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POINTWISE REPRESENTATION METHOD

VLADIMIR MIHAJLOVICH OSIPOV, VLADIMIR VLADIMIROVICH OSIPOV

Abstract. This article suggests an approximate analytical apparatus for modeling linear dynamic system of various types. This apparatus uses spline step models and point depictions of functions and operators.

1. Introduction

The method of pointwise representations is rather efficient, analytically powerful, and constructive for mathematical modelling of dynamical systems. This is due to special algebraic properties of the analytic methods used for the description of pointwise representations as finite dimensional models of functions and operators.

The method is based on the following simple idea. To any function $f(\tau)$ in the space of continuous on $[0,1]$, which is an element of the Hilbert space $L^2(0,1)$, the following N-dimensional vector is assigned:

$$f_T = \mathrm{col}[f(\tau_1^{(N)}), \dots, f(\tau_\nu^{(N)}), \dots, f(\tau_N^{(N)})] \tag{1.1}$$

which consists of the samples of this function at the nodes of an orthogonal N-grid:

$$\left\{\tau_\nu^{(N)} : \cos\left(N\pi\tau_\nu^{(N)}\right) = 0\right\}. \tag{1.2}$$

Note that $\tau_\nu^{(N)} = \frac{2\nu-1}{2N}$ ($\nu = \overline{1,N}$). The vector f_T is called a pointwise representation vector of the function $f(\tau)$, associated with the N-grid (1.2) which is the Chebyshev grid.

It is known that such a grid is the best among all possible orthogonal grids from many points of view. This means that using various types of interpolation, the function $f(\tau)$ can be restored by its pointwise representation N-vector (1.1) with highest accuracy.

Let us consider now the space $M(0,1)$ of all piecewise continuous functions defined in $[0,1]$. We normalize it by introducing a sup-norm:

$$\|f\| = \sup_{\tau\in[0,1]} |f(\tau)|, \quad f(\tau) \in M(0,1) \tag{1.3}$$

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Then the space $C(0,1)$ of all continuous functions on [0,1] becomes a subspace in $M(0,1)$, and

$$\|\varphi\| = \sup_{\tau\in[0,1]} |\varphi(\tau)| = \max_{\tau\in[0,1]} |\varphi(\tau)|, \quad \varphi(\tau) \in C(0,1) \subset M(0,1) \tag{1.4}$$

With this norm $M(0,1)$ and $C(0,1)$ are complete, i.e., Banach spaces. We note that $M(0,1)$ is also a subset of the Hilbert space $L_2(0,1)$. Since the product of two piecewise continuous and bounded functions on [0,1] is again a piecewise continuous function bounded in the same segment, the set $M(0,1)$ is closed with respect to the operation of multiplication. Due to the properties of the norm (1.3),

$$\|f\varphi\| \le \|f\| \cdot \|\varphi\|, \quad f(\tau), \varphi(\tau) \in M(0,1); \tag{1.5}$$

$$\|1\| = 1, \tag{1.6}$$

it is not only a Banach space, but also a commutative Banach algebra with an identity. Let's denote it by $AM(0,1)$. Obviously the $AC(0,1)$, which is the Banach algebra of all continuous in [0,1] functions is a subalgebra of this algebra.

2. Results

Let us define the value at a discontinuity point as an average of left and right limits. Then any function in $M(0,1)$ is defined on any orthogonal N-grid and, in particular, on the Chebyshev grid:

$$\tau_\nu^{(N)} = \frac{2\nu - 1}{2N} \quad (\nu = \overline{1,N}). \tag{2.1}$$

Hence its pointwise representation N-vector will be also determined:

$$f_T = \mathrm{col}[f(\tau_1^{(N)}), \dots, f(\tau_\nu^{(N)}), \dots, f(\tau_N^{(N)})] \to f(\tau) \in M(0,1) \tag{2.2}$$

The set of all pointwise images defined on the N-grid (2.1), is the linear N-dimensional space R_T^N which is complete for any norm. Let's supply it with the norm

$$\|f_T\| = \sup_\nu |f(\tau_\nu^{(N)})| < \|f\| = \sup_{\tau\in[0,1]} |f(\tau)| \tag{2.3}$$

This N-vector can be presented in the form of the integral transformation

$$T_N f(\tau) = \int_0^1 f(\tau)\delta_T(\tau)d\tau = f_T, \quad f(\tau) \in M(0,1) \tag{2.4}$$

where the δ-kernel is defined by the N-grid (2.1)

$$\delta_T(\tau) = \mathrm{col}[\delta(\tau - \tau_1^{(N)}), \dots, \delta(\tau - \tau_\nu^{(N)}), \dots, \delta(\tau - \tau_N^{(N)})] \tag{2.5}$$

Homomorhism $T_N : M(0,1) \to R_T^N$ means that the pointwise representation N-vector f is an image of not only one function $f(\tau)$ in $M(0,1)$, but of the whole class of functions, and differences of any two functions of this class are functions of the type

$$r_T(\tau) = \alpha_N(\tau)\cos(N\pi\tau), \quad \alpha_N(\tau) \in M(0,1), \tag{2.6}$$

with zeroes in nodes of the N-grid (2.1); therefore, their pointwise transformations have the zero image in R_T^N. The set of functions (2.6) forms the kernel $\ker T_N$ of homomorphism T_N:

$$\ker T_N / T_N r_T(\tau) = 0 \tag{2.7}$$

Any function $f(\tau)$ in $M(0,1)$ extended to an even periodic function using some method, has an approximated model of $M_N(f;\tau)$ in the form of the discrete Fourier

N-sum constructed by the values $f(\tau_\nu^{(N)})$ in the nodes of the N-grid (2.1), therefore the pointwise representation N-vectors of the function $f(\tau) \in M(0,1)$ and its model are identical, i.e. the transformation T_N maps them into the same element $f_T \in R_T^N$. Their difference belongs to the kernel (2.7) of the homomorphism T_N:

$$f(\tau) - M_N(f;\tau) = r_N(\tau) \in \ker T_N,$$

and, as a result,

$$f(\tau) = M_N(f;\tau) + r_N(\tau), \tag{2.8}$$

so any function in $M(0,1)$ is represented as the sum of its interpolated model and an element from $\ker T_N$. The latter one plays the role of an error of approximation of the interpolating model.

As $N \to \infty$ the error tends to zero in the $L_2(0,1)$ norm, since

$$\lim_{N\to\infty} \int_0^1 [f(\tau) - M_N(f;\tau)]^2 d\tau = \lim_{N\to\infty} \int_0^1 [r_N(\tau)]^2 d\tau = 0, \tag{2.9}$$

and, hence, we have the convergence

$$\lim_{N\to\infty} M_N(f;\tau) = f(\tau) \quad \text{and} \quad \lim_{N\to\infty} r_N(\tau) = 0, \tag{2.10}$$

almost everywhere in [0,1] (Carleson theorem). The set $S_N(0,1)$ of interpolated models $M_N(0,1)$ is a space which is a N-dimensional subspace of $M(0,1)$. The mapping $P_N : M(0,1) \to S_N(0,1)$ is a homomorphism with the kernel (2.7).

Sets $S_N(0,1)$ and R_T^N are equivalent, since there is a one-to-one correspondence of their elements. Moreover, they are isometrically isomorphic. Thus, it is possible to illustrate the relation of these spaces by the diagram

$$\begin{tikzcd} M(0,1) \arrow[r, "P_N"] \arrow[d, "T_N"'] & {} \arrow[d, "P_N"'] \\ S_N(0,1) \arrow[r, equal] & R_T^N \end{tikzcd} \tag{2.11}$$

Arrows T_N and P_N show homomorphisms. The double line marks the isometric isomorphism of spaces $S_N(0,1)$ and R_T^N.

As N grows, due to the convergence (2.10), homomorphisms tend to isometric isomorphisms. This does not describe all algebraic properties of point vector images of functions of $M(0,1)$ and all relations of the appropriate spaces as sets. It was already specified, that the space $M(0,1)$ is also a commutative Banach algebra $AM(0,1)$ with the usual operation of multiplication.

The space R_T^N of pointwise vector images as a homomorphic image of $M(0,1)$ space has the following property. Let us define in R_T^N a commutative pointwise multiplication operation for vectors. Let $f(\tau)$ and $\varphi(\tau)$ be two functions in $M(0,1)$ and f_T and φ_T be their pointwise vector images in R_T^N, i.e.,

$$\begin{gathered} f(\tau) \overset{T_N}{\to} f_T = \operatorname{col}\left[f(\tau_1^{(N)}), \dots, f(\tau_\nu^{(N)}), \dots, f(\tau_N^{(N)})\right] \\ \varphi(\tau) \overset{T_N}{\to} \varphi_T = \operatorname{col}\left[\varphi(\tau_1^{(N)}), \dots, \varphi(\tau_\nu^{(N)}), \dots, \varphi(\tau_N^{(N)})\right] \end{gathered} \tag{2.12}$$

Then the N-vector

$$\Phi_T = \operatorname{col}\left[f(\tau_1^{(N)}) \cdot \varphi(\tau_1^{(N)}), \dots, f(\tau_\nu^{(N)}) \cdot \varphi(\tau_\nu^{(N)}), \dots, f(\tau_N^{(N)}) \cdot \varphi(\tau_N^{(N)})\right], \tag{2.13}$$

whose coordinates are products of the respective coordinates of vectors (2.12), which can be symbolically written as $f_T \otimes \varphi_T$, then an N-vector is a pointwise representation vector of the product $f_T \otimes \varphi(\tau)$ of functions in $M(0,1)$:

$$f(\tau)\varphi(\tau) \xrightarrow{T_N} f_T \otimes \varphi_T = \Phi_T \in R_T^N, \tag{2.14}$$

and, according to (2.3)

$$\|\Phi_T\| = \|f_T \otimes \varphi_T\| \le \|f_T\| \cdot \|\varphi_T\| \le \|f\| \cdot \|\varphi\|. \tag{2.15}$$

In R_T^N the identity element is defined as

$$1_T = \mathrm{col}[1, \dots, 1, \dots, 1] \to 1 \in M(0,1) \tag{2.16}$$

with the unit norm $\|1_T\| = 1$ which enjoys the property

$$f_T \otimes 1_T = 1_T \otimes f_T = f_T, \quad f_T \in M(0,1) \tag{2.17}$$

Thus, the set R_T^N of pointwise vector images with the introduced operation of multiplication and with the sup-norm (2.3) is a commutative Banach algebra with the identity for any N. Let us denote it by AR_T^N. Since for any $f(\tau)$ and $\varphi(\tau)$ in $M(0,1)$ the following equation is valid

$$T_N[f(\tau)\varphi(\tau)] = T_N[f(\tau)] \cdot T_N[\varphi(\tau)] = f_T \otimes \varphi_T, \tag{2.18}$$

then the pointwise transformation T_N for any N is a continuous homomorphism not only of $M(0,1)$ space to space R_T^N , but also of the Banach algebra AM to the algebra AR_T^N:

$$AM \xrightarrow{T_N} AR_T^N$$

However, the mapping $P_N : M(0,1) \to S_N(0,1)$ does not enjoy this property since $S_N(0,1)$ is a space of N-dimensional interpolated models of $M(0,1)$ functions which are quadrature cosine Fourier sums, which is not an algebra with a usual multiplication.

Besides,

$$P_N[f(\tau)\varphi(\tau)] \ne P_N[f(\tau)] \cdot P_N[\varphi(\tau)]$$

and apparently the relation of $M(0,1)$ and $S_N(0,1)$ can be comprehensively described by their homomorphism as a mapping of linear spaces. If we consider zero degree splines as approximation models for the functions in $M(0,1)$, the situation changes significantly.

The point is that the set of spline models

$$Sp_N^0(f_T;\tau) = \sum_{\nu=1}^{N} f(\tau_\nu^{(N)})\pi_N(\tau - \tau_\nu^{(N)}), \quad f(\tau) \in M(0,1) \tag{2.19}$$

with interpolation elements

$$\pi_N(\tau - \tau_\nu^{(N)}) = \begin{cases} 1 & \tau \in (\tau_\nu^{(N)} - \frac{1}{2N}, \tau_\nu^{(N)} + \frac{1}{2N}) \\ 0 & \tau \notin (\tau_\nu^{(N)} - \frac{1}{2N}, \tau_\nu^{(N)} + \frac{1}{2N}) \end{cases} \quad (\nu = \overline{1,N}) \tag{2.20}$$

which look like rectangular pulses of the unit height, is not only a sup-normalized N-dimensional subspace of step interpolation forms of $M(0,1)$, but also is a commutative Banach algebra ASp_N^0 with an identity, with the usual operation of multiplication. It is a subalgebra of the algebra AM. In fact, the following property of elements (2.20),

$$\pi_N(\tau-\tau_\nu^{(N)})\cdot\pi_N(\tau-\tau_m^{(N)})=\begin{cases}\pi_N(\tau-\tau_\nu^{(N)}) & \nu=m\\ 0 & \nu\neq m\end{cases}\quad(\nu,m=\overline{1,N})\tag{2.21}$$

implies

$$Sp_N^0(f_T;\tau)\cdot Sp_N^0(\varphi_T;\tau)=\sum_{\nu=1}^{N}f(\tau_\nu^{(N)})\varphi(\tau_\nu^{(N)})\pi_N(\tau-\tau_\nu^{(N)})=Sp_N^0(f_T\otimes\varphi_T;\tau)$$

i.e., the product of two step spline models $f(\tau)$ and of dimension N of functions $f(\tau)$ and $\varphi(\tau)$ from $M(0,1)$ is a spline model of the same dimension of the product of these functions. In other words, the space $Sp_N^0(0,1)$ as the set of step interpolation forms, is closed with respect to the operation of multiplication. Furthermore,

$$Sp_N^0(1_T;\tau)=1\in M(0,1)\tag{2.22}$$

Thus, the homomorphic mapping π_N of $M(0,1)$ space to its subspace $Sp_N^0(0,1)$ of spline models can be considered as a homomorphism of algebra AM algebra to algebra ASp_N^0 , and the latter one is isometrically isomorphic to algebra AR_T^N for any N:

$$\begin{tikzcd} AM \arrow[r, "T_N"] \arrow[d, "\pi_N"'] & {} \arrow[d, "T_N"] \\ ASp_N^0 \arrow[r, equal] & AR_T^N \end{tikzcd}\tag{2.23}$$

As $N\to\infty$ the sequence $Sp_N^0(f_T,\tau)$ of step interpolation forms converges almost everywhere to any function $f(\tau)\in M(0,1)$ and if the latter one is continuous, then the convergence is uniform.

At the same time functions in $M(0,1)$, which form the kernel $\ker T_N$ of T_N, converge almost everywhere to zero, and homomorphisms π_N and T_N become isometric isomorphisms of algebras.

Now let us consider the linear bounded operator A_τ acting from $M(0,1)$ to $M(0,1)$ or to some subspace $M_y(0,1)\overline{\subset}M(0,1)$, in particular, in $C(0,1)$ to the space of continuous in [0,1] functions. It is possible that the range is finite dimensional. The operator A_τ

$$A_\tau x(\tau)=y(\tau);\quad x(\tau)\in M(0,1);\quad y(\tau)\in M_y(0,1)\overline{\subset}M(0,1)\tag{2.24}$$

is linear

$$A_\tau[\alpha x_1(\tau)+\beta x_2(\tau)]=\alpha\cdot A_\tau x_1(\tau)+\beta\cdot A_\tau x_2(\tau);\quad x_1(\tau),x_2(\tau)\in M(0,1);\ \alpha,\beta\in\mathbb{R}$$

and its boundedness means the following inequality for sup-norms holds:

$$\|y\|=\|A_\tau x\|\leq C\cdot\|x\|,\tag{2.25}$$

where the least possible value of a positive constant C is the norm $\|A_\tau\|$ of the operator A_τ:

$$\|A_\tau\|=\sup_{x\neq 0}\frac{\|A_\tau x\|}{\|x\|}.\tag{2.26}$$

The boundedness of the linear operator is equivalent to its continuity is in the following sense: images $A_\tau x_1(\tau)$ and $A_\tau x_2(\tau)$ of two close elements $x_1(\tau)$ and $x_2(\tau)$ of $M(0,1)$ are also close, i.e. for every $\varepsilon>0$ there is a $\delta>0$, such that $\|\tau x_1-\tau x_2\|<\delta$ implies $\|A_\tau x_1-A_\tau x_2\|<\varepsilon$.

Homomorphic T_N and π_N which were introduced above

$$\begin{gathered} T_N x(\tau) = X_T; \quad x(\tau) \in M(0,1); \quad X_T \in R_T^N \\ \pi_N x(\tau) = Sp_N^0(X_T;\tau); \quad Sp_N^0(X_T;\tau) \in Sp_N^0(0,1) \end{gathered} \tag{2.27}$$

are linear bounded operators. Their domain is space $M(0,1)$.

Operator T_N maps functions of $M(0,1)$ into their pointwise representation vectors of R_T^N, associated with the Chebyshev N-grid (2.1), and the operator π_N maps them into interpolated spline models constructed on the same N-grid which form an N-dimensional subspace $Sp_N^0(0,1)$ of step forms of the space $M(0,1)$.

Let us note that because of the obvious equation

$$\pi_N[\pi_N x(\tau)] = \pi_N^2 x(\tau) = \pi_N Sp_N^0(X_T;\tau) = Sp_N^0(X_T;\tau) \Rightarrow \pi_N^2 = \pi_N \tag{2.28}$$

the operator π_N is a projecting operator. Inequalities for sup-norms which were mentioned above

$$\|\pi_N x(\tau)\| = \|Sp_N^0(X_T;\tau)\| = \|X_T\| = \|T_N x(\tau)\| \le \|x(\tau)\|; \quad x(\tau) \in M(0,1)$$

imply the boundedness of operators T_N and π_N; their norms are equal to one: $\|T_N\| = \|\pi_N\| = 1$ at any N.

Let us apply the operator of pointwise transformation T_N to the operational equation (2.24):

$$T_N[A_\tau x(\tau)] = T_N y(\tau) = Y_T^N \tag{2.29}$$

As a result we obtain the vector-matrix equation

$$A_T^N \cdot X_T^{(N)} = Y_T^{(N)}, \tag{2.30}$$

generally speaking, approximate, which is a homomorphic image of equation (2.24) in R_T^N (the N-dimensional space of point images). There may be more than one pointwise matrix representation $A_T^{(N)}$ $(N \times N)$ which is assigned to the linear operator A_τ of $M(0,1)$.

The problem is to find the general method of an explicit definition (choice) of a matrix pointwise representations of the set of all possible representations for any linear bounded operator A_τ mapping $M(0,1)$ into some its subspace $M_y(0,1)$.

In this connection, let us note the following important property. N-dimensional space $Sp_N^0(0,1)$ of approximating spline models of functions in $M(0,1)$ has a basis of N rectangular impulse functions

$$\pi_N(\tau - \tau_\nu^{(N)}) = \begin{cases} 1 & \tau \in (\tau_\nu^{(N)} - \frac{1}{2N}, \tau_\nu^{(N)} + \frac{1}{2N}) \\ 0 & \tau \notin (\tau_\nu^{(N)} - \frac{1}{2N}, \tau_\nu^{(N)} + \frac{1}{2N}) \end{cases} \quad (\nu = \overline{1,N}) \tag{2.31}$$

of the unit height, the support of $\frac{1}{N}$ and axes of symmetry in the nodes of the Chebyshev N-grid (2.1).

Any element of $Sp_N^0(0,1)$ can be represented as a linear combination of basis elements, with components of a pointwise representation N-vector of the modeled function of $M(0,1)$ as coefficients:

$$Sp_N^0(X_T;\tau) = \sum_{\nu=1}^{N} x(\tau_\nu^{(N)})\pi_N(\tau - \tau_\nu^{(N)}) \quad x(\tau) \in M(0,1). \tag{2.32}$$

We form a basis N-vector using basis elements (2.31) as components:

$$\Pi_N(\tau) = \operatorname{col}[\pi_N(\tau - \tau_1^{(N)}), \dots, \pi_N(\tau - \tau_\nu^{(N)}), \dots, \pi_N(\tau - \tau_N^{(N)})],$$

Then spline model (2.32) of any function (i.e., the element of the space $Sp_N^0(0,1)$) can be written as an inner product of the pointwise representation vector X_T of function $x(\tau)$ by the basis vector $\Pi_N(\tau)$:

$$x(\tau) \overset{\pi_N}{\to} Sp_N^0(X_T;\tau) = \sum_{\nu=1}^{N} x(\tau_\nu^{(N)})\pi_N(\tau - \tau_\nu^{(N)}) = (X_T, \Pi_N(\tau)) = X_T^+\Pi_N(\tau) \tag{2.33}$$

In particular, let us find spline models of functions $A_\tau\pi_N(\tau - \tau_\nu^{(N)})$ $(\nu = \overline{1,N})$ which for any linear bounded operator A_τ acting in $M(0,1)$, are also elements of this space. Using operator π_N we project them onto the N-dimensional space of spline models $Sp_N^0(0,1)$, i.e., we represent them as a combination of basis elements (2.31). Thus we obtain

$$\pi_N A_\tau \pi_N(\tau - \tau_\nu^{(N)}) = \sum_{k=1}^{N} \alpha_{k\nu}\pi_N(\tau - \tau_k^{(N)}) \quad (\nu = \overline{1,N}) \tag{2.34}$$

The coefficients of these decomposition are components of pointwise vector images of functions $A_\tau\pi_N(\tau - \tau_\nu^{(N)})$ $(\nu = \overline{1,N})$.

It should be noted that the $(\nu - 1)$ first coefficients are equal to zero and the stepwise representation (2.34) begins with the ν-th step since the original function $A_\tau\pi_N(\tau - \tau_\nu^{(N)})$ is equal to zero up to the moment when the ν-th rectangular finite impulse $\pi_N(\tau - \tau_\nu^{(N)})$ occurs. Thus, the decomposition has the form

$$\begin{aligned} \pi_N A_\tau \pi_N(\tau - \tau_\nu^{(N)}) &= \sum_{k=1}^{N} \alpha_{k\nu}\pi_N(\tau - \tau_k^{(N)}) \\ &= [0,\dots,0,\dots\alpha_{\nu\nu},\dots,\alpha_{k\nu},\dots,\alpha_{N\nu}]\cdot\Pi_N(\tau) \end{aligned} \tag{2.35}$$

$(\nu = \overline{1,N})$ which is an inner product of a N-vector of coefficients with $(\nu - 1)$ zero first components by the basis vector $\Pi_N(\tau)$.

This implies the vector-matrix representation for a vector function,

$$\begin{aligned} &\pi_N A_\tau \Pi_N(\tau) \\ &= \pi_N \begin{bmatrix} A_\tau\Pi_N(\tau - \tau_1^{(N)}) \\ \vdots \\ A_\tau\Pi_N(\tau - \tau_\nu^{(N)}) \\ \vdots \\ A_\tau\Pi_N(\tau - \tau_N^{(N)}) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{\nu 1} & \cdots & \alpha_{N1} \\ & \ddots & \vdots & & \vdots \\ & & \alpha_{\nu\nu} & \cdots & \alpha_{N\nu} \\ & 0 & & \ddots & \vdots \\ & & & & \alpha_{NN} \end{bmatrix} \Pi_N(\tau) \\ &= (A_T^{(N)})^+\Pi_N \end{aligned} \tag{2.36}$$

The symbol $(A_T^{(N)})^+$ denotes an upper triangular matrix with coefficients of decomposition (2.35) as components:

$$(A_T^{(N)})^+ = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{\nu 1} & \cdots & \alpha_{N1} \\ & \ddots & \vdots & & \vdots \\ & & \alpha_{\nu\nu} & \cdots & \alpha_{N\nu} \\ & 0 & & \ddots & \vdots \\ & & & & \alpha_{NN} \end{bmatrix} \tag{2.37}$$

This $N \times N$ matrix is the result of transposing the lower triangular matrix

$$A_T^{(N)} = \begin{bmatrix} \alpha_{11} & & & & \\ \vdots & \ddots & & & \\ \alpha_{\nu 1} & \cdots & \alpha_{\nu\nu} & & \\ \vdots & & \vdots & \ddots & \\ \alpha_{N1} & \cdots & \alpha_{N\nu} & \cdots & \alpha_{NN} \end{bmatrix} \tag{2.38}$$

Now let us find the spline representation $\pi_N A_\tau Sp_N^0(X_T,\tau)$ as an approximate model of function $A_\tau Sp_N^0(X_T,\tau) \in M(0,1)$.

As $\|A_\tau Sp_N^0(X_T,\tau) - \pi_N A_\tau Sp_N^0(X_T,\tau)\| \to 0$ when $N \to \infty$, then for N large enough values of these functions will differ by less than any prescribed positive value for any linear positive operator A_τ. Taking into account (2.33) and (2.36) and also the property of the inner product we obtain:

$$\begin{aligned} A_\tau Sp_N^0(X_T,\tau) &\approx \pi_N A_\tau Sp_N^0(X_T,\tau) = (X_T, \pi_N A_\tau \Pi_N(\tau)) \\ &= (X_T, (A_T^{(N)})^+ \Pi_N(\tau)) = (A_T^{(N)} X_T, \Pi_N(\tau)) \\ &= Sp_N^0(A_T^{(N)} X_T, \tau) \end{aligned} \tag{2.39}$$

where $A_T^{(N)}$ is the matrix in (2.38). It is necessary to make the final step. By the property of a norm the following inequality

$$\|A_\tau x(\tau) - A_\tau Sp_N^0(X_T,\tau)\| \le \|A_\tau\| \cdot \|x(\tau) - Sp_N^0(X_T,\tau)\| \tag{2.40}$$

holds for any bounded (continuous) linear operator A_τ, any $x(\tau)$ of $M(0,1)$ and any N. Since

$$\|x(\tau) - Sp_N^0(X_T,\tau)\| \to \quad \text{as } N \to \infty \tag{2.41}$$

i.e., the sequence $Sp_N^0(X_T,\tau)$ of spline approximation models converges by norm to any modeled function $x(\tau)$ in $M(0,1)$ (and there is even a uniform convergence, if $x(\tau)$ is a continuous function), then the inequality (2.40) implies

$$\|A_\tau x(\tau) - A_\tau Sp_N^0(X_T,\tau)\| \to 0 \quad \text{as } N \to \infty\,. \tag{2.42}$$

The latter convergence implies that for N large enough, taking into account (2.39), we have any prescribed accuracy for approximating spline models

$$\begin{aligned} y(\tau) = A_\tau x(\tau) &\approx A_\tau Sp_N^0(X_T,\tau) \approx \pi_N A_\tau Sp_N^0(X_T,\tau) \\ &= (A_T^{(N)} X_T, \Pi_N(\tau)) = Sp_N^0(A_T^{(N)} X_T, \tau) \end{aligned} \tag{2.43}$$

Hence for pointwise images we obtain

$$T_y y(\tau) = Y_T^{(N)} = [A_\tau x(\tau)]_T = A_T^{(N)} X_T^{(N)}; \quad X_T^{(N)} = T_N x(\tau). \tag{2.44}$$

Thus, any linear bounded operator acting from $M(0,1)$ to any of its subspaces $M_y(0,1)\overline{\subset}M(0,1)$ under the homomorphic mapping into the N-dimensional subspace $Sp_N^0(0,1) \subset M(0,1)$ of spline models (with basis (2.31)) has a pointwise representation by a lower triangular matrix (2.38). The equality (2.44) for pointwise images, generally speaking, approximate, corresponds to the operator equation (2.24). In practice components of pointwise matrix representation of a linear operator operator A_τ can be found using the projection of function $A_\tau \pi_N(\tau - \tau_\nu^{(N)})$ ($\nu = \overline{1,N}$)) onto the subspace of spline models and their decompositions as linear combinations of basis elements (2.31) (decomposition of the type (2.35)). Coefficients of these decompositions form rows of the matrix which, after transposition, is the matrix of pointwise representation $A_T^{(N)}$ of the operator A_τ.

Further this method is used to find pointwise matrix representations of various linear operators which are necessary, in particular, for the solution of linear differential equations of various types, which are transformed into algebraic (vector-matrix) equations by pointwise representations. This can be treated as a special operator calculus.

In particular, the pointwise matrix representation is found for the operator which shifts the function $x(\tau) \in M(0,1)$ along axis "τ" by a fixed step which is equal to the distance between Chebyshev nodes of the grid (1.2), i.e., by the value of $\frac{1}{N}$.

Let us call this operator the pointwise shift operator and denote it by Z_τ. The image of this operator for any bounded function $x(\tau)$ with a support in [0,1] (obviously $x(\tau) \in M(0,1)$), will mean a shift of the function variable by $\frac{1}{N}$:

$$Z_\tau x(\tau) = x(\tau - \frac{1}{N}) \quad \tau \in (\frac{1}{N}, 1 + \frac{1}{N}) \tag{2.45}$$

Using the general method, described above, to obtain pointwise matrix representations for linear bounded operators, the linear operator Z_τ of pointwise shift in the space R_T^N of pointwise images has the matrix representation

$$Z = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & 0 & \\ & 1 & 0 & & \\ 0 & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \tag{2.46}$$

This $N \times N$ matrix is called the canonical right shift matrix.

The degrees of initial matrix of shift

$$E = Z^0, Z^1, Z^2, \dots Z^k, \dots Z^{N-1} \tag{2.47}$$

form a linearly independent system of matrices, since their linear combinations, i.e. matrix polynomials of degree $N-1$ with real coefficients

$$P_{N-1}(Z) = \sum_{k=0}^{N-1} A_k Z^k = \begin{bmatrix} A_0 & & & & \\ A_1 & A_0 & & 0 & \\ A_2 & A_1 & A_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ A_{N-1} & \cdots & \cdots & A_1 & A_0 \end{bmatrix} \tag{2.48}$$

are triangular matrices $(N \times N)$ can be identically equal to zero only if all coefficients are equal to zero.

A Toeplitz type matrix (2.48) is called a polynomial shift matrix.

First of all, let us note that the matrix polynomial (2.48) is a homomorphic image in R_T^N of the polynomial operator

$$P_{N-1}(z_\tau) = \sum_{k=0}^{N-1} A_k z_\tau^k \tag{2.49}$$

with the shift operator Z_τ as a variable, which is described as

$$P_{N-1}(z_\tau)x(\tau) = \sum_{k=0}^{N-1} A_k x(\tau - \frac{k}{N}); \quad x(\tau) \in M(0,1) \tag{2.50}$$

and thus sums up all successive shifts of a finite function $x(\tau)$ in $M(0,1)$ with the appropriate weight coefficients $\{A_k\}$. Obviously, we have

$$P_{N-1}(z_\tau)x(\tau) \overset{T_N}{\to} P_{N-1}(Z)X_T \tag{2.51}$$

The set $P_{N-1}(Z)$ of all possible polynomial shift matrices is a subspace of the linear space of lower tridiagonal matrices $(N \times N)$, which is an N-dimensional linear space with basis (2.47) of the first N degrees of $N \times N$ matrix Z. A wide class of such matrixes arises as a set of functions of matrix Z, which is a canonical shift matrix.

Formally polynomial shift matrices appear after the change of complex variable z of the whole rational function (polynomial) $P_{N-1}(z)$ of degree $N-1$ by the matrix argument Z $(N \times N)$:

$$P_{N-1}(z) = \sum_{k=0}^{N-1} A_k z^k \overset{z \to Z}{\to} \sum_{k=0}^{N-1} A_k Z^k = P_{N-1}(Z) \quad (N \times N) \tag{2.52}$$

Obviously, we have a one-to-one correspondence between the above polynomials and matrices (2.52). Polynomial $P_{N-1}(z)$ of a complex variable z will be called a generating polynomial of matrix $P_{N-1}(Z)$ $(N \times N)$.

Let us further call polynomial shift matrices P-matrixes $(N \times N)$.

Every P-matrix is completely defined by an ordered set of N real numbers which are coefficients of a generating polynomial. Evidently properties of these sets of numbers define both properties of generating polynomials and properties of appropriate P-matrices.

Besides, if we assume that all degrees of a variable "z", exceeding $N-1$, vanish (these are degrees $z^N, z^{N+1} \dots$) i.e. impose the condition "z" is nilpotent with index N, as it is valid for matrix argument Z, then we have described one more binary operation (in addition to summation) in the space of polynomials of degree less than or equal to $N-1$, such that the space is closed with respect to this operation. This is an operation of polynomial multiplication which satisfies all usual properties (usual axioms of multiplication) and, in particular, is commutative.

Thus the set of generating polynomials is a more complicated algebraic structure than a linear space. It is a commutative algebra with an identity. The set of appropriate P-matrixes $(N \times N)$ is the same algebra, since for any pair of these matrixes a commutative operation of multiplication with P-matrix $(N \times N)$ as a result is described. Besides, the set of P-matrices is a linear space. The identity matrix E $(N \times N)$ is an identity in the matrix algebra.

Obviously these algebras are isomorphic (as sets they are simply equivalent) and all operations follow the same rules and can be reduced to operations over coefficients of generating polynomials (elements of one algebra), which are at the same time entrees of appropriate P-matrices (which are elements of the other algebra).

It is also possible to prove the following three statements.

Statement 2.1. *The set of functions of a complex variable z, which are defined and continuous in the unit circle $|z| \leq 1$ and analytic inside this circle, form a*

Banach algebra with the identity AF, with the norm

$$\|\varphi(z)\| = \max_{|z|\le 1} |\varphi(z)|, \quad \varphi(z) \in AF, \tag{2.53}$$

which coincides with the l_1-norm of the appropriate power series of functions in AF:

$$\|\varphi(z)\| = \|\sum_{k=0}^{\infty} \varphi_k z^k\| = \max_{|z|\le 1} |\sum_{k=0}^{\infty} \varphi_k z^k| = \sum_{k=0}^{\infty} |\varphi_k| \tag{2.54}$$

The set of such power series is also a Banach algebra with the identity AGF, which is isometrically isomorphic to algebra AF. This statement is illustrated by the diagram

$$AF \;=\!=\!=\; AGF \tag{2.55}$$

Statement 2.2. *There exists projector $\Pi^{(N)}$, which is a homomorphism of normed algebras AF, AGF and l_1-normed N-algebra $AGF^{(N)}$ of partial sums of power series of degree N, which are treated as elements of algebra AGF.*

This statement is illustrated by the diagram

$$\begin{tikzcd} AGF^{(N)} & \arrow[l, "\Pi^{(N)}"'] \\ AGF \arrow[u, "\Pi^{(N)}"] \arrow[r, equal] & AF \arrow[u, "\Pi^{(N)}"'] \end{tikzcd} \tag{2.56}$$

Statement 2.3. *The change of variable z by the canonical shift matrix Z $(N \times N)$ leads to the homomorphism of algebra AF of analytical in the circle $|z| \le 1$ functions to the algebra $AGF^{(N)}(Z)$ of polynomial shift $N \times N$ matrices (P-matrices) and also to the isometric isomorphism of an N-algebra $AGF^{(N)}$ of generating polynomials to the matrix algebra $AGF^{(N)}(Z)$.*

The diagram has the final form

$$\begin{tikzcd} AGF^{(N)} \arrow[r, equal] & AGF^{(N)}(Z) \\ AGF \arrow[u, "\Pi^{(N)}"] \arrow[r, equal] & AF \arrow[u, "\Pi^{(N)}"'] \end{tikzcd} \tag{2.57}$$

Besides, using the above general method to obtain pointwise matrix representations of linear operators, it is possible to deduce P-matrix representation of the integral operator J_τ defined as

$$y(\tau) = J_\tau x(\tau) = \int_0^\tau x(\tau) d\tau \quad \tau \in [0,1]. \tag{2.58}$$

It is a linear bounded operator. Its domain is the space $M(0,1)$; its range is a subset of the space of continuous functions from $C(0,1)$ vanishing at $t = 0$. Operator J_τ in the space R_T^N of pointwise vector images has the matrix representation J_T $(N \times N)$:

$$y(\tau) = J_\tau x(\tau) = \int_0^\tau x(\tau) d\tau \overset{T_N}{\to} Y_T = J_T X_T. \tag{2.59}$$

The representation J_T corresponds to two-step mapping

$$y(\tau) = J_\tau x(\tau) \overset{\pi_N}{\to} Sp_N(J_T X_T; \tau) \overset{T_N}{\to} Y_T = J_T X_T \tag{2.60}$$

which is the first step approximation.

The matrix representation J_T of the integration operator has the form:

$$J_\tau \xrightarrow{T_N} J_T = \frac{1}{N}\begin{bmatrix} 1/2 & & & & \\ 1 & 1/2 & & 0 & \\ \vdots & 1 & 1/2 & & \\ 1 & & \ddots & \ddots & \\ 1 & 1 & \cdots & 1 & 1/2 \end{bmatrix} \quad (N \times N) \tag{2.61}$$

It is necessary to note, that any method, which improves the accuracy of approximate equations, changes significantly the structure of the representing matrix of the integration operator. Compared to (2.61), this not only makes the structure more complicated, but also essentially influences analytical structure and efficiency of the developed applied theory based on pointwise representations.

Mainly this is due to the fact that the matrix of integration (2.61) is a polynomial shift matrix (P-matrix) and consequently for any N it can be presented as a linear combination of the first N degrees of the canonical shift matrix Z $(N \times N)$ which is reduced to a rational function of the matrix variable Z:

$$\begin{aligned} J_T &= \frac{1}{N}\big[\frac{1}{2}E + \sum_{k=1}^{N-1} Z^k\big] = \frac{1}{2N}\big[E + 2\sum_{k=1}^{N-1} Z^k\big] \\ &= \frac{1}{2N}\big[E + 2Z\sum_{k=1}^{N-1} Z^k\big] = \frac{1}{2N}\big[E + 2Z(E-Z)^{-1}\big] \\ &= \frac{1}{2N}(E-Z)^{-1}(E+Z) \end{aligned} \tag{2.62}$$

Considered functions of a dimensionless variable τ stand for functions of time variable "t", defined in a finite interval $[0,T]$. After the substitution $t = T\tau$ the equation is transformed, [0,1] is the domain for variable "τ", while T is a parameter. In the notation of function $x(\tau) \in M(0,1)$ this parameter is not included explicitly. However we assume

$$x(\tau) = x(T\tau) = x(t) \quad t \in [0,T], \tag{2.63}$$

and components $x(\tau_\nu^{(N)})$ $(\nu = \overline{1,N})$ of the pointwise representation vector X_T of the function $x(\tau)$ are function values $x(T\tau_\nu^{(N)}) = x(t_\nu^{(N)})$ $(\nu = \overline{1,N})$ in the nodes of the Chebyshev time N-grid

$$t_\nu^{(N)} = T\tau_\nu^{(N)} = \frac{T(2\nu-1)}{2N} \quad (\nu = \overline{1,N}). \tag{2.64}$$

The operator of integration over time variable "t" also involves the factor T, since

$$J_t x(t) = \int_0^t x(t)dt = T\int_0^{t/T} x(T\tau)d\tau = T\int_0^\tau x(\tau)d\tau = TJ_\tau x(\tau). \tag{2.65}$$

For the pointwise matrix of integration we have

$$\begin{aligned} TJ_T &= \frac{T}{N}\begin{bmatrix} 1/2 & & & & \\ 1 & 1/2 & & 0 & \\ \vdots & 1 & 1/2 & & \\ 1 & & \ddots & \ddots & \\ 1 & 1 & \cdots & 1 & 1/2 \end{bmatrix} \\ &= \frac{T}{2N}(E-Z)^{-1}(E+Z) = \lambda_0(E-Z)^{-1}(E+Z) \quad (N\times N) \end{aligned} \tag{2.66}$$

The scalar factor

$$\lambda_0 = \frac{T}{2N} \tag{2.67}$$

is a half of the time distance between two adjacent nodes of the N-grid (2.64) and at the same time is an N-multiple eigenvalue of the matrix (2.66) with the determinant which equals λ_0^N.

The parameter (2.67) plays an important role in the research of time processes by the method of pointwise representations: it connects the width of the spectral characteristics (frequency ω_{cp}.) and the characteristic time "T" in the time process with the dimension of these representations. Really, by Kotelnikov theorem we have

$$\lambda_0 = \frac{T}{2N} = \frac{1}{2}\frac{\pi}{\omega_{cp}} \Rightarrow \lambda_0\omega_{cp} = \frac{\pi}{2} \tag{2.68}$$

For the fixed frequency ω_{cp} the parameter λ_0 also should be fixed by the relation (2.68). Thus any change of T should lead to the change of the dimension N, such that the ratio $\frac{T}{N}$ is constant. Thus Chebyshev nodes of the time N-grids are also fixed

$$t_\nu^{(N)} = \frac{T(2\nu-1)}{2N} = \lambda_0(2\nu-1) \quad (\nu=\overline{1,N}). \tag{2.69}$$

as well as the values of the function $x(t)t\in[0,T]$, as components of its pointwise representation vector X_T.

Consequently, the increase of the dimension N (with the increase of T) will mean the addition of new components of the pointwise representation vector without any change of all previous components, i.e. this leads to the property well known for the Fourier coefficients. Let us introduce a polynomial shift matrix, as a function of matrix variable Z:

$$J(Z) = (E-Z)^{-1}(E+Z) = E + 2\sum_{k=1}^{N-1} Z^k \tag{2.70}$$

Then the matrix of integration (2.66) can be rewritten as

$$\begin{aligned} TJ_T &= \lambda_0 J(Z) = \lambda_0(E-Z)^{-1}(E+Z) \\ &= \lambda_0\big[E + 2\sum_{k=1}^{N-1} Z^k\big] = \lambda_0\begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ \vdots & 2 & 1 & & \\ 2 & & \ddots & \ddots & \\ 2 & 2 & \cdots & 2 & 1 \end{bmatrix} \end{aligned} \tag{2.71}$$

In the framework of the general approach, the problem of the pointwise image of the convolution operator is investigated. The latter is treated as a compact integration operator with the difference kernel and as a commutative binary operation which is closed in L_1 - norm, which transforms $M(0,1)$ into a convolution normed algebra with an identity (with the δ - function as an identity).

In the space of pointwise images R_T^N a convolution of functions is mapped into a convolution of vector images of these functions. The latter convolution is closed in l_1 -norm, which makes R_T^N a convoluted algebra and ASR_T^N becomes a homomorphic image of a functional convoluted algebra ASM. With the growth of N the homomorphism tends to the isomorphism.

Convolution operators are very important for the theory of linear dynamical systems, they connect an input and an output. Therefore the following fact is significant. Pointwise modelling of convolution operators leads to the application of usual functions of dynamic systems: transfer functions as Laplace transforms of the kernels of convolution operators, with the role of impulse transfer characteristics of appropriate dynamical systems.

Connections of convolution algebras with algebraic structures in the sets of functions of a complex variable play an essential role when studying properties of linear dynamical systems by their pointwise models. It is proved that

$$g * x = \int_0^t g(t-\eta)x(\eta)d\eta \overset{T_N}{\to} Y_T = W_g^*(Z)X_T, \tag{2.72}$$

while the P-matrix $W_g^*(Z)$ can be explicitly determined by the Laplace inverse transform $G^*(\lambda)$ of the kernel $g(t)$:

$$G(p) \overset{p \to \frac{1}{\lambda}}{\to} G^* \overset{\lambda TJ}{\to} = G^*(TJ) = G^*\big[\lambda_0(E-Z)^{-1}(E+Z)\big] = W_g^*(Z). \tag{2.73}$$

Here TJ is the pointwise representation of the Volterra, which is a P-matrix. Thus input - output connection for the linear (stationary) dynamical system is modeled in the pointwise representation of the vector-matrix equation

$$Y_T = W_g^*(Z)X_T. \tag{2.74}$$

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Krasnoyarsk State Academy for Non-Ferrous Metals and Gold, 95 "Krasnoyarskiy rabochiy" prospect, Krasnoyarsk 660025, Russia

E-mail address: gafur@rol.ru, ffomia@rol.ru, Osipova@color.krasnoyarsk.su