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APPROXIMATE CONTROLLABILITY OF DISTRIBUTED SYSTEMS BY DISTRIBUTED CONTROLLERS

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Abstract. Approximate controllability problem for a linear distributed control system with possibly unbounded input operator, connected in a series to another distributed system without control is investigated. An initial state of the second distributed system is considered as a control parameter. Applications to control partial equations governed by hyperbolic controller, and to control delay systems governed by hereditary controller are considered.

1. Statement of the Problem

Research in control theory started for single control systems. However, many technical applications use control systems interconnected in many ways. The goal of the present paper is to establish approximate controllability conditions for a control system interconnected in a series with a second homogeneous system without control in such a way that a control function of the first control system is an output of the second one, so a control is considered as an initial state of a second system.

Let X, U, Z be Hilbert spaces, and let A, C be infinitesimal generators of strongly continuous C_0-semigroups $S_A(t)$ in X and $S_C(t)$ in Z correspondingly in the class C_0 [5, 8]. Consider the abstract evolution control equation

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ u(t) = Kz(t), \quad 0 \le t < +\infty, \end{gathered} \tag{1.1}$$

where $z(t)$ is a mild solution of another evolution equation of the form

$$\dot{z}(t) = Cz(t), \quad z(0) = z_0, \quad 0 \le t < +\infty. \tag{1.2}$$

Here $x(t), x_0 \in X$, $u(t) \in U$, $z(t), z_0 \in Z$, $B : U \to X$ is a linear possibly unbounded operator, $K : Z \to U$ is a linear possibly unbounded onto operator.

Equation (1.2) is said to be a controller equation. A control $u(t)$ is defined by $u(t) = Kz(t)$ as an output of controller equation (1.2).

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Let $x(t,0,x_0,u(\cdot))$ be a mild solution of (1.1) with the initial condition $x(0)=x_0$, and let $u(t,0,z_0)=Kz(t,0,z_0)$, where $z(t,0,z_0)$ is a mild solution of equation (1.2) with the initial condition $z(0)=z_0$.

The initial data $z_0\in Z$ of equation (1.2) is considered as a control.

Let $\mu\notin\sigma_A$. We will consider the spaces W and V defined as follows: W is the domain of the operator A with the norm $\|x\|_\mu=\|(\mu I-A)x\|$; V is the closure of X with respect to the norm $\|x\|_{-\mu}=\|(R_A(\mu)x)\|$, where $R_A(\mu)=(\mu I-A)^{-1}$.

Obviously $W\subset X\subset V$ with continuous dense imbeddings. The following facts are well known, see for example [5, 9, 8, 14, 15, 21],

- For each $t\ge 0$ the operator $S_A(t)$ has a unique continuous extension $\mathcal{S}_{\mathcal{A}}(t)$ on the space V. The family of operators $\mathcal{S}_{\mathcal{A}}(t):V\to V$ is the semigroup in the class C_0 with respect to the norm of V. The corresponding infinitesimal generator $\mathcal{A}$ of the semigroup $\mathcal{S}_{\mathcal{A}}(t)$ is the closed dense extension of the operator A on the space V with domain $D(\mathcal{A})=X$.
- The sets of the generalized eigenvectors of operators $\mathcal{A},\mathcal{A}^*$ and A, A^* are the same.
- For each $\mu\notin\sigma_A$ the operator $R_A(\mu)$ has a unique continuous extension to the operator $\mathcal{R}_A(\mu):V\to X$.
- A mild solution $x(t,0,x_0,u(\cdot))$ of (1.1) with initial condition (1.2) is defined by the representation formula
$$x(t,0,x_0,u(\cdot))=S(t)x_0+\int_0^t\mathcal{S}(t-\tau)Bu(\tau)d\tau, \tag{1.3}$$
where the integral in (1.3) is understood in the Bochner sense [5] with respect to the topology of V.

Denote
$$u_{t_2}(t,0,z_0)=\begin{cases}u(t,0,z_0) & \text{if } 0\le t\le t_2,\\ 0 & \text{if } t>t_2.\end{cases} \tag{1.4}$$

Definition 1.1. Equation (1.1) is said to be approximately controllable on $[0,t_1]$ in the class of controls vanishing after time moment $t_2, 0<t_2<t_1$, if for each $x_1\in X$ and $\varepsilon>0$ there exists a control $u(\cdot)\in L_2([0,t_2],U), u(t)=0$ a.e. on $[t_2,+\infty)$, such that
$$\|x_1-x(t_1,0,0,u(\cdot))\|<\varepsilon.$$

Definition 1.2. Equation (1.1) is said to be approximately controllable on $[0,t_1]$ by controller (1.2) if for each $x_1\in X$ and $\varepsilon>0$ there exists $z_0\in Z$, such that
$$\|x_1-x(t_1,0,0,u_{t_2}(\cdot,0,z_0))\|<\varepsilon.$$

2. Assumptions

(1) The operators A and C have purely point spectrum σ_A and σ_C with no finite limit points. Eigenvalues of both A and C have finite multiplicities.

(2) Let the spectrum σ_A of the operator A be infinite and consists of numbers λ_j, $j=1,2,\dots$, with multiplicities α_j, enumerated in such a way that their absolute values are non-decreasing with respect to j (i.e. $|\lambda_j|\ge|\lambda_{j+1}|$). The sequence
$$t^k\exp\lambda_j t,\quad j=1,2,\dots,\quad k=0,\dots,\alpha_j-1 \tag{2.1}$$

is minimal on $[0,\delta]$ for some $\delta > 0$, i. e., there exists a sequence biorthogonal to the above sequence with respect to the scalar product in $L_2[0,\delta]$.

(3) There exists $T \geq 0$ such that all mild solutions of the equation $\dot{x}(t) = Ax(t)$ are expanded in a series of generalized eigenvectors of the operator A converging (in the topology of X) for any $t > T$ uniformly in each segment $[T_1, T_2], T < T_1 < T_2$ ($\sum_{j=1}^{\infty}$ is considered with respect to the topology of V).

(4) The unbounded operator B is bounded as an operator from U to V.

(5) $\int_0^t \mathcal{S}_A(t-\tau)Bu(\tau)d\tau \in X$ for any $u(\cdot) \in L_2([0,t],U)$, and the operator $\Phi(t) : L_2([0,t],U) \to X$ defined by

$$\Phi(t)u(\cdot) = \int_0^t \mathcal{S}_A(t-\tau)Bu(\tau)d\tau \tag{2.2}$$

is bounded for each $t \geq 0$. The integral $\int_0^t \mathcal{S}_A(t-\tau)Bu(\tau)d\tau$ is considered in the topology of the space V.

(6) We consider the operator $K : Z \to U$ with domain $D(K)$ such that $z(t) \in D(K)$ for a.e. $t > 0$ and $(Kz)(\cdot) \in L_2([0,t_1],U), \forall t_1 > 0$. The operator

$$Q : Z \to L_2([0,t_1],U), \quad Qz = u(t), \quad t \in [0,t_1]$$

is bounded for all $t_1 > 0$.

3. Main results

Denote

$$\begin{aligned}&\mathrm{Range}\{\lambda I - A, \mathcal{R}_A(\mu)B\}\\ &= \{y \in X : \exists x \in X,\, \exists u \in U,\ y = (\lambda I - A)x + \mathcal{R}_A(\mu)Bu\}.\end{aligned}$$

Theorem 3.1. *For equation* (1.1) *to be approximately controllable on* $[0,t_1]$, $t_1 > T + \delta$, *in the class of controls vanishing after time moment* $t_1 - T$, *it is necessary and sufficient that*

(1) *The linear span of the generalized eigenvectors of the operator* A *is dense in* X.

(2) *The condition*

$$\overline{\mathrm{Range}\{\lambda I - A, \mathcal{R}_A(\mu)B\}} = X, \quad \forall\lambda \in \sigma_A,\ \forall\mu \notin \sigma_A, \tag{3.1}$$

holds.

Theorem 3.2. *For equation* (1.1) *to be approximately controllable on* $[0,t_1]$, $t_1 > T$ *by distributed controller* (1.2), *it is necessary that all the conditions of Theorem 3.1 hold.*

If these conditions hold and the subspace $KS_C(\cdot)Z$ *of* $L_2([0,t_2],U)$ *is dense in* $L_2([0,t_2],U)$ *for some* $t_2 > 0$, *then equation* (1.1) *is approximately controllable on* $[0,t_1]$, $t_1 > T + \delta$, *by controller* (1.2).

4. Approximate controllability of abstract boundary control problem by abstract boundary controller

Let X, U, Z, Y_1, Y_2 be Hilbert spaces. Consider the abstract boundary control problem

$$\begin{gathered}\dot{x}(t) = Lx(t),\\ \Gamma x(t) = Bu(t),\\ x(0) = x_0,\\ u(t) = Kz(t),\end{gathered} \tag{4.1}$$

where $z(t)$ is a mild solution of the boundary-value problem

$$\dot{z}(t) = Mz(t), \tag{4.2}$$
$$Hz(t) = 0,$$
$$z(0) = z_0. \tag{4.3}$$

Equation (4.2)-(4.3) is called boundary controller.

Here $L : X \to X$ and $M : Z \to Z$ are linear unbounded operators with dense domains $D(L)$ and $D(M)$; $B : U \to Y_1$ is a linear bounded one-to-one operator, $K : Z \to U$ is a linear (possibly unbounded) onto operator, $\Gamma : X \to Y_1$ and $H : Z \to Y_2$ are linear operators satisfying the following conditions:

(1) Γ and H are onto, $\ker \Gamma$ is dense in X, $\ker H$ is dense in Z.
(2) There exists a $\mu \in \mathbb{R}$ such that $\mu I - L$ is onto and $\ker(\mu I - L) \cap \ker \Gamma = \{0\}$.
(3) There exists a $\mu \in \mathbb{R}$ such that $\mu I - M$ is onto and $\ker(\mu I - M) \cap \ker H = \{0\}$.

Problems (4.1) and (4.2)-(4.3) are assumed to be well-posed. Problem (4.1) is an abstract model for classical control problems described by linear partial differential equations of both parabolic and hyperbolic type when a control acts through the boundary. The control process is released by initial condition (4.3) which is considered as a control.

Now consider the space $W_1 = \ker \Gamma$. We have $W_1 \subset D(L) \subset X$ with continuous dense injection. Define the operator $A : W_1 \to X$ by

$$Ax = Lx \quad \text{for } x \in W_1. \tag{4.4}$$

For $y \in Y_1$ define

$$\hat{B}y = Lx - Ax,\ x \in \Gamma^{-1}(y) = \{z \in D(L) : \Gamma x = y\}. \tag{4.5}$$

Given $u \in U$ denote $\tilde{B}u = \hat{B}Bu$. The operator $B : U \to V$ is bounded, but the operator $\hat{B} : Y_1 \to X$ defined by (4.5) is unbounded, so the operator $\tilde{B} : U \to X$ is unbounded. It follows from (4.5) that

$$Lx = Ax + \tilde{B}u, \tag{4.6}$$
$$\Gamma x = Bu. \tag{4.7}$$

The same way is applied to the space $W_2 = \ker H$. Again, we have $W_2 \subset D(M) \subset Z$ with continuous dense injection. Define the operator $C : W_2 \to Z$ by

$$Cz = Mz \quad \text{for } z \in W_2. \tag{4.8}$$

Hence

$$\begin{gathered}\dot{z}(t) = Cz(t),\\ z(0) = z_0,\end{gathered} \tag{4.9}$$

We assumed all the hypotheses in section 2 for the above operators A, C, K hold true. Together with equation (4.1) consider the abstract boundary-value problem

$$Lx = \mu x, \tag{4.10}$$

$$\Gamma x = y. \tag{4.11}$$

Since problem (4.1) is uniformly well-posed then for any $y \in Y_1$ there exists the solution $x_\mu = D_A(\mu)y$ of equation (4.10)-(4.11), where $D_A(\mu) : Y_1 \to X$ is a linear bounded operator (The operator D_μ is defined by well-known Green formula for given boundary problem).

The next theorems follow from Theorems 3.1-3.2.

Theorem 4.1. *For equation* (4.1) *to be approximately controllable on* $[0, t_1]$, $t_1 > T + \delta$, *in the class of controls vanishing after time moment* $t_1 - T$, *it is necessary and sufficient that*

(1) *The linear span of the generalized eigenvectors of the operator* A *(i.e. eigenfunctions of the boundary problem* $Lx = \lambda x$, $Gx = 0$*) is dense in* X

(2)
$$\overline{\operatorname{Range}\{\lambda I - A, R_A(\mu)\hat{B}B\}} = X, \quad \forall \mu \notin \sigma_A,\ \forall \lambda \in \sigma_A, \tag{4.12}$$

Theorem 4.2. *For equation* (4.1) *to be approximately controllable on* $[0, t_1]$ *by boundary controller* (4.2)-(4.3), *it is necessary that*

(1) *The linear span of the generalized eigenvectors of the operator* A *is dense in* X.

(2) *The condition (4.12) holds.*

If these conditions hold and the set of functions $u(\cdot), u(t) = Kz(t)$ *with* $z(t)$ *a solution of boundary-value problem* (4.2)-(4.3), *is dense in* $L_2([0, t_2], U)$ *for some* $t_2 > 0$, *then equation* (4.1) *is approximately controllable on* $[0, t_1], t_1 > T + \delta$, *by boundary controller* (4.2)-(4.3).

Theorem 4.3. *For equation* (4.1) *to be approximately controllable on* $[0, t_1]$ *by boundary controller* (4.2)-(4.3), *it is necessary that*

(1) *All generalized eigenvectors of the operator* A *defined by* (4.4) *are dense in* X.

(2)
$$\overline{\operatorname{Range}\{\lambda I - A, D_A(\mu)B\}} = X, \quad \forall \mu \notin \sigma_A,\ \forall \lambda \in \sigma_A. \tag{4.13}$$

If these conditions hold and the set of functions $u(\cdot)$, $u(t) = Kz(t)$ *with* $z(t)$ *a mild solution of boundary-value problem* (4.2)-(4.3), *is dense in* $L_2([0, t_2], U)$ *for some* $t_2 > 0$, *then equation* (4.1) *is approximately controllable on* $[0, t_1], t_1 > T + \delta$, *by boundary controller* (4.2)-(4.3).

5. Approximate controllability of partial differential equations by a hyperbolic controller

The results of the previous section can be applied to the investigation of approximate controllability of linear partial differential control equation with boundary control governed by distributed controller described by partial differential equations.

Consider the parabolic partial differential equation

$$\frac{\partial y}{\partial t}(t, x) = \frac{\partial}{\partial x}\Big(p_1(x)\frac{\partial y}{\partial x}(t, x)\Big) + p_2(x)y(t, x), \quad t \ge 0,\ 0 \le x \le l, \tag{5.1}$$

with non-homogeneous regular boundary conditions [7], [13]

$$a_0 y(t,0) + a_1 \frac{\partial y}{\partial x}(t,0) = a_2 u(t), t \geq 0, \tag{5.2}$$

$$b_0 y(t,l) + b_1 \frac{\partial y}{\partial x}(t,l) = b_2 u(t), t \geq 0, \tag{5.3}$$

subject to the initial conditions

$$y(0,x) = \varphi_0(x), \quad 0 \leq x \leq l, \tag{5.4}$$

where $p_1(x)$ and $p_2(x)$ are real functions, continuous in the segment $[0,l]$;

$$\begin{gathered} p_1(x) > 0, \quad p_2(x) \leq 0, \quad x \in [0,l]; \\ \varphi_0(\cdot), \quad \varphi_1(\cdot) \in L_2[0,l]; \\ a_j, b_j \in \mathbb{R}, j = 0,1; \\ |a_0| + |a_1| \neq 0, \\ |b_0| + |b_1| \neq 0, \\ a_0 a_1 \leq 0, \quad b_0 b_1 \geq 0. \end{gathered}$$

Here

$$u(t) = z(t,\alpha), \quad t \geq 0, \quad \alpha \in [0,m],$$

where $z(t,x), \quad t \geq 0, \quad 0 \leq x \leq m$, is a mild solution of the hyperbolic partial differential equation

$$\frac{\partial^2 z}{\partial t^2}(t,x) = \frac{\partial}{\partial x}\Big(q_1(x)\frac{\partial z}{\partial x}(t,x)\Big) + q_2(x) z(t,x), \quad t \geq 0,\ 0 \leq x \leq m, \tag{5.5}$$

with homogeneous regular boundary conditions

$$\alpha_0 z(t,0) + \alpha_1 \frac{\partial z}{\partial x}(t,0) = 0, \tag{5.6}$$

$$\beta_0 z(t,m) + \beta_1 \frac{\partial z}{\partial x}(t,m) = 0 \tag{5.7}$$

subject to the initial conditions

$$z(0,x) = \psi_0(x), \quad \frac{\partial y}{\partial x}(0,x) = \psi_1(x), \quad 0 \leq x \leq m, \tag{5.8}$$

where $q_1(x)$ and $q_2(x)$ are real functions, continuous in the segment $[0,m]$;

$$\begin{gathered} q_1(x) > 0, \quad q_2(x) \leq 0, \quad x \in [0,m]; \\ \psi_0(\cdot), \psi_1(\cdot) \in L_2[0,m]; \\ \alpha_j, \beta_j \in \mathbb{R}, \quad j = 0,1; \\ |\alpha_0| + |\alpha_1| \neq 0, \\ |\beta_0| + |\beta_1| \neq 0, \\ \alpha_0\alpha_1 \leq 0, \beta_0\beta_1 \geq 0. \end{gathered}$$

Partial differential equation (5.5) with boundary condition (5.6)-(5.7) will be called a hyperbolic controller. The pair

$$(\psi 0(x), \psi_1(x)), \quad 0 \leq x \leq m,$$

where $\psi_0(x)$ and $\psi_1(x)$ are defined by (5.8), is considered as a control of equation (5.1)-(5.3) governed by hyperbolic controller (5.5)-(5.7).

One can rewrite equation (5.1)-(5.3) in the form of (4.1) with the state space $X = L_2[0,l] \times L_2[0,l]$; the corresponding operator A generates a C_0-semigroup.

By the same way one can rewrite equation (5.5)-(5.7) in the form of (4.2)-(4.3) with the state space $Z = L_2[0,m] \times L_2[0,m]$; the corresponding operator C generates a C_0-semigroup.

Here, conditions 1-4 of section 2 are valid for A and C with $T = 0$. The linear span of the eigenvectors of the corresponding selfadjoint operator A is dense in $L_2[0,l]$. The eigenvalues of the operator A are negative and the corresponding functions (2.1) are minimal on $[0,\delta]$ for all $\delta > 0$ [3]. We have

$$D_\mu Bu = \int_0^l G(x,\xi,\mu)(\omega_0(\xi)a_2 + \omega_l(\xi)b_2)ud\theta, \tag{5.9}$$

where $G(x,\xi,\mu)$ is the Green function of the boundary value problem

$$\begin{gathered}(p_1(x)y'(x))' + p_2(x)y(x) = \mu y(x), 0 \le x \le l,\\ a_0y(0) + a_1y'(0) = a_2u,\\ b_0y(l) + b_1y'(l) = b_2u,\end{gathered} \tag{5.10}$$

and

$$\omega_0(\xi) = \begin{cases} -\frac{p_1(0)}{\alpha_1}\delta(\xi), & \text{if } a_1 \neq 0,\\ \frac{p_1(0)}{\alpha_0}\delta'(\xi), & \text{if } a_0 \neq 0,\end{cases} \tag{5.11}$$

$$\omega_l(\xi) = \begin{cases} -\frac{p_1(l)}{b_1}\delta(\xi - l), & \text{if } b_1 \neq 0,\\ \frac{p_1(l)}{b_0}\delta'(\xi - l), & \text{if } b_0 \neq 0.\end{cases} \tag{5.12}$$

We have here $U = \mathbb{R}^2$; the operator $K : Z \mapsto U$ is defined for given $\alpha \in [0,m]$ by

$$Kz(\cdot) = z_1(\alpha), \quad \forall z(\cdot) = \begin{pmatrix} z_1(\cdot)\\ z_2(\cdot)\end{pmatrix} \in L^2[0,m] \times L^2[0,m] \tag{5.13}$$

Theorem 5.1. *Condition* (4.12) *holds if and only if for each* λ *the boundary-value problem*

$$(p_1(x)\varphi')' + p_2(x)\varphi - \lambda\varphi = 0, \quad x \in [0,l], \lambda \in \sigma_A \tag{5.14}$$

subject to the boundary conditions

$$a_0\varphi(0) + a_1\varphi'(0) = 0, \tag{5.15}$$

$$b_0\varphi(l) + b_1\varphi'(l) = 0 \tag{5.16}$$

$$\int_0^l \varphi(\xi)(\omega_0(\xi)a_2 + \omega_l(\xi)b_2)d\xi = 0, \tag{5.17}$$

has only trivial solution.

Using Theorems 4.1-4.3 and 5.1 one can prove the following statement.

Theorem 5.2. *Let* $\frac{a}{l}$ *be an irrational number. For* (5.1) *to be approximately controllable on* $[0,t_1]$*, for all* $t_1 > l$ *by boundary controller* (5.2)-(5.3)*, it is necessary and sufficient that for each* $\lambda \in \sigma_A$ *the boundary-value problem* (5.14) *subject the* *boundary conditions* (5.15)-(5.16) *and the boundary conditions:*

$$
\begin{aligned}
p(0)\varphi(0)\frac{a_2}{a_1} + p(l)\varphi(l)\frac{b_2}{b_1} &= 0, \quad a_1 \neq 0 \& b_1 \neq 0, \\
p(0)\varphi'(0)\frac{a_2}{a_0} - p(l)\varphi(l)\frac{b_2}{b_1} &= 0, \quad a_0 \neq 0 \& b_1 \neq 0, \\
p(0)\varphi(0)\frac{a_2}{a_1} - p(l)\varphi'(l)\frac{b_2}{b_0} &= 0, \quad a_1 \neq 0 \ \& b_0 \neq 0, \\
p(0)\varphi'(0)\frac{a_2}{a_0} + p(l)\varphi'(l)\frac{b_2}{b_0} &= 0, \quad a_0 \neq 0 \& b_0 \neq 0,
\end{aligned}
$$

has only trivial solution.

Remarks. 1. The problem of approximate controllability of equation (4.1) by parabolic controller (4.2)-(4.3) is still open. It means that if there exists a possibility to choose a distributed controller for construction then it is worthwhile to construct a hyperbolic controller.

2. The results of this section can be extended to the case of partial differential hyperbolic equation

$$
\frac{\partial^2 y}{\partial t^2}(t,x) = \frac{\partial}{\partial x}\Big(p_1(x)\frac{\partial y}{\partial x}(t,x)\Big) + p_2(x)y(t,x), t \geq 0, 0 \leq x \leq l,
$$

subject to boundary conditions (5.2)-(5.3) governed by hyperbolic controller (5.5)-(5.8).

6. Approximate controllability of linear differential control systems with delays by hereditary controller

In this section we will investigate linear differential control systems with delays governed by hereditary controller. These objects can be considered as a particular case of equation (1.1) with a bounded input operator [4, 8, 10, 15, 16] subject to the distributed controller of the form (1.2), so the results of the previous section can be applied.

Consider a linear differential-difference system [2]

$$
\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - h_{1k}) + B_0 u(t), \tag{6.1}
$$

$$
0 = h_{10} < h_{11} < \dots < h_{1m},
$$

$$
x(0) = x^0, \, x(\tau) = \varphi(\tau) \quad \text{a. e. on } [-h_{1m}, 0] \tag{6.2}
$$

where

$$
\dot{u}(t) = \sum_{k=0}^{p} C_k u(t - h_{2k}), \tag{6.3}
$$

$$
0 = h_{20} < h_{21} < \dots < h_{2p},
$$

$$
u(0) = u^0, u(\tau) = \psi(\tau) \quad \text{a. e. on } [-h_{2m}, 0]. \tag{6.4}
$$

System (6.3) is said to be a hereditary controller. Here

$$
\begin{aligned}
x(t), x^0 \in \mathbb{R}^n, &\quad \varphi(\cdot) \in L_2([-h_{1m}, 0], \mathbb{R}^n), \\
u(t), u^0 \in \mathbb{R}^r, &\quad \psi(\cdot) \in L_2([-h_{2m}, 0], \mathbb{R}^r);
\end{aligned}
$$

$A_j, j = 0, 1, \dots, m_1$, are constant $n \times n$ matrices, B_0 is a constant $n \times r$ matrix, C_j, $j = 0, 1, \dots, m_2$ are constant $r \times r$ matrices. We consider the Hilbert spaces [10]

$$X = \mathbb{R}^n \times L_2([-h_{1m}, 0], \mathbb{R}^n),$$
$$Z = \mathbb{R}^r \times L_2([-h_{1m}, 0], \mathbb{R}^r)$$

as the state spaces of systems (6.1) and (6.3) respectively; $U = \mathbb{R}^r, K(u_0, \psi(\cdot)) = u_0, \forall (u_0, \psi(\cdot)) \in Z$. Denote the identity $n \times n$ matrix by I.

Definition 6.1. System (6.1) is said to be approximately controllable on $[0, t_1]$ by hereditary controller (6.3) if for any $\varepsilon > 0$ and for any final state $(x_1, \psi(\cdot)) \in X$ there exists $(u_0, \xi(\cdot)) \in Z$ such that the corresponding solution $x(t)$ of system (6.1) satisfies the inequality

$$\|(x_1, \psi(\cdot)) - (x(t_1), x(t_1 + \cdot))\| < \varepsilon, -h_{1m} \le \tau \le 0$$

(The norm is considered in the space X).

It is well-known [8, 4, 10, 15, 16] that systems (6.1) and (6.3) can be written in the form (1.1)-(1.2) with the state spaces X and U defined above, and the linear space of the eigenvectors of the corresponding operator A is dense in X if and only if $\operatorname{rank} A_p = r$.

Here Assumptions 1-4 of section 2 for the corresponding operators A, C and K are valid with $T = nh$ [2, 1, 17, 18]; the corresponding functions (2.1) are minimal on $[0, \delta]$, for all $\delta > 0$ [19]. It was proved [18] that condition (3.1) for equation (1.1) is equivalent to the condition

$$\operatorname{rank}\{\lambda I - \sum_{k=0}^{m} A_k e^{-h_{1k}}, B_0\} = n, \quad \forall \lambda \in \sigma_A.$$

and the density of the linear span of the generalized eigenvectors of operator C implies the density of the corresponding subspace $KS_C(\cdot)Z$ in $L_2([0, t_1 - T], U)$.

Theorem 6.2. *For equation* (6.1) *to be approximately controllable on* $[0, t_1]$ *by boundary controller* (6.3), *it is necessary that*

$$\operatorname{rank} A_m = n. \tag{6.5}$$

$$\operatorname{rank}\{\lambda I - \sum_{k=0}^{m} A_k e^{-h_{1k}}, B_0\} = n, \quad \forall \lambda \in \mathbb{C}. \tag{6.6}$$

When these conditions hold and $\operatorname{rank} C_p = r$, *system* (6.1) *is approximately controllable on* $[0, t_1]$, $t_1 > nh_m$, *by hereditary controller* (6.3).

Approximate controllability of linear differential control systems with delays by scalar hereditary controller. Consider system (6.1) with one delay and one input, subject to scalar hereditary regulator ($r = 1$) with one delay, namely

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + B_0 u(t), \tag{6.7}$$
$$h_1 > 0,$$
$$x(0) = x^0, x(\tau) = \varphi(\tau) \quad \text{a.e. on } [-h_1, 0], \tag{6.8}$$

where

$$\dot{u}(t) = C_0 u(t) + C_1 u(t-h_2), \tag{6.9}$$
$$h_2 > 0,$$
$$u(0) = u^0, u(\tau) = \psi(\tau) \quad \text{a.e. on } [-h_2, 0]. \tag{6.10}$$

Here

$$x(t), x^0 \in \mathbb{R}^n, \quad \varphi(\cdot) \in L_2([-h_1, 0], \mathbb{R}^n),$$
$$u(t), u^0 \in \mathbb{R}, \quad \psi(\cdot) \in L_2[-h_2, 0];$$

A_j, $j = 0, 1$, are constant $n \times n$ matrices, B_0 is a constant column-vector, $C_j, j = 0, 1$, are scalars.

We consider the Hilbert spaces

$$X = \mathbb{R}^n \times L_2([-h_1, 0], \mathbb{R}^n), \quad Z = \mathbb{R} \times L_2[-h_2, 0]$$

as the state spaces of systems (6.7) and (6.9) respectively; $U = \mathbb{R}, K(u_0, \psi(\cdot)) = u_0, \forall (u_0, \psi(\cdot)) \in Z$.

Corollary 6.3. *System* (6.7) *is approximately controllable on* $[0, t_1], t_1 > nh$, *by hereditary controller* (6.9) *if and only if*

1. $\operatorname{rank}\left\{\lambda I - A_0 - A_1 e^{-h_1}, B_0\right\} = n$, *for all* $\lambda \in \mathbb{C}$.
2. $\operatorname{rank} A_1 = n$ *and* $C_1 \neq 0$.

Remark. Many ideas of the proofs of the theorems presented above, are imported from [11, 20], where closed problems of approximate null-controllability for distributed equations governed by distributed controller were considered. Complete proofs of the theorems will be presented in the full version of the paper.

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