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OSCILLATION CRITERIA FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Consider the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

and the second-order linear delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

where p and τ are continuous functions on $[t_0, \infty)$, $p(t) > 0$, $\tau(t)$ is non-decreasing, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. Several oscillation criteria are presented for the first-order equation when

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1,$$

and for the second-order equation when

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds < 1\,.$$

1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the first-order differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

and to the second-order equation

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.2}$$

where $p \in C([t_0,\infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau \in C([t_0, \infty),\ \mathbb{R})$, $\tau(t)$ is non-decreasing, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example the references in this article and the references cited therein.

By a solution of (1.1) (resp. (1.2)) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1.1) (resp. (1.2)) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory.*

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In this paper our main purpose is to present the state of the art on the oscillation of all solutions to (1.1) especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \le \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1,$$

and for (1.2) when

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds \le \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds < 1.$$

2. Oscillation criteria for the first-order equation

In this section we study the delay equation (1.1). The first systematic study for the oscillation of all solutions to (1.1) was made by Myshkis. In 1950 [42] he proved that every solution of (1.1) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t-\tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{2.1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [33] proved that the same conclusion holds if

$$A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{2.2}$$

In 1979, Ladas [32] established integral conditions for the oscillation of (1.1) with constant delay. Tomaras [54-56] extended this result to (1.1) with variable delay. For related results see Ladde [36-38]. The following most general result is due to Koplatadze and Canturija [25]. If

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{2.3}$$

then all solutions of (1.1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{2.4}$$

then (1.1) has a nonoscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [35] and in 1984 Fukagai and Kusano [13] established oscillation criteria (of the type of conditions (2.2) and (2.3)) for (1.1) with *oscillating* coefficient $p(t)$.

It is obvious that there is a gap between the conditions (2.2) and (2.3) when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [12] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of (1.1). Their result says that all the solutions of (1.1) are oscillatory, if $0 < \alpha \le \frac{1}{e}$ and

$$A > 1 - \frac{\alpha^2}{4}. \tag{2.5}$$

Since then, several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991, Jian [20] derived the condition

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)}, \tag{2.6}$$

while in 1992, Yu and Wang [63] and Yu, Wang, Zhang and Qian [64] obtained the condition

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \tag{2.7}$$

In 1990, Elbert and Stavroulakis [8] and in 1991 Kwong [30], using different techniques, improved (2.5), in the case where $0 < \alpha \leq \frac{1}{e}$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{2.8}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{2.9}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [26] improved (2.7), while in 1998, Philos and Sficas [45] and in 1999, Zhou and Yu [65] and Jaroš and Stavroulakis [19] derived the conditions

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \tag{2.10}$$

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{2.11}$$

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.12}$$

respectively.

Consider (1.1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [22] and in 2003, Sficas and Stavroulakis [46] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{(1-\alpha)^2 - 4\Theta}}{2} \tag{2.13}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1} \tag{2.14}$$

respectively, where

$$\Theta = \frac{e^{\lambda_1\theta\alpha} - \lambda_1\theta\alpha - 1}{(\lambda_1\theta)^2}$$

and

$$M = \frac{1 - \alpha - \sqrt{(1-\alpha)^2 - 4\Theta}}{2}.$$

Remark 2.1 ([22, 46]). Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1 - \lambda_1\alpha - 1}{{\lambda_1}^2}$, and (2.13) reduces to

$$A > 2\alpha + \frac{2}{\lambda_1} - 1, \tag{2.15}$$

while in this case it follows that $M = 1 - \alpha - \frac{1}{\lambda_1}$ and (2.14) reduces to

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha\lambda_1}}{\lambda_1}. \tag{2.16}$$

In the case where $\alpha = \frac{1}{e}$, then $\lambda_1 = e$, and (2.16) leads to

$$A > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\alpha \to 0$, then all the previous conditions (2.5)-(2.15) reduce to the condition (2.2), i.e. $A > 1$. However, the condition (2.16) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (2.16) improves all the above conditions when $0 < \alpha \le 1/e$ as well. Note that the value of the lower bound on A can not be less than

$$\frac{1}{e} \approx 0.367879441.$$

Thus the aim is to establish a condition which leads to a value *as close as possible to* $1/e$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = 1/e$.

Condition	Lower bound
(2.5)	0.966166179
(2.6)	0.892951367
(2.7)	0.863457014
(2.8)	0.845181878
(2.9)	0.735758882
(2.10)	0.709011646
(2.11)	0.708638892
(2.12)	0.599215896
(2.15)	0.471517764
(2.16)	0.459987065

We see that the condition (2.16) essentially improves all the known results in the literature.

Example 2.2 ([46]). Consider the delay differential equation

$$x'(t) + px\big(t - q\sin^2\sqrt{t} - \frac{1}{pe}\big) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46\,.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (2.5)-(2.15) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \ge \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1995, by Elbert and Stavroulakis [9], by Kozakiewicz [28], Li [40, 41], and in 1996, by Domshlak and Stavroulakis [6].

3. Oscillation criteria for the second-order equation

In this section we study the second-order delay equation (1.2):

$$x''(t) + p(t)x(\tau(t)) = 0,\ t \geq t_0,$$

For the case of ordinary differential equations, i.e. when $\tau(t) \equiv t$, the history of the problem began as early as in 1836 by the work of Sturm [47] and was continued in 1893 by Kneser [21]. Essential contribution to the subject was made by Hille, Wintner, Hartman, Leighton, Nehari, and others (see the monograph by C. Swanson [48] and the references cited therein). In particular, in 1948 Hille [17] obtained the following well-known oscillation criteria. Let

$$\limsup_{t\to\infty} t \int_t^{+\infty} p(s)ds > 1 \tag{3.1}$$

or

$$\liminf_{t\to\infty} t \int_t^{+\infty} p(s)ds > \frac{1}{4}, \tag{3.2}$$

the conditions being assumed to be satisfied if the integral diverges. Then (1.2) with $\tau(t) \equiv t$ is oscillatory.

For the delay differential equation (1.2) earlier oscillation results can be found in the monographs by Myshkis [43] and Norkin [44]. In 1968 Waltman [57] and in 1970 Bradley [1] proved that (1.2) is oscillatory if

$$\int^{+\infty} p(t)dt = +\infty.$$

Proceeding in the direction of generalization of Hille's criteria, in 1971 Wong [60] showed that if $\tau(t) \geq \alpha t$ for $t \geq 0$ with $0 < \alpha \leq 1$, then the condition

$$\liminf_{t\to\infty} t \int_t^{+\infty} p(s)ds > \frac{1}{4\alpha} \tag{3.3}$$

is sufficient for the oscillation of (1.2). In 1973, Erbe [10] generalized this condition to

$$\liminf_{t\to\infty} t \int_t^{+\infty} \frac{\tau(s)}{s} p(s)ds > \frac{1}{4} \tag{3.4}$$

without any additional restriction on τ. In 1987, Yan [61] obtained some general criteria improving the previous ones.

An oscillation criterion of different type is given in 1986 by Koplatadze [23] and in 1988 by Wei [59], where it is proved that (1.2) is oscillatory if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > 1 \tag{3.5}$$

or

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > \frac{1}{e}. \tag{3.6}$$

The conditions (3.5) and (3.6) are analogous to the oscillation conditions (2.2) and (2.3) respectively, for the first order delay equation (1.1). The essential difference between (3.3), (3.4) and (3.5), (3.6) is that the first two can guarantee oscillation for ordinary differential equations as well, while the last two work only for delay equations. Unlike first-order differential equations, where the oscillatory character is due to the delay only, equation (1.2) can be oscillatory without any delay at all, i.e., in the case $\tau(t) \equiv t$. Figuratively speaking, two factors contribute to the oscillatory character of (1.2): the presence of the delay and the second order nature of the equation. The conditions (3.3), (3.4) and (3.5), (3.6) illustrate the role of these factors taken separately.

In what follows it will be assumed that the condition

$$\int^{+\infty} \tau(s)p(s)ds = +\infty \tag{3.7}$$

is fulfilled. As it follows from [24, Lemma 4.1], this condition is necessary for (1.2) to be oscillatory. The study being devoted to the problem of oscillation of (1.2), the condition (3.7) does not affect the generality.

In this section oscillation results are obtained for (1.2) by reducing it to a first order equation. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay equations only and do not work in the ordinary case.

Theorem 3.1 ([27]). *Let* (3.7) *be fulfilled and the differential inequality*

$$x'(t) + \Big(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\Big)p(t)x(\tau(t)) \le 0$$

have no eventually positive solution. Then (1.2) *is oscillatory.*

Note that Theorem 3.1 reduces the question of oscillation of (1.2) to that of the absence of eventually positive solutions of the differential inequality

$$x'(t) + \Big(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\Big)p(t)x(\tau(t)) \le 0. \tag{3.8}$$

So oscillation results for first order delay differential equations can be applied since the oscillation of the equation

$$u'(t) + g(t)u(\delta(t)) = 0 \tag{3.9}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$u'(t) + g(t)u(\delta(t)) \le 0. \tag{3.10}$$

This fact is a simple consequence of the following comparison theorem deriving the oscillation of (3.9) from the oscillation of the equation

$$v'(t) + h(t)v(\sigma(t)) = 0. \tag{3.11}$$

We assume that $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ are locally integrable, $\delta, \sigma : \mathbb{R}^+ \to \mathbb{R}$ are continuous, $\delta(t) \le t, \sigma(t) \le t$ for $t \in \mathbb{R}^+$, and $\delta(t) \to +\infty$, $\sigma(t) \to +\infty$ as $t \to +\infty$.

Theorem 3.2. *Let $g(t) \ge h(t)$ and $\delta(t) \le \sigma(t)$ for $t \in R^+$ and let* (3.11) *be oscillatory. Then* (3.9) *is also oscillatory.*

Corollary 3.3. *Let* (3.9) *be oscillatory. Then the inequality* (3.10) *has no eventually positive solution.*

Turning to applications of Theorem 3.1, we will use it together with the criteria (2.2) and (2.3) to get the following result.

Theorem 3.4 ([27]). *Let*

$$K := \limsup_{t\to\infty} \int_{\tau(t)}^t \Big(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\Big)p(s)ds > 1, \tag{3.12}$$

or

$$k := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \Big(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\Big)p(s)ds > \frac{1}{e}. \tag{3.13}$$

Then (1.2) *is oscillatory.*

To apply Theorem 3.1 it suffices to note that: (i) (3.7) is fulfilled since otherwise $k = K = 0$; (ii) since $\tau(t) \to +\infty$ as $t \to +\infty$, the relations (3.12), (3.13) imply the same relations with 0 changed by any $T \geq 0$.

Remark 3.5 ([27])**.** Theorem 3.4 improves the criteria (3.5), (3.6) by Koplatadze [23] and Wei [59] mentioned above. This is directly seen from (3.12), (3.13) and can be easily checked if we take $\tau(t) \equiv t - \tau_0$ and $p(t) \equiv p_0/(t - \tau_0)$ for $t \geq 2\tau_0$, where the constants $\tau_0 > 0$ and $p_0 > 0$ satisfy

$$\tau_0 p_0 < 1/e.$$

In this case neither of (3.5), (3.6) is applicable for (1.2) while both (3.12), (3.13) give the positive conclusion about its oscillation. Note also that this is exactly the case where the oscillation is due to the delay since the corresponding equation without delay is non-oscillatory.

Remark 3.6 ([27])**.** Criteria (3.12), (3.13) look like (2.2), (2.3) but there is an essential difference between them as pointed out in the introduction. The condition (2.3) is close to be the necessary one, since according to [25] if $A \leq 1/e$, then (3.9) is nonoscillatory. On the other hand, for an oscillatory equation (1.2) without delay we have $k = K = 0$. Nevertheless, the constant $1/e$ in Theorem 3.4 is also the best possible in the sense that for any $\varepsilon \in (0, 1/e]$ it can not be replaced by $1/e - \varepsilon$ without affecting the validity of the theorem. This is illustrated as follows.

Example 3.7 ([27])**.** Let $\varepsilon \in (0, 1/e]$, $1 - e\varepsilon < \beta < 1$, $\tau(t) \equiv \alpha t$ and $p(t) \equiv \beta(1-\beta)\alpha^{-\beta}t^{-2}$, where $\alpha = \exp(\frac{1}{\beta-1})$. Then (3.13) is fulfilled with $1/e$ replaced by $1/e - \varepsilon$. Nevertheless (1.2) has a nonoscillatory solution, namely $u(t) \equiv t^{\beta}$. Indeed, denoting $c = \beta(1-\beta)\alpha^{-\beta}$, we see that the expression under the limit sign in (3.13) is constant and equals

$$\alpha c|\ln\alpha|(1 + \alpha c) = (\beta/e)(1 + (\beta(1-\beta))/e) > \beta/e > 1/e - \varepsilon.$$

Note that there is a gap between conditions (3.12) and (3.13) when $0 \leq k \leq 1/e$, $k < K$. In the case of first order equations the conditions (2.5)–(2.16) attempt to fill this gap. Using results in this direction, one can derive various sufficient conditions for the oscillation of (1.2). According to Remark 2, neither of them can be optimal in the above sense but, nevertheless, they are of interest since they cannot be derived from other known results in the literature. We combine Theorem 3.1 and [19, Corollary 1] to obtain the following result.

Theorem 3.8 ([27])**.** *Let K and k be defined by* (3.12), (3.13), $0 \leq k \leq 1/e$ *and*

$$K > k + \frac{1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{3.14}$$

where $\lambda(k)$ is the smaller root of the equation $\lambda = \exp(k\lambda)$. Then (1.2) *is oscillatory.*

Note that condition (3.14) is analogous to condition (2.12).

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