omega du\,,$$

where $d\Omega$ is the element of surface area on S^2 and we have to write σ as a function of g and θ. The Enskog equation has the same structure of the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = E(f)\,,$$

where $E(f)$ is the Enskog's collision operator defined by $E(f) = E^+(f) - E^-(f)$. The left-hand side defines the total derivative of f that is equated by the Enskog's collision operator, which is expressed by the difference between the gain and loss terms respectively defined by

$$E^+(f)(t,x,v) = a^2 \int_{\mathbb{R}^3\times S^2_+} Y(f)\sigma(s,\theta)f(t,x+a\eta,w')d\eta dw$$
$$E^-(f)(t,x,v) = a^2 f(t,x,v)\int_{\mathbb{R}^3\times S^2_+} Y(f)\sigma(s,\theta)f(t,x-a\eta,w')d\eta dw\,,$$

where Y is a functional on M, and $S^2_+ = \{\eta \in \mathbb{R}^3 : |\eta| = 1,\ \sigma(s,\theta) \geq 0\}$, and a is the diameter of hard sphere.

A survey of mathematical results on the existence theory for the Cauchy problem for small initial data decay to zero at infinity in the phase space is proposed in [1] as well as in papers [2, 11, 12, 13]. Several other papers have been published about this type of results. Nevertheless, the main results are contained in the papers which have been cited above.

Specifically, paper [12] refers to a hard sphere gas and to initial conditions which tend exponentially to zero at infinity in the phase space. Paper [11] generalizes the result of [12]. The main result concerning the existence of solutions to the classical Boltzmann equation is a theorem by Diperna and Lions [4] that proves existence, but not uniqueness of renormalized solutions; i.e, solutions in a weak sense, which are even more general than distributional solutions. An analogous result holds in the relativistic case, as was shown by Dudynsky and Ekiel-Jezewska [5]. Regarding classical solutions, Illner and Shinbrot [12] have shown global existence of solutions to the nonrelativistic Boltzmann equation for small initial data(close to the vacuum), Galeano, Vasquez and Orozco [6] shown a result for the relativistic Boltzmann equation. When the data are close to equilibrium, global existence of classical solutions has been proved by Glassey and Strauss [8] in the relativistic case and by Ukay [14] in the nonrelativistic case. In the case of the relativistic Enskog equation we don't known results and this would be a first one. The paper is divide in two sections, we build the functional setting in the first, and we prove a lemma and the theorem of existence and uniqueness in the second one.

2. Functional Setting

For a given $\beta > 0$, let

$$\begin{aligned} M = \big\{ f \in C([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3) &: \text{there exists } c>0 \text{ such that} \\ |f(t,x,v)| &\leq ce^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)}\big\}\,. \end{aligned}$$

This space is a Banach space (See [4])

$$\|f\| = \sup_{t,x,v} e^{\beta(\sqrt{1+|v|^2}+|x+tv|^2)}|f(t,x,v)|\,.$$

We introduce the notation

$$f^\#(t,x,v) = f(t,x+tv,v),.$$

Then the Enskog equation can be written as

$$\frac{d}{dt}f^\#(t,x,v) = E^\#(f)\,.$$

Therefore $f^{\#}(t,x,v) = f_0(x,v) + \int_0^t E^{\#}(f)d\tau$. Now

$$E^{+}(f)(t,x,v) = a^2 \int_{\mathbb{R}^3\times S_+^2} Y(f)\sigma(s,\theta)f(t,x,v')f(t,x+a\eta,w')d\eta dw$$

and

$$\begin{aligned}
&E^{+}(f^{\#})(t,x,v)\\
&= a^2 \int_{\mathbb{R}^3\times S_+^2} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x,v')f^{\#}(t,x+a\eta,w')d\eta dw\\
&= a^2 \int_{\mathbb{R}^3\times S_+^2} Y(f^{\#})\sigma(s,\theta)f(t,x+tv,v')f(t,x+a\eta+tv,w')d\eta dw\\
&= a^2 \int_{\mathbb{R}^3\times S_+^2} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x+t(v-v'),v')f^{\#}(t,x+a\eta+t(v-w'),w')d\eta dw\,.
\end{aligned}$$

Analogously

$$\begin{aligned}
&E^{-}(f^{\#})(t,x,v)\\
&= a^2 f(t,x+tv,v)\int_{\mathbb{R}^3\times S_+^2} Y(f^{\#})\sigma(s,\theta)f(t,x-a\eta+tv,w)d\eta dw\\
&= a^2 f^{\#}(t,x,v)\int_{\mathbb{R}^3\times S_+^2} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x-a\eta+t(v-w),w)d\eta dw\,.
\end{aligned}$$

3. Relativistic Enskog Equation

Lemma 3.1. *Suppose that $\sigma(s,\theta)\in L^1_{\rm loc}(\Omega)$ and that there is a constant $c>0$ such that $|Y(f^{\#})|\le c\|f^{\#}\|$ for every $f^{\#}\in M$. Then for some constant $L>0$,*

$$\begin{gathered}
\int_0^t |E^{+}(f^{\#})|d\tau \le \frac{4acL\pi^2}{\beta^4|v|}e^{\beta\sqrt{1+|v|^2}}\|f^{\#}\|^3\\
\int_0^t |E^{-}(f^{\#})|d\tau \le \frac{4acL\pi^2}{\beta^4|v|}e^{\beta\sqrt{1+|v|^2}}\|f^{\#}\|^3\,.
\end{gathered}$$

Proof. Note that

$$\begin{aligned}
|E^{+}(f^{\#})| \le a^2 \int_{\mathbb{R}^3\times S_+^2} &c\|f^{\#})\|\,|\sigma(s,\theta)|\,|f^{\#}(t,x+t(v-v'),v')|e^{\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}\\
&\times |f^{\#}(t,x+a\eta+t(v-w'),w')|e^{\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}\\
&\times e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}d\eta dw\,.
\end{aligned}$$

Since $\sigma(s,\theta)\in L^1_{\rm loc}(\Omega)$, there is a constant $L>0$ such that

$$\begin{aligned}
&|E^{+}(f^{\#})|\\
&\le a^2 \int_{\mathbb{R}^3\times S_+^2} cL\|f^{\#})\|^3 e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}d\eta dw\,.
\end{aligned}$$

Applying the conservation of energy law, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\times S^2_+} cL\|f^{\#})\|^3 e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}d\eta dw\\&= cL\|f^{\#})\|^3\int_{\mathbb{R}^3\times S^2_+} e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)}d\eta dw\,.\end{aligned}$$

Moreover,

$$\begin{aligned}&cL\|f^{\#})\|^3\int_{\mathbb{R}^3\times S^2_+} e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)}d\eta dw\\&= cL\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\int_{\mathbb{R}^3\times S^2_+} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)}d\eta dw\,.\end{aligned}$$

By Fubbini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^3\times S^2_+} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)}d\eta dw &= \int_{\mathbb{R}^3} e^{-\beta\sqrt{1+|w|^2}}\Big(\int_{S^2_+} e^{-\beta|x+a\eta+tv|^2}d\eta\Big)dw\\&\le \frac{4\pi}{\beta^3}\sqrt{\frac{\pi}{\beta}}\cdot\frac{1}{a}\,.\end{aligned}$$

Therefore,

$$|E^+(f^{\#})| \le \frac{4a^2cL\pi^{\frac32}}{\beta^{\frac72}a}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\,.$$

Hence

$$\begin{aligned}\int_0^t |E^+(f^{\#})|d\tau &\le \frac{4acL\pi^{\frac32}}{\beta^{\frac72}}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}}\int_0^\infty e^{-\beta|x+tv|^2}d\tau\\&\le \frac{4acL\pi^2}{\beta^4|v|}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}}\,.\end{aligned}$$

Then

$$\begin{aligned}|E^-(f^{\#})| &\le a^2|f(t,x+tv,v)|e^{\beta(\sqrt{1+|v|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)}\\&\quad\times\int_{\mathbb{R}^3\times S^2_+}|Y(f^{\#})\|\sigma(s,\theta)|f^{\#}(t,x-a\eta+tv-tw,w)\\&\quad\times e^{\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)}e^{-\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)}d\eta dw\\&\le a^2\|f^{\#}\|^3 cLe^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\int_{\mathbb{R}^3\times S^2_+} e^{-\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)}d\eta dw\\&\le a^2\|f^{\#}\|^3 cLe^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\int_{\mathbb{R}^3} e^{-\beta\sqrt{1+|w|^2}}dw\int_{S^2_+} e^{-\beta|x-a\eta+tv|^2}d\eta\\&\le \frac{4a^2cL\pi}{\beta^3}e^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\sqrt{\frac{\pi}{\beta}}\frac{1}{a}\|f^{\#}\|^3\\&\le \frac{4acL\pi^{3/2}}{\beta^{7/2}}e^{-\beta\sqrt{1+|v|^2}}e^{-\beta|x+tv|^2}\|f^{\#}\|^3\end{aligned}$$

and

$$\begin{aligned}\int_0^t |E^-(f^\#)|d\tau &\le \frac{4acL\pi^{3/2}}{\beta^{7/2}} e^{-\beta\sqrt{1+|v|^2}} \int_0^t e^{-\beta|x+\tau v|^2} d\tau \|f^\#\|^3 \\ &\le \frac{4acL\pi^2}{\beta^4|v|} e^{-\beta\sqrt{1+|v|^2}} \|f^\#\|^3 .\end{aligned}$$

So that

$$\int_0^t |E^-(f^\#)|d\tau \le \frac{4acL\pi^2}{\beta^4|v|} \|f^\#)\|^3 e^{-\beta\sqrt{1+|v|^2}}$$

which completes the proof □

Theorem 3.2. *Suppose that $\sigma(s,\theta) \in L^1_{\rm loc}(\Omega)$ and there exists $c > 0$ such that $|Y(f^\#| \le c\|f^\#)\|$ for every $f^\# \in M_R = \{f \in M : \|f\| \le R\}$ with $R^2 < \frac{\beta^4|v|}{16\pi^2 cLa}$ and $\|f_0\| < \frac{R}{2e^{-\beta|x|^2}}$. Then the Enskog relativistic equation has solution in M_R.*

Proof. We define the operator $\boldsymbol{\mathcal{F}}$ on M by

$$\boldsymbol{\mathcal{F}} f^\# = f_0(x,v) + \int_0^t |E^\#(f)|d\tau .$$

Then

$$\begin{aligned}|\boldsymbol{\mathcal{F}} f^\#| &\le |f_0(x,v)| + \Big|\int_0^t E^\#(f)d\tau\Big| \\ &\le |f_0(x,v)| e^{\beta(\sqrt{1+|v|^2}+|x|^2)} e^{-\beta(\sqrt{1+|v|^2}+|x|^2)} + \Big|\int_0^t E^+(f^\#) - E^-(f^\#)d\tau\Big| \\ &\le \|f_0\| e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x|^2} + \int_0^t |E^+(f^\#) - E^-(f^\#)|d\tau \\ &\le \|f_0\| e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x|^2} + \frac{8acL\pi^2}{\beta^4|v|}\|f^\#)\|^3 e^{-\beta\sqrt{1+|v|^2}} \\ &\le \Big[\frac{R}{2} + \frac{8acL\pi^2}{\beta^4|v|} R^3\Big] e^{-\beta\sqrt{1+|v|^2}} \\ &= Re^{-\beta\sqrt{1+|v|^2}} \Big[\frac{1}{2} + \frac{8acL\pi^2}{\beta^4|v|} R^2\Big] \\ &\le Re^{-\beta\sqrt{1+|v|^2}} \Big[\frac{1}{2} + \frac{8acL\pi^2}{\beta^4|v|} \frac{\beta^4|v|}{16\pi^2 cLa}\Big] \\ &\le Re^{-\beta\sqrt{1+|v|^2}} \Big[\frac{1}{2} + \frac{1}{2}\Big] \\ &= Re^{-\beta\sqrt{1+|v|^2}} < R .\end{aligned}$$

Therefore, $\boldsymbol{\mathcal{F}}$ maps M_R into itself. Similarly, we show that $\boldsymbol{\mathcal{F}}$ is a contraction on M_R. Since elements of M_R are continuous, the continuity of $\boldsymbol{\mathcal{F}} f^\#$ is evident. □

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