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CONTROLLABILITY, APPLICATIONS, AND NUMERICAL SIMULATIONS OF CELLULAR NEURAL NETWORKS

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Abstract. In this work we consider the model of cellular neural network (CNN) introduced by Chua and Yang in 1988. We impose the Von-Neumann boundary conditions and study the controllability of corresponding system, then these results are used in image detection by means of numerical simulations.

1. Introduction

Since its introduction ([2, 3]) Cellular neural networks (CNN) have been used in numerous problems. Among them we have: Chua's circuit ([6]), Hopf bifurcation model ([12]), Cellular Automata and systolic arrays ([9]), image detection ([4]), population growth model ([5]). In none of these works the Von-Neumann boundary conditions have been imposed; only in ([7]) periodic boundary conditions were considered.

The system obtained, after some changes ([2, 3]) is

$$\dot{v} = -v + AG(v) + Bu + f(u, v) \tag{1.1}$$

where, $u, v \in \mathbb{R}^{mn\times 1}$, are column vectors; A, B are matrices in $\mathbb{R}^{mn\times mn}$, $f(u, v)$ is a nonlinear perturbation, and $G(v)$ is a function which can be either linear or non-linear. In this paper we set the Von-Neumann boundary conditions, consider $G(v) = v$ and study the controllability of the resulting system which is

$$\dot{v} = (A - I)v + Bu + I \tag{1.2}$$

where $A, B \in \mathbb{R}^{mn\times mn}$, after using the boundary conditions, are tridiagonal matrices, I is the identity matrix in $\mathbb{R}^{mn\times mn}$.

Also, we implement some numerical simulations of these results to show image detection; specifically Chinese characters.

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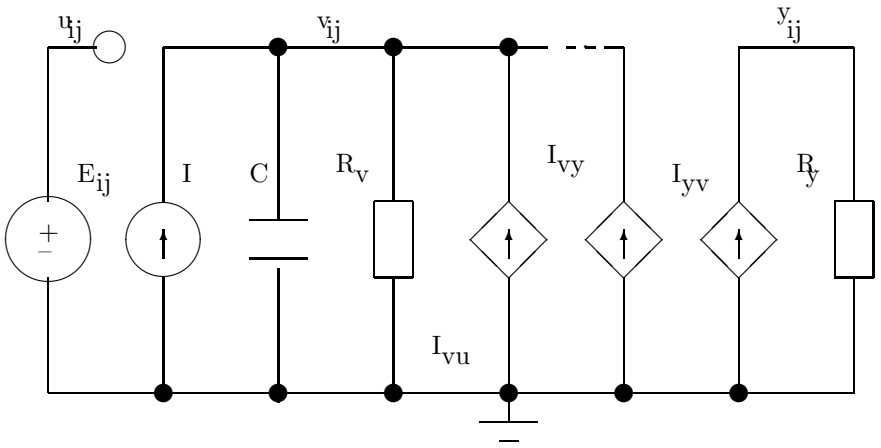

FIGURE 1. Typical circuit of CNN ij-position.

2. Cellular Neural Networks

A CNN consists, basically, in a collection of non linear circuit displayed in a 2-dimensional array. The basic circuit of CNN is called cell. A cell is made of elements of linear and non-linear circuit which usually are linear capacitors, linear resistors, linear and non linear controlled sources, and independents sources. Each cell receives external signals through its input. The state voltage of a give cell is influenced no only by its own input through a feedback, its output; but also by the input and output of the neighboring cells. These interactions are implemented by voltage-controlled current sources. In the initial papers ([2, 3]) any cell in CNN is connected only to its neighbor cells; this is accomplished by using the so called 1-neighborhood or simply neighborhood and consequently 3×3-cloning templates. The adjacent cells can interact directly with each other in the sense that are made of a massive aggregate of regularity spaced cells which communicate with each other directly only through its nearest neighbors. In the figure 1 the basic circuit of a CNN of a cell (located at, say, position ij of the array) is depicted. Here v_{ij} is the voltage across the cell (state of the cell) with its initial condition satisfying $|v_{ij}(0)| \leq 1$. E_{ij} is an independent voltage source, and $u_{ij} = E_{ij}$ is called the input or control, also assumed to satisfy $|u_{ij}| \leq 1$. I is an independent current source, C is a linear capacitors, R_v and R_y are linear resistors. I_{vu}, I_{vy} are linear voltage-controlled currents sources such that at each neighbor cell, say kl, $I_{vy} = (I_{vy})_{kl} = a_{kl} g(v_{kl})$ are current source; $I_{vu} = (I_{vy})_{kl} = b_{kl} u_{kl}$; is nonlinear voltage-controlled source give by $I_{vy} = \frac{1}{R_y} g(v_{ij})$ where, a_{kl}, $b_{kl} \in \mathbb{R}$ and g is an output sigmoid function.

3. Dynamics of CNN

Definition 3.1 (r-neighborhood)**.** The r-neighborhood of a cell c_{ij}, in a cellular neural network is defined by

$$N^{ij} = \{c_{i_1 j_1} : \max\{|i - i_1|; |j - j_1|\} \leq r;\ 1 \leq i_1 \leq m,\ 1 \leq j_1 \leq n\} \tag{3.1}$$

where r is a positive integer.

We consider the case $r = 1$ which produces a couple of 3×3-matrices (cloning templates); the feedback and control operator, given as

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{3.2}$$

The output feedback depends on the interactive parameters a_{ij} and the input control depends an parameters b_{ij}, $v \in \mathbb{R}^{mn}$ is the voltage and represents the state vector, and $u = (u_{11}, u_{12}, \dots, u_{mn})^T \in \mathbb{R}^{mn}$ is the control (input), and the output $y = G(v)$

$$G : \mathbb{R}^{mn} \to \mathbb{R}^{mn}; \qquad G(v) = (g(v_{11}), g(v_{12}), \dots, g(v_{mn}))^T \tag{3.3}$$

g is differentiable, bounded and $\|g\| \le 1$ (in the most general case $\|g\| \le K$) and non decreasing ($g' \ge 0$); that is a sigmoid function. We also assume $\|u\| \le 1$, $\|u(0)\| \le 1$.

Definition 3.2. Let K and L be two square matrices of the same size and elements k_{ij}, l_{ij} respectively; we define $\odot$ product

$$K \odot L = \sum_{i,j} k_{ij} l_{ij}. \tag{3.4}$$

By imposing the Von-Neumann boundary conditions

$$\begin{aligned} &\left.\begin{aligned} v_{ik} &= v_{ik+1} \\ v_{ik-1} &= v_{ik+2} \end{aligned}\right\} \quad i = -1, \dots, n+2, \quad k = 0, m \\ &\left.\begin{aligned} v_{kj} &= v_{k+1j} \\ v_{k-1j} &= v_{k+2j} \end{aligned}\right\} \quad j = -1, \dots, m+2, \quad k = 0, n; \end{aligned} \tag{3.5}$$

and applying the Kirchhoff Law of Voltage and Current, we obtain the equation at cell c_{ij},

$$\dot{v}_{ij} = -v_{ij} + \widetilde{A} \odot \widehat{G}(v_{ij}) + \widetilde{B} \odot \widehat{u}_{ij} + I, \tag{3.6}$$

and in its vector form, by taking the row order in this vector, that is, the first n-elements are formed by the first row of matrix and so on, the resulting system is

$$\dot{v} = -v + AG(v) + Bu + I, \tag{3.7}$$

where

$$\begin{aligned} AG(v) &= (\widetilde{A} \odot \widehat{G}(v_{11}), \dots, \widetilde{A} \odot \widehat{G}(v_{mn}))^T, \\ Bu &= (\widetilde{B} \odot \widehat{u}_{11}, \dots, \widetilde{B} \odot \widehat{u}_{mn})^T, \quad I = (I, \dots, I)^T \end{aligned}$$

matrices A, B are block tridiagonal, $AG(v) = (\widehat{A} + \mathring{A})v$ and $Bu = (\widehat{B} + \mathring{B})u$ with

$$\widehat{A} = \begin{pmatrix} A_2 & A_3 & 0 & \dots & 0 & 0 \\ A_1 & A_2 & A_3 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & A_1 & A_2 & A_3 & 0 \\ 0 & 0 & \dots & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & \dots & A_1 & A_2 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} B_2 & B_3 & 0 & \dots & 0 & 0 \\ B_1 & B_2 & B_3 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & B_1 & B_2 & B_3 & 0 \\ 0 & 0 & \dots & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & \dots & B_1 & B_2 \end{pmatrix},$$

$$A_i = \begin{pmatrix} a_{i2} & a_{i3} & 0 & \dots & 0 \\ a_{i1} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{i3} \\ 0 & \dots & \dots & a_{i1} & a_{i2} \end{pmatrix}, \quad i = 1, 2, 3.$$

The matrix $\widehat{B}$ has the same blocks. The perturbation matrices look like,

$$\overset{\circ}{A} = \begin{pmatrix} L_1 + \Gamma_2 & \Gamma_3 & 0 & \dots & 0 & 0 \\ \Gamma_1 & \Gamma_2 & \Gamma_6 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & 0 \\ \vdots & \ddots & \Gamma_1 & \Gamma_2 & \Gamma_3 & 0 \\ \vdots & \dots & \ddots & \Gamma_1 & \Gamma_2 & \Gamma_3 \\ 0 & \dots & \dots & L_2 & \Gamma_1 & \Gamma_2 \end{pmatrix}, \quad \begin{cases} L_1 = A_1 + \Gamma_1 \\ L_2 = A_3 + \Gamma_3 \end{cases}.$$

$$\Gamma_i = \begin{pmatrix} a_{i1} & 0 & & \dots & 0 \\ 0 & 0 & & & \\ 0 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & a_{i3} & 0 & 0 \end{pmatrix} \quad i = 1, 2, 3.$$

The matrix $\overset{o}{B}$ is defined similarly.

Remark 3.3. Other types of order were tested but they produce the same type of matrix, block tridiagonal.

Lemma 3.4. *If A, B are two arbitrary square matrices of size $l \times l$ and real entries, then $(A \otimes B)^n = A^n \otimes B^n$, for all $n \in \mathbb{N}$.*

Corollary 3.5. *If A is a matrix of order $n \times n$ and $\Pi = \operatorname{circ}(0, 1, 0, \dots, 0)$ is circulant matrix, then $(A \otimes \Pi)^k = A^k \otimes \Pi^k$; for $k = 1, \dots, m$.*

4. CNN and Controllability

In this section we study the controllability of the general system (3.7) by means of the properties of block tridiagonal matrices Instead of (3.7) we study the linear case

$$\dot{v} = (A - I)v + Bu + I\,. \tag{4.1}$$

The study of the controllability of (4.1) is equivalent to study the controllability of

$$\dot{v} = (A - I)v + Bu. \tag{4.2}$$

Note that $A - I$ is tridiagonal matrix same type as A.

Lemma 4.1. *Any block tridiagonal matrix*

$$A = \begin{pmatrix} A_2 & A_3 & 0 & \dots & 0 & 0 \\ A_1 & A_2 & A_3 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & A_1 & A_2 & A_3 & 0 \\ 0 & 0 & \dots & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & \dots & A_1 & A_2 \end{pmatrix}$$

can be written as $A = A_3 \otimes \Pi + A_1 \otimes \Pi^{n-1} + A_2 \otimes \Pi^n$.

Lemma 4.2. *For every block tridiagonal matrix* A*, the following takes place*

$$A^k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (A_3^{k-i-j} A_1^{nj-j} A_2^i \otimes \Pi)^{k-i-2j}; \quad k \in \mathbb{N}.$$

Proof. For $l \in \mathbb{N}$ fixed

$$\begin{aligned} A^l &= [A_3 \otimes \Pi + A_1 \otimes \Pi^{n-1} + A_2 \otimes \Pi^n]^l \\ &= \sum_{i=0}^{l} \sum_{j=0}^{l-i} \binom{l}{i} \binom{l-i}{i} (A_3 \otimes \Pi)^{l-i-j} (A_1 \otimes \Pi^{n-1})^j (A_2 \otimes \Pi^n)^i \\ &= \sum_{i=0}^{l} \sum_{j=0}^{l-i} \binom{l}{i} \binom{l-i}{i} A_3^{l-i-j} A_1^j A_2^i \otimes \Pi^{l-i-2j}. \end{aligned}$$

□

Theorem 4.3. *Let* A *and* B *be two* $n \times n$ *block tridiagonal matrices. Then*

$$\begin{aligned} A^k B = &\sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (A_3^{k-i-j} A_1^j A_2^i) \\ &\times [B_3 \otimes \Pi + B_1 \otimes \Pi^{n-1} + B_2 \otimes \Pi^n] \Pi^{k-(i+2j)} \end{aligned}$$

for $k \in \mathbb{N}$.

Proof. By induction: for $k = 1$,

$$AB = \sum_{i=0}^{1} \sum_{j=0}^{1-i} \binom{1}{i} \binom{1-i}{j} (A_3^{1-i-j} A_1^j A_2^i)[B_3 \otimes \Pi + B_1 \otimes \Pi^{n-1} + B_2 \otimes \Pi^n] \Pi^{1-(i+2j)}.$$

Assume the statement of the theorem is true for $k = m$. Then for for $k = m + 1$, we have

$$\begin{aligned} A^{m+1} B &= A A^m B \\ &= \sum_{i=0}^{m+1} \sum_{j=0}^{m+1-i} \binom{m+1}{i} \binom{m+1-i}{j} (A_3^{m+1-i-j} A_1^j A_2^i) \\ &\times [B_3 \otimes \Pi + B_1 \otimes \Pi^{n-1} + B_2 \otimes \Pi^n] \Pi^{m+1-(i+2j)} \end{aligned}$$

□

According to [10, Theorem 3], the controllability of (4.2) depends on the rank of (A, B). However,

$$\begin{aligned}\mathbf{Rg}[\mathbb{R}(A,B)] &= \mathbf{Rg}([B, AB, \dots, A^{n-1}B]) \\ &= \mathbf{Rg}\left[\left(\begin{bmatrix} C_1 & C_2 & \dots & C_{n-1} \\ C_1 & C_2 & \dots & C_{n-1} \\ \vdots & \vdots & & \vdots \\ C_1 & C_2 & \dots & C_{n-1} \end{bmatrix} \otimes \mathbf{B}\right)\mathbf{D}\right],\end{aligned}$$

where

$$\mathbf{C} = \begin{bmatrix} C_1 & C_2 & \dots & C_{n-1} \\ C_1 & C_2 & \dots & C_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ C_1 & C_2 & \dots & C_{n-1} \end{bmatrix},$$

$\mathbf{B} = B_3 \otimes \Pi + B_1 \otimes \Pi^{n-1} + B_2 \otimes \Pi^n$,

$$C_1 = \sum_{i=0}^{0}\sum_{j=0}^{0-i} \binom{0}{i}\binom{0-i}{j}(A_3^{0-i-j}A_1^jA_2^i)$$

$$C_2 = \sum_{i=0}^{1}\sum_{j=0}^{1-i} \binom{1}{i}\binom{1-i}{j}(A_3^{1-i-j}A_1^jA_2^i)$$

$$\vdots$$

$$C_{n-1} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i} \binom{n-1}{i}\binom{n-1-i}{j}(A_3^{n-1-i-j}A_1^jA_2^i),$$

and

$$\mathbf{D} = \begin{bmatrix} \Pi^n & 0 & 0 & 0 \\ 0 & \Pi^{1-(i+2j)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Pi^{(n-1)-(i+2j)} \end{bmatrix}.$$

Proposition 4.4. *Let*

$$\mathbf{D} = \begin{bmatrix} \Pi^n & 0 & 0 & 0 \\ 0 & \Pi^{1-(i+2j)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Pi^{(n-1)-(i+2j)} \end{bmatrix}.$$

Then $|\det(\mathbf{D})| = 1$.

The proof of the above proposition can be found in [1] We are now ready to give the main result of this section, which is quite technical, but applicable to several situations discussed later.

Theorem 4.5. *The system (4.2) is controllable if and only if* $\mathbf{Rg}(C \otimes B) = n$.

Proof. By [10, Theorem 3], the system (4.2) is controllable if and only if

$$\mathbf{Rg}[\mathbb{R}(A,B)] = \mathbf{Rg}[B, AB, \dots, A^{n-1}B].$$

By the above proposition this is true if and only if $\mathbf{Rg}(C \otimes B) = n$. □

Example. Let $m = 3$ and $n = 3$; let matrices $\widetilde{A}$ and $\widetilde{B}$ be as in (3.2); let the output $y = G_2(v)$, with $G_2 : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 27}$ given as

$$G_2(v) = \big(G(v_{11}), G(v_{12}), G(v_{13}), G(v_{21}), G(v_{22}), G(v_{23}), G(v_{31}), G(v_{32}), G(v_{33})\big)^T .$$

We impose Von-Neumann the boundary conditions and get

$$G(v_{11}) = \begin{pmatrix} v_{11} & v_{11} & v_{12} \\ v_{11} & v_{11} & v_{12} \\ v_{21} & v_{21} & v_{22} \end{pmatrix}, \quad G(v_{12}) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix},$$
$$G(v_{13}) = \begin{pmatrix} v_{12} & v_{13} & v_{11} \\ v_{12} & v_{13} & v_{11} \\ v_{22} & v_{23} & v_{21} \end{pmatrix}, \quad G(v_{21}) = \begin{pmatrix} v_{11} & v_{11} & v_{12} \\ v_{21} & v_{21} & v_{22} \\ v_{31} & v_{31} & v_{32} \end{pmatrix},$$
$$G(v_{22}) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}, \quad G(v_{23}) = \begin{pmatrix} v_{12} & v_{13} & v_{11} \\ v_{22} & v_{23} & v_{21} \\ v_{32} & v_{33} & v_{31} \end{pmatrix},$$
$$G(v_{31}) = \begin{pmatrix} v_{21} & v_{21} & v_{22} \\ v_{31} & v_{31} & v_{32} \\ v_{11} & v_{11} & v_{12} \end{pmatrix}, \quad G(v_{32}) = \begin{pmatrix} v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \\ v_{11} & v_{12} & v_{13} \end{pmatrix},$$
$$G(v_{33}) = \begin{pmatrix} v_{22} & v_{23} & v_{21} \\ v_{32} & v_{33} & v_{31} \\ v_{12} & v_{13} & v_{11} \end{pmatrix} .$$

Now $AG_2(v)$ has the form

$$\begin{pmatrix} (a_{11}+a_{12}+ \\ a_{21}+a_{22}) & a_{13}+a_{23} & 0 & a_{31}+a_{32} & a_{33} & 0 & 0 & 0 & 0 \\ a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{13}+a_{23} & a_{11}+a_{21} & a_{12}+a_{22} & a_{33} & a_{31} & a_{32} & 0 & 0 & 0 \\ a_{11}+a_{12} & a_{13} & 0 & a_{21}+a_{22} & a_{23} & 0 & a_{31}+a_{32} & a_{33} & 0 \\ a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \\ a_{13} & a_{11} & a_{12} & a_{23} & a_{21} & a_{22} & a_{33} & a_{31} & a_{32} \\ a_{31}+a_{32} & a_{33} & 0 & a_{11}+a_{12} & a_{13} & 0 & a_{21}+a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \\ a_{33} & a_{31} & a_{32} & a_{13} & a_{11} & a_{12} & a_{23} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} .$$

We write $AG_2(v)$ as

$$AG_2(v) = (\widehat{A} + \overset{\circ}{A})G_2(v) = \widehat{A}G_2(v) + \overset{\circ}{A}G_2(v).$$

Then we do the same for matrix Bu. Now (4.1) becomes

$$\dot{v} = -v + \widehat{A}G_2(v) + \widehat{B}u + f(u, v) \tag{4.3}$$

Note that A and B are tridiagonal matrices and $f(u, v) = I + \overset{\circ}{A}G_2(v) + \overset{\circ}{B}u$ is a perturbation of (4.2); if $f(u, v) = 0$, (4.3) is controllable, for [8] (Theorem 11),

then (4.3) also is controllable, where $\overset{\circ}{A}G_2(v)$ and $\overset{\circ}{B}u$ have the form, respectively,

$$\overset{\circ}{A}G_2(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13}+a_{23} & 0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & 0 & 0 & a_{23} & 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{33} & 0 & 0 & a_{13} & 0 & 0 & a_{23} & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{pmatrix},$$

$$\overset{\circ}{B}u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{13}+b_{23} & 0 & 0 & b_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{13} & 0 & 0 & b_{23} & 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{33} & 0 & 0 & b_{13} & 0 & 0 & b_{23} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{pmatrix}.$$

Remark 4.6. *So far we have studied the case where $G(v) = \alpha v$, $\alpha > 0$; that is, the linear case. The non-linear case*

$$\dot{v} = -v + AG(v) + Bu \tag{4.4}$$

can be attacked just writing down

$$\dot{v} = (A-I)v + Bu + (AG(v) - Av) = (A-I)v + Bu + A(G(v) - v) \tag{4.5}$$

and imposing the condition of $A(G(v) - v)$ being globally Lipschitz. In this case we guarantee controllability of (4.4) if (4.5) is controllable.

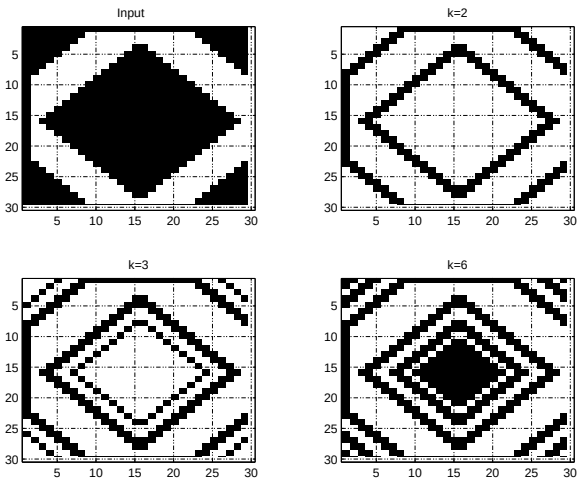

FIGURE 2. Input and some iterations by a 30×30 matrix.

5. Numerical Simulations

In this section we use our model of CNN in image detection; most of our examples are Chinese characters. The idea is input an image and iterate equation (1.2) by using Runge-Kutta 4-order method. We shall use the corner detecting CNN since in [11], but taking $b_{22} = 5$; in other words

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -7/20 & -1/4 & -7/20 \\ -1/4 & 5 & -1/4 \\ -7/20 & -1/4 & -7/20 \end{pmatrix}, \quad I = 3 \times 10^{-4}\text{Amp.}$$

First, we consider figure 2 a diamond as input and some iterations, we detect the main character of the stroke in the first three steps of this process. In a 30 array; and after a few iterations we reach the maximum detections.

In figure 3, we find the same behavior as in the figure 2; by taking now k (number of iterations) a little bigger.

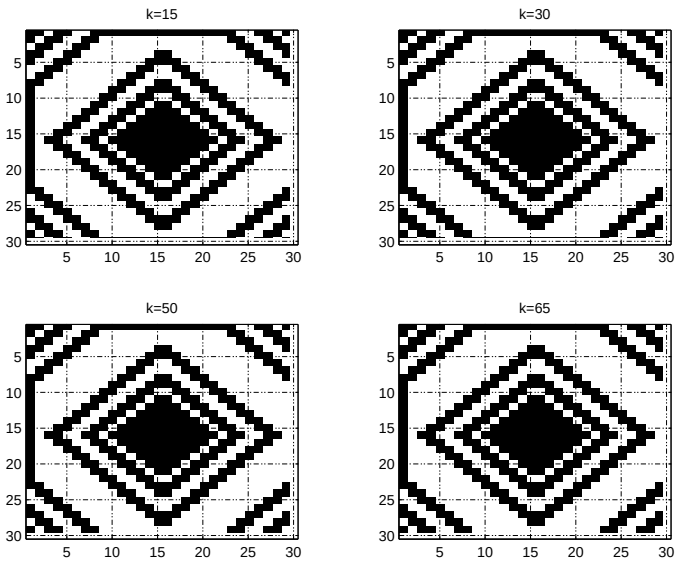

Figure 3. More iterations in case of the diamond.

Figure 4 is a Chinese character with an 35×35 array. After some iterations for $k = 3$ and $k = 10$ maximum detection is achieved.

Figure 5 is made by an iteration of the input in figure 4 with k bigger, the output is the same as in the previous figure.

As a concluding remark, we want to mention that the two input figures chosen here are the same as two of the chosen in ([2], [7]), but now we are imposing Von-Neumann boundary conditions. In our case maximum detection is attained in fewer steps that the ones in the mentioned papers.

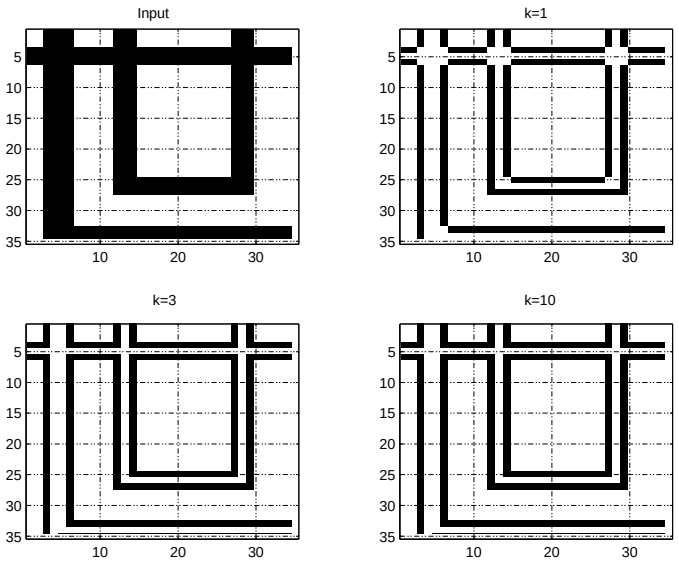

FIGURE 4. Input and some iterations for a 35×35 matrix with an ideogram.

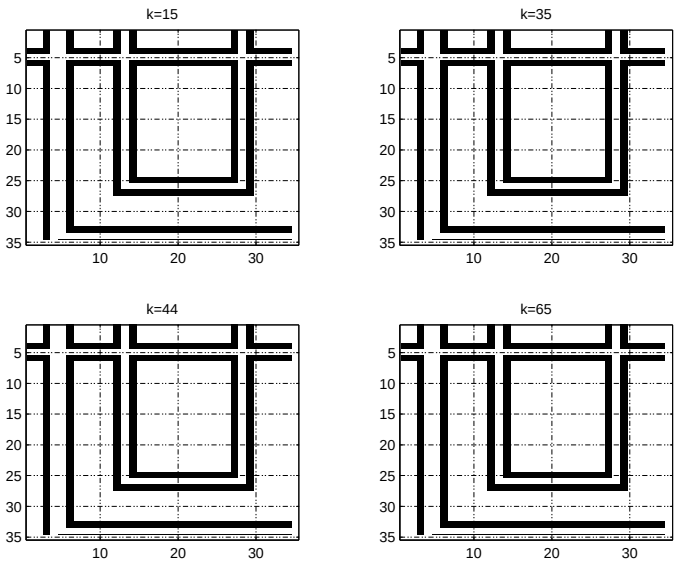

FIGURE 5. 35×35-array; some more iterations.

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