2004 Conference on Diff. Eqns. and Appl. in Math. Biology, Nanaimo, BC, Canada.
Electronic Journal of Differential Equations, Conference 12, 2005, pp. 87–101.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

USING LIE SYMMETRIES IN EPIDEMIOLOGY

MARIA CLARA NUCCI

ABSTRACT. Lie symmetry method has been and still is successfully applied in different problems of physics for about a hundred years, but its application in epidemiology has been rare perhaps because the ordinary differential equations studied in this field are generally of first-order in contrast with those in physics which are usually of second-order. Here we exemplify the use of Lie symmetry method in the study of mathematical models in epidemiology, and show how it complements the mathematical techniques (qualitative and numerical analysis) traditionally used.

1. INTRODUCTION

In January 2001, the first Whiteman prize for notable exposition on the history of mathematics was awarded to Thomas Hawkins by the American Mathematical Society. In the citation, published in the Notices of AMS **48** 416-417 (2001), one reads that Thomas Hawkins "...has written extensively on the history of Lie groups. In particular he has traced their origins to [Lie's] work in the 1870s on differential equations ... the *idée fixe* guiding Lie's work was the development of a Galois theory of differential equations ...[Hawkins's book [16]] highlights the fascinating interaction of geometry, analysis, mathematical physics, algebra and topology ...". Also Hawkins had established "the nature and extent of Jacobi's influence upon Lie" [17]. This is particularly noteworthy since 2004 marks two hundred years since Jacobi's birth. "Given the fact that the Jacobi Identity is fundamental to the theory of Lie groups, Jacobi's influence upon Lie will come as no surprise. But the bald fact that he inherited the Identity from Jacobi fails to convey fully or accurately the historical dimension of the impact of Jacobi's work on partial differential equations" [17].

In the Introduction of his book [48] Olver wrote that "it is impossible to overestimate the importance of Lie's contribution to modern science and mathematics. Nevertheless anyone who is already familiar with [it] ...is perhaps surprised to know that its original inspirational source was the field of differential equations".

Lie's monumental work on transformation groups, [29], [30] and [31], and in particular contact transformations [32], led him to achieve his goal [33]. Many books have been dedicated to this subject and its generalizations [1, 6, 49, 48, 7, 52, 53, 18, 21, 22, 23, 19, 5].

2000 *Mathematics Subject Classification.* 34A05, 34C14, 92D30.
Key words and phrases. Lie symmetries; mathematical epidemiology.

Published April 20, 2005.

Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations. Any known integration technique (We mean those taught in most undergraduate courses on ordinary differential equations.) can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, i.e. the Lie symmetry algebra, which can be easily derived by a straightforward although lengthy procedure. As computer algebra software becomes widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is becoming easier to perform. A major drawback of Lie's method is that it is useless when applied to systems of n first-order equations[1], because they admit an infinite number of symmetries, and there is no systematic way to find even an one-dimensional Lie symmetry algebra, apart from trivial groups like translations in time admitted by autonomous systems. One may try to derive an admitted n-dimensional solvable Lie symmetry algebra by making an ansatz on the form of its generators but when successful (rarely) it is just a lucky guess.

However, in [39] we have remarked that any system of n first-order equations could be transformed into an equivalent system where at least one of the equations is of second-order. Then the admitted Lie symmetry algebra is no longer infinite-dimensional, and nontrivial symmetries of the original system could be retrieved [39]. This idea has been successfully applied in several instances. In [39] it was shown that Krause's symmetries [27] for the Kepler problem are actually Lie symmetries, and in [45] how to derive the harmonic oscillator from the Kepler problem by using Lie symmetries. The Kepler problem and MICZ-Kepler problem were also shown to be equivalent to an isotropic two-dimensional system of linear harmonic oscillators in [28] thanks to Lie symmetries. In [35] Lie group analysis – when applied to Euler-Poisson equations as obtained from the reduction method [39] – unveiled the Kowalevski top [26] and its peculiar integral without making use of either Noether's theorem [38] or the Painlevé method [26]. In [41] Lie group analysis related the famous Lorenz system [34] to the Euler equations of a rigid body moving about a fixed point and subjected to a torsion depending on time and angular velocity, namely Lie group analysis transformed the "butterfly" into a "tornado". In [42] a solvable many-body problem introduced by Calogero [10] was shown to be intrinsically linear by means of Lie symmetries. In [43] a three-body problem derived and solved up to a quadrature by Jacobi [24] was shown to be reducible to the equation of motion of a single free particle on the line.

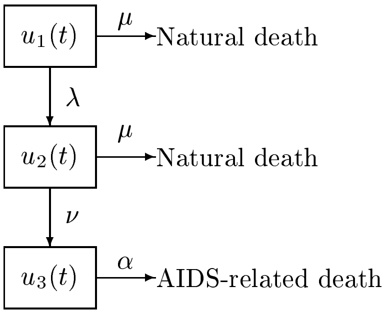
Lie group analysis is successfully applied in different problems of physics (and has been for about a hundred years), but rarely in biology (or epidemiology) maybe because the ordinary differential equations studied in these fields are generally of first-order in contrast with those in physics which are usually of second-order. Yet when Lie group analysis is successfully applied to epidemiological models then several instances of integrability even linearity are found which lead to the general solution of the model. Thus the dynamics of epidemics can be exactly described.

[1]Any undergraduate science/engineering student knows that a n-order ordinary differential equation can be transformed into an equivalent system of n first-order equations. Less well-known to students but common knowledge among experts in Lie group analysis is the dramatic consequence that that transformation has on the dimension of the admitted Lie symmetry algebra. In fact while the maximum Lie symmetry algebra admitted by a single n-order equation is finite [13] the dimension of the Lie symmetry algebra admitted by a system of n first-order equations is infinite.

The purpose of this paper is to promote the use of Lie symmetry method among bio-mathematical practitioners. We present three examples [54], [12], [51] where Lie symmetries have been found, and the general solution of the epidemiological model consequently derived whenever appropriate conditions among the involved parameters are satisfied. In [46] and [44] one can find an instance where Lie group analysis leads to the general solution of a SIS model formulated in [8] without any condition on the involved parameters. Moreover each example epitomizes a different situation, i.e. hidden linearity of the model, an anomalous behavior in the dynamics of the infectives, and a general periodic solution in apparent contrast with prediction by qualitative analysis, respectively. In section 2 we show that for a certain relationship among the involved parameters, Lie group analysis unveils the hidden linearity [54] of a seminal model given by Anderson, which describes HIV transmission in male homosexual/bisexual cohorts [2]. In section 3 we show that for an appropriate relationship among the involved parameters, Lie group analysis leads to the general solution of a core group model for sexually transmitted disease formulated by Hadeler and Castillo-Chavez [15], and gives a deeper insight on the strange behavior of the number of infectives [12]. In section 4 we show that for a certain relationship among the involved parameters Lie group analysis, when applied to a SIRI disease transmission model formulated by Derrick and van den Driessche [11], leads to a periodic general solution [51] in apparent contrast to the qualitative analysis performed in [11]. In section 5 we conclude with some final remarks.

2. An HIV-transmission model

In [54], Lie group analysis was applied to a seminal model formulated by Anderson, which describes HIV transmission in male homosexual/bisexual cohorts [2]. This compartmental model divides the population at time t into susceptibles (HIV negatives), infecteds (HIV positives), and AIDS patients, represented by $u_1(t)$, $u_2(t)$, and $u_3(t)$, respectively. HIV infecteds are individuals who test positive for specific antibodies to the virus [3]. AIDS patients are persons exhibiting characteristic clinical manifestations of full-blown AIDS, the end-stage of the disease [36]. In this model, the population is not subject to recruitment and individuals are removed only by death. An individual may belong to only one compartment at any specified time. However, individuals move from one compartment to the next according to the following flow diagram:



The parameter μ is the per capita natural death rate (non-AIDS related) of both susceptibles and infecteds, and α is the AIDS-related death rate. The term λ is the per capita force of infection and is defined as:

$$\lambda = \frac{\beta c u_2(t)}{u_1(t) + u_2(t) + u_3(t)},$$

where β is the average probability that an infected individual will infect a susceptible partner over the duration of their relationship [2], [14], and c is the effective rate of partner change within the specified risk category [2].

In the model, all infecteds are supposed to develop AIDS with an average incubation period $1/\nu$ [2], [36]. The system of nonlinear ordinary differential equations derived from this model is

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{-\beta c u_1 u_2}{u_1 + u_2 + u_3} - \mu u_1 \tag{2.1}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = \frac{\beta c u_1 u_2}{u_1 + u_2 + u_3} - (\nu + \mu) u_2 \tag{2.2}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = \nu u_2 - \alpha u_3. \tag{2.3}$$

We can easily transform this system into a system of one equation of second order in u_1, and one of first order in u_2. Indeed, if we derive u_3 from (2.1), i.e.:

$$u_3 = \frac{-\frac{\mathrm{d}u_1}{\mathrm{d}t}(u_2 - u_1) + \beta c u_1 u_2 + \mu u_1^2 + \mu u_1 u_2}{\frac{\mathrm{d}u_1}{\mathrm{d}t} + \mu u_1} \tag{2.4}$$

then we obtain the following system in u_1 and u_2:

$$\begin{aligned} \frac{\mathrm{d}^2u_1}{\mathrm{d}t^2} =& \Big[\alpha\beta c\mu u_1^2 u_2 + \alpha\beta c u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} u_2 + \alpha\mu^2 u_1^3 + \alpha\mu^2 u_1^2 u_2 \\ & + 2\alpha\mu u_1^2 \frac{\mathrm{d}u_1}{\mathrm{d}t} + 2\alpha\mu u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} u_2 + \alpha u_1 \big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 + \alpha\big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 u_2 \\ & - \beta c\mu^2 u_1^3 - \beta c\mu^2 u_1^2 u_2 - \beta c\mu\nu u_1^2 u_2 - 2\beta c\mu u_1^2 \frac{\mathrm{d}u_1}{\mathrm{d}t} \\ & - \beta c\mu u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} u_2 - \beta c\nu\, u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} u_2 - \beta c u_1 \big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 + \beta c\big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 u_2 \\ & - \mu^3 u_1^3 - \mu^3 u_1^2 u_2 - 2\mu^2 u_1^2 \frac{\mathrm{d}u_1}{\mathrm{d}t} - 2\mu^2 u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} u_2 \\ & - \mu u_1 \big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 - \mu\big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2 u_2\Big] / (\beta c u_1 u_2) \end{aligned} \tag{2.5}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = -\left(\mu u_1 + \mu u_2 + \nu u_2 + \frac{\mathrm{d}u_1}{\mathrm{d}t}\right) \tag{2.6}$$

When Lie group analysis of this system is performed using [40], a linear partial differential equation of parabolic structure is obtained. Its characteristic curve is given by $u_1 + u_2$. Consequently, we introduce the new dependent variable:

$$v_2 = u_1 + u_2 \tag{2.7}$$

to obtain a new system, which in the case $\alpha = \mu + \beta c$ admits an eight-dimensional Lie symmetry algebra. Actually, it becomes separable, i.e.:

$$\begin{aligned} \frac{\mathrm{d}^2u_1}{\mathrm{d}t^2} =& \big[\beta c\mu u_1^2 + \beta c u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} + \mu^2 u_1^2 - \mu\,\nu u_1^2 + 2\mu u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} - \\ & - \nu u_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} + 2\big(\frac{\mathrm{d}u_1}{\mathrm{d}t}\big)^2\big] / u_1 \end{aligned} \tag{2.8}$$

$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = -(\mu+\nu)v_2 + \nu u_1 \tag{2.9}$$

Therefore, equation (2.8) is linearizable by means of a point transformation [33] because it admits an eight-dimensional Lie symmetry algebra generated by the following eight operators:

$$\begin{gathered}
X_1 = e^{-(\beta c+\mu-\nu)t}\big(\frac{1}{u_1}\,\partial_t - \mu\,\partial_{u_1}\big), \quad X_2 = e^{-\mu t}\big(\frac{1}{u_1}\,\partial_t - (\beta c+\mu-\nu)\,\partial_{u_1}\big), \\
X_3 = e^{(\beta c+\mu-\nu)t}u_1^2\,\partial_{u_1}, \quad X_4 = e^{\mu t}u_1^2\,\partial_{u_1}, \quad X_5 = u_1\,\partial_{u_1}, \quad X_6 = \partial_t, \\
X_7 = e^{(\beta c-\nu)t}\left(-\partial_t + (\beta c+\mu-\nu)u_1\,\partial_{u_1}\right), \quad X_8 = -e^{(\beta c-\nu)t}\left(-\partial_t + \mu u_1\,\partial_{u_1}\right),
\end{gathered}$$

if $\beta c \neq \nu$, or

$$\begin{gathered}
\hat{X}_1 = e^{-\mu t}\big(\frac{1}{u_1}\,\partial_t - \mu\,\partial_{u_1}\big), \quad \hat{X}_2 = e^{-\mu t}\big(\frac{t}{u_1}\,\partial_t - (\mu t+1)\,\partial_{u_1}\big), \\
\hat{X}_3 = e^{\mu t}u_1^2\,\partial_{u_1}, \quad \hat{X}_4 = e^{\mu t}tu_1^2\,\partial_{u_1}, \quad \hat{X}_5 = u_1\,\partial_{u_1}, \quad \hat{X}_6 = \partial_t, \\
\hat{X}_7 = t\left(t\,\partial_t - (\mu t+1)u_1\,\partial_{u_1}\right), \quad \hat{X}_8 = t\left(\partial_t - \mu u_1\,\partial_{u_1}\right),
\end{gathered}$$

if $\beta c = \nu$. To find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra, and, following Lie's classification of two-dimensional algebras in the real plane [33], we have to transform it into the canonical form

$$\partial_{\tilde{u}}, \quad \tilde{x}\partial_{\tilde{u}} \tag{2.10}$$

with $\bar{u}$ and $\bar{t}$ the new dependent and independent variables, respectively. We find that one such subalgebra is that generated by X_3 and X_4, if $\beta c \neq \nu$, or $\hat{X}_3$ and $\hat{X}_4$, if $\beta c = \nu$. Then it is easy to derive that the transformation which changes (2.8) into a linear ordinary differential equations is either:

$$\bar{t} = e^{(\nu-\beta c)t}, \quad \bar{u} = -\frac{e^{(\nu-\beta c-\mu)t}}{u_1} \tag{2.11}$$

if $\beta c \neq \nu$, or

$$\bar{t} = t, \quad \bar{u} = -\frac{e^{-\mu t}}{u_1} \tag{2.12}$$

if $\beta c = \nu$. Thus, equation (2.8) becomes:

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\bar{t}^2} = 0 \tag{2.13}$$

and its general solution is trivially[2]

$$\bar{u} = a_1\bar{t} + a_2, \tag{2.14}$$

which yields the following general solution of system (2.1)-(2.3):

$$u_1 = \frac{e^{\nu t}c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\,c-\nu\right)c_1 + e^{\beta\,c\,t}\beta\,c\right]}, \tag{2.15}$$

$$u_2 = \frac{\left(\beta\,c-\nu\right)\int \frac{e^{\beta\,c\,t+2\nu t}}{\left(e^{\beta\,c\,t}\beta\,c+e^{\nu t}\beta\,c\,c_1-e^{\nu t}c_1\nu\right)^2}\,dt\ \beta\,c\,c_2 + c_3}{e^{\mu t+\nu t}}, \tag{2.16}$$

[2]Here a_1, a_2 are arbitrary constants.

$$\begin{aligned} u_3 = & \frac{\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\nu\right] c_3}{e^{\beta\, c\, t+\mu t+\nu t}\left(\beta\, c-\nu\right)}+\frac{-e^{\nu t}\, c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\beta\, c\right]} \\ & +\frac{\beta\, c\, c_2\left[e^{\nu t}\left(\beta\, c-\nu\right) c_1+e^{\beta\, c\, t}\nu\right]\int\frac{e^{\beta\, c\, t+2\nu t}}{\left(e^{\beta\, c\, t}\beta\, c+e^{\nu t}\beta\, c\, c_1-e^{\nu t}c_1\nu\right)^2}\, dt}{e^{\beta\, c\, t+\mu t+\nu t}}. \end{aligned} \tag{2.17}$$

where c_1, c_2, c_3 are arbitrary constants. If $\beta c = 2\nu$, the general solution assumes a simpler form:

$$u_1 = \frac{c_1}{e^{\mu t}\left(2e^{\nu t}+c_1c_2\right)}, \tag{2.18}$$

$$\begin{aligned} u_2 = & \left[2e^{\nu t}\log\left(2e^{\nu t}+c_1c_2\right)c_1-2e^{\nu t}c_1+4e^{\nu t}c_3+\right. \\ & \left.+\log\left(2e^{\nu t}+c_1c_2\right)c_1^2c_2+2c_1c_2c_3\right] / \left[2e^{\mu t+\nu t}\left(2e^{\nu t}+c_1c_2\right)\right], \end{aligned} \tag{2.19}$$

$$\begin{aligned} u_3 = & \left[e^{\nu t}\log\left(2e^{\nu t}+c_1c_2\right)c_1-2e^{\nu t}c_1+2e^{\nu t}c_3+\right. \\ & \left.+\log\left(2e^{\nu t}+c_1c_2\right)c_1^2c_2+2c_1c_2c_3\right] / \left[2e^{\mu t+2\nu t}\right]. \end{aligned} \tag{2.20}$$

In [54] the solution was tested on data from three U.S. epidemiologic studies, and found to closely match observed epidemic data.

3. A core group model

In [15] Hadeler and Castillo-Chavez presented a model for sexually transmitted diseases which takes into consideration an active and relatively small core group of constant size. The core group recruits individuals from the non-core group, and the rate of recruitment may depend on the state of the core group. The non-core group is completely inactive. The total population has size $P(t)$, and the non-core group has size A. The population of the core group C is further divided into susceptibles S, educated (or vaccinated) V, and infecteds I. The birth rate is $b > 0$, the birth rate of infecteds is $\tilde{b} \le b, \tilde{b} \ge 0$, the death rate is $\mu > 0$, the death rate of infecteds is $\tilde{\mu} \ge \mu$, the recovery rate is $\alpha \ge 0$, the education (vaccination) rate is $\psi \ge 0$, the transmission rate from infecteds to susceptibles is $\beta \ge 0$, the transmission rate from infecteds to educated (vaccinated) is $\tilde{\beta}, 0 \le \tilde{\beta} \le \beta$. At recovery individuals may either pass into the educated class at the rate $\alpha\gamma, 0 \le \gamma \le 1$, or return to the susceptible class at the rate $\alpha(1-\gamma)$. Recruitment into the core group is described by a function $r(I, C)$. Hadeler and Castillo-Chavez focused on the situation where the disease has no demographic effects and population size is constant, i.e. $P =$const, $b = \tilde{b} = \mu = \tilde{\mu}$. Thus their model assumes the form

$$\dot{A} = \mu P - Ar(I, C) - \mu A, \tag{3.1}$$

$$\dot{S} = Ar(I, C) - \beta\frac{SI}{C} - \psi S + \alpha(1-\gamma)I - \mu S, \tag{3.2}$$

$$\dot{V} = \psi S - \tilde{\beta}\frac{VI}{C} + \alpha\gamma I - \mu V, \tag{3.3}$$

$$\dot{I} = \frac{\beta SI + \tilde{\beta}VI}{C} - \alpha I - \mu I. \tag{3.4}$$

where the overdot denotes differentiation with respect to t. They point out that this system is closely related to a model for an isolated population of constant size

$C = 1$, i.e.

$$\dot{S} = \mu - \beta SI - \psi S + \alpha(1-\gamma)I - \mu S, \tag{3.5}$$
$$\dot{V} = \psi S - \tilde{\beta}VI + \alpha\gamma I - \mu V, \tag{3.6}$$
$$\dot{I} = \beta SI + \tilde{\beta}VI - \alpha I - \mu I. \tag{3.7}$$

In [15] the stationary solutions of system (3.5)-(3.7) are found and their qualitative features discussed. Then the stationary solutions of system (3.1)-(3.4) are also discussed. Qualitative conclusions are finally drawn.
In [12] Lie group analysis is applied to system (3.5)-(3.7) in order to determine under which physical conditions on the parameters Lie point symmetries exist and, when possible, deduce the general solution in closed form. Also in [12] a discussion of the solutions that have been found is presented to show how Lie group analysis complements Hadeler and Castillo-Chavez's qualitative analysis.

System (3.5)-(3.7) is composed of three first order ordinary differential equations which can be easily reduced to two equations by using the following condition

$$C \equiv 1 = S + V + I. \tag{3.8}$$

Then we can easily transform the system of two equations so obtained into one equation of second order. We derive I from (3.8), i.e.

$$I = 1 - S - V, \tag{3.9}$$

and then deduce V from equation (3.5), i.e.

$$V = \frac{\dot{S} - \mu S + \psi S - \mu}{\alpha\gamma - \alpha + \beta S} - S + 1. \tag{3.10}$$

Consequently a second order equation for S is obtained. When we apply Lie group analysis to this equation[3] then we obtain a first-order linear partial differential equation for $v(t, S)$; its characteristic curve suggests to make the following simplifying transformation

$$S = \frac{-\alpha\gamma + \alpha + u}{\beta}, \tag{3.11}$$

where $u(t)$ is the new dependent variable. Then (3.10) transforms into

$$V = 1 + \frac{\dot{u} + \alpha(1-\gamma)(\mu+\psi) + (\alpha\gamma - \alpha + \mu + \psi - u)u - \beta\mu}{\beta u}, \tag{3.12}$$

[3]We look for Lie operators of the form $\Gamma = v(t, S)\partial_t + G(t, S)\partial_S$.

and we have to study the following equation in $u = u(t)$:

$$\begin{aligned}
\ddot{u} =&(\alpha^2\beta\gamma^2\mu u + \alpha^2\beta\gamma^2\psi u - \alpha^2\beta\gamma\mu u - \alpha^2\beta\gamma\psi u + \alpha^2\tilde{\beta}\gamma^2\mu^2 + 2\alpha^2\tilde{\beta}\gamma^2\mu\psi \\
&- \alpha^2\tilde{\beta}\gamma^2\mu u + \alpha^2\tilde{\beta}\gamma^2\psi^2 - \alpha^2\tilde{\beta}\gamma^2\psi u - 2\alpha^2\tilde{\beta}\gamma\mu^2 - 4\alpha^2\tilde{\beta}\gamma\mu\psi + 2\alpha^2\tilde{\beta}\gamma\mu u \\
&- 2\alpha^2\tilde{\beta}\gamma\psi^2 + 2\alpha^2\tilde{\beta}\gamma\psi u + \alpha^2\tilde{\beta}\mu^2 + 2\alpha^2\tilde{\beta}\mu\psi - \alpha^2\tilde{\beta}\mu u + \alpha^2\tilde{\beta}\psi^2 \\
&- \alpha^2\tilde{\beta}\psi u + \alpha\beta^2\gamma\mu u + 2\alpha\beta\tilde{\beta}\gamma\mu^2 + 2\alpha\beta\tilde{\beta}\gamma\mu\psi - 2\alpha\beta\tilde{\beta}\gamma\mu u - \alpha\beta\tilde{\beta}\gamma\psi u \\
&- 2\alpha\beta\tilde{\beta}\mu^2 - 2\alpha\beta\tilde{\beta}\mu\psi + 2\alpha\beta\tilde{\beta}\mu u + \alpha\beta\tilde{\beta}\psi u + \alpha\beta\gamma\mu^2 u \\
&+ \alpha\beta\gamma\mu\psi u - 2\alpha\beta\gamma\mu u^2 - \alpha\beta\gamma\mu\dot{u} - 2\alpha\beta\gamma\psi u^2 - \alpha\beta\gamma\psi\dot{u} - \alpha\beta\gamma u\dot{u} - \alpha\beta\mu^2 u \\
&- \alpha\beta\mu\psi u + \alpha\beta\mu u^2 + \alpha\beta\mu\dot{u} + \alpha\beta\psi u^2 + \alpha\beta\psi\dot{u} - 2\alpha\tilde{\beta}\gamma\mu^2 u - 4\alpha\tilde{\beta}\gamma\mu\psi u \\
&+ 2\alpha\tilde{\beta}\gamma\mu u^2 - 2\alpha\tilde{\beta}\gamma\mu\dot{u} - 2\alpha\tilde{\beta}\gamma\psi^2 u + 2\alpha\tilde{\beta}\gamma\psi u^2 - 2\alpha\tilde{\beta}\gamma\psi\dot{u} + \alpha\tilde{\beta}\gamma u\dot{u} \\
&+ 2\alpha\tilde{\beta}\mu^2 u + 4\alpha\tilde{\beta}\mu\psi u - 2\alpha\tilde{\beta}\mu u^2 + 2\alpha\tilde{\beta}\mu\dot{u} + 2\alpha\tilde{\beta}\psi^2 u \\
&- 2\alpha\tilde{\beta}\psi u^2 + 2\alpha\tilde{\beta}\psi\dot{u} - \alpha\tilde{\beta}u\dot{u} + \beta^2\tilde{\beta}\mu^2 \\
&- \beta^2\tilde{\beta}\mu u + \beta^2\mu^2 u - \beta^2\mu u^2 - \beta^2\mu\dot{u} - 2\beta\tilde{\beta}\mu^2 u - 2\beta\tilde{\beta}\mu\psi u \\
&+ 2\beta\tilde{\beta}\mu u^2 - 2\beta\tilde{\beta}\mu\dot{u} + \beta\tilde{\beta}\psi u^2 + \beta\tilde{\beta}u\dot{u} - \beta\mu^2 u^2 - \beta\mu\psi u^2 + \beta\mu u^3 - \beta\mu u\dot{u} \\
&+ \beta\psi u^3 + \beta u^2\dot{u} + \beta\dot{u}^2 + \tilde{\beta}\mu^2 u^2 + 2\tilde{\beta}\mu\psi u^2 - \tilde{\beta}\mu u^3 + 2\tilde{\beta}\mu u\dot{u} \\
&+ \tilde{\beta}\psi^2 u^2 - \tilde{\beta}\psi u^3 + 2\tilde{\beta}\psi u\dot{u} - \tilde{\beta}u^2\dot{u} + \tilde{\beta}\dot{u}^2)/(\beta u).
\end{aligned}$$

In [12] Lie group analysis was applied to (3.13) and non-trivial Lie point symmetries were obtained in five cases: in the first case an eight-dimensional Lie symmetry algebra was obtained, which means that equation (3.13) is linearizable, while in the other four cases a two-dimensional Lie symmetry algebra was found. Here we present Case (5) of [12] in order to show how to get the general solution if a two-dimensional Lie symmetry algebra is found and which new insights – not detected by qualitative analysis - on the dynamics of the epidemics can be obtained from it. Case (5) corresponds to the following relationship among the involved parameters:

$$\tilde{\beta} = 0, \quad \psi = \alpha\gamma, \quad \beta = \frac{\alpha(1-\gamma)(\alpha\gamma+\mu)}{\mu}$$

Then (3.13) admits a two-dimensional Lie algebra generated by the following operators:

$$\Gamma_1 = e^{(\mu+\alpha\gamma)t}\Big(\partial_t - (\mu+\alpha\gamma)u\partial_u\Big), \quad \Gamma_2 = \partial_t \tag{3.13}$$

A basis of the differential invariants of order ≤ 1 for operator Γ_1 in (3.13) is

$$\tilde{t} = u\, e^{(\mu+\alpha\gamma)t}, \quad \tilde{u} = (\dot{u} + u\mu + u\alpha\gamma)\, e^{2(\mu+\alpha\gamma)t}, \tag{3.14}$$

and therefore (3.13) becomes the first order equation:

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{t}} = \frac{\tilde{u} + \tilde{t}^2}{\tilde{t}}, \tag{3.15}$$

which can be easily integrated, i.e.:

$$\tilde{u} = a_1\tilde{t} + \tilde{t}^2. \tag{3.16}$$

The integration of (3.15) is not haphazard but derives from Lie symmetry method itself [33]. In fact replacing (3.14) into (3.16) yields the following first order equation:

$$\dot{u} = \frac{u\left(-e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu-u)+a_1\right)}{e^{(\alpha\gamma+\mu)t}} \tag{3.17}$$

which is easy to integrate because it admits the Lie symmetry generated by Γ_1 in (3.13). Lie proved that if one knows a symmetry $\tau(t,u)\partial_t+\xi(t,u)\partial_u$ of a first-order ordinary differential equation, say $\dot{u}=f(t,u)$, then an integrating factor for the corresponding linear differential form, say $\mathrm{d}u-f(t,u)\mathrm{d}t=0$, is $1/(\xi-f(t,u)\tau)$ [33]. Thus the general solution of (3.13) is

$$u = \frac{a_1}{e^{(\alpha\gamma+\mu)t}(e^{a_1/(e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu))}a_1a_2-1)}. \tag{3.18}$$

and consequently the general solution of (3.5)-(3.7) is:

$$S = \frac{\left(e^{(\alpha\gamma+\mu)t}\left(e^{a_1/\left(e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu)\right)}a_1a_2-1\right)(\gamma-1)\alpha-a_1\right)\mu}{e^{(\alpha\gamma+\mu)t}\left(e^{a_1/\left(e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu)\right)}a_1a_2-1\right)(\alpha\gamma+\mu)(\gamma-1)\alpha}, \tag{3.19}$$

$$V = \frac{e^{(\alpha\gamma+\mu)t}\alpha^2\gamma(\gamma-1)-a_1\mu}{e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu)(\gamma-1)\alpha}, \tag{3.20}$$

$$I = \frac{e^{a_1/\left(e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu)\right)}a_1^2a_2\mu}{e^{(\alpha\gamma+\mu)t}\left(e^{a_1/\left(e^{(\alpha\gamma+\mu)t}(\alpha\gamma+\mu)\right)}a_1a_2-1\right)(\alpha\gamma+\mu)(\gamma-1)\alpha}. \tag{3.21}$$

In [12] the effectiveness of a disease management program in the core group was simulated by plotting the solutions with the help of the graphing capability of MAPLE 7. In some instances a temporary increase of the number of infecteds followed by a decrease occurs despite the presence of the vaccination/education program. This outcome is in agreement with the qualitative description by Hadeler and Castillo-Chavez. The quantitative description in [12] provides a further insight on the strange behavior of the number of infecteds in the core group as can be seen in several instances. In fact as it was shown in [12] if the prevalence of infecteds is initially small in the core group then an increase occurs before the number of infecteds actually decreases.

The same numerical value of α, μ and γ as given in [15] are used, i.e.:

$$\alpha = 4, \quad \mu = 0.2, \quad \gamma = 0.025. \tag{3.22}$$

The numerical values of the other parameters are derived from the relationships that Lie group analysis has discerned.

The dynamics of the core group is simulated by taking into consideration two different initial conditions at time $t=0$:

(A) no vaccinated/educated are present and the prevalence of infecteds is relatively small, i.e. $S(0)=0.9, V(0)=0, I(0)=0.1$;

(B) no vaccinated/educated are present and the prevalence of infecteds is high being nearly half the size of the core group, i.e. $S(0)=0.6, V(0)=0, I(0)=0.4$

In the figures, the solid line represents the plot of S, the lighter dashed line represents the plot of V, and the darker dashed line represents the plot of I. In this case the numerical values of the remaining parameters are as follows:

$$\tilde{\beta}=0, \quad \psi=\alpha\gamma=0.1, \quad \beta=\frac{\alpha(1-\gamma)(\alpha\gamma+\mu)}{\mu}=5.85$$

Note that if the initial prevalence of infecteds is small ($I(0) = 0.1$), then there is a delay in the effectiveness of the vaccination/education program as can be seen in Figure 1, even in this case when there is no exchange between vaccinated/educated and infecteds (i.e., $\tilde{\beta} = 0$). Instead if the initial prevalence of infecteds is high ($I(0) = 0.4$), then the vaccination/education program immediately takes effect (Figure 2) as expected. Further discussion can be found in [12].

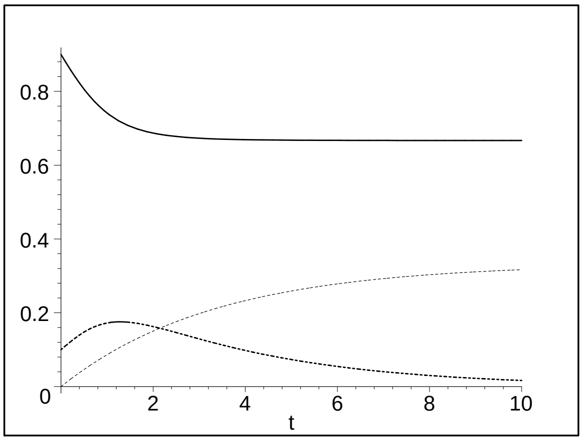


FIGURE 1. $\tilde{\beta} = 0$, $\psi = 0.1$, $\beta = 5.85$, $S(0) = 0.9$, $V(0) = 0$, $I(0) = 0.1$

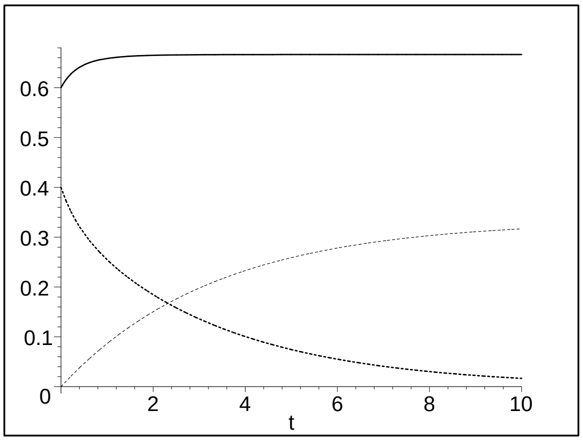
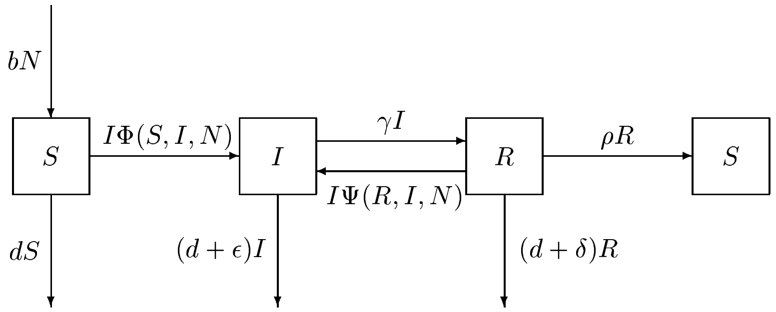


FIGURE 2. $\tilde{\beta} = 0$, $\psi = 0.1$, $\beta = 5.85$, $S(0) = 0.6$, $V(0) = 0$, $I(0) = 0.4$

4. A SIRI MODEL

In [11] Derrick and van den Driessche formulated a model of disease transmission in a nonconstant population of size N divided into three classes: susceptibles (S), infectives (I) and recovereds (R). Individuals move from one compartment to the next according to the following flow diagram:



The parameter b is per capita birth-rate, d per capita disease free death rate, ϵ excess per capita death rate of infectives, δ excess per capita death rate of recovereds, γ per capita recovery rate of infectives, and ρ per capita loss of immunity rate of recovereds. The incidence of disease in the susceptible class is given by the function $I\Phi(S, I, N)$, while $I\Psi(R, I, N)$ is the transfer rate of the recovered class into the infective class. The above hypotheses lead to the following differential equations, where $'$ denotes differentiation with respect to t,

$$S' = bN - dS + \varrho R - I\Phi(S, I, N) \tag{4.1}$$
$$I' = I[\Phi(S, I, N) + \Psi(R, I, N) - (d + \epsilon + \gamma)] \tag{4.2}$$
$$R' = \gamma I - (d + \delta + \varrho)R - I\Psi(R, I, N) \tag{4.3}$$

The analysis in [11] was mainly dedicated to show existence (or nonexistence) of periodic solutions for the SIRS model (4.1)-(4.3) when proportions of individuals in the three epidemiological classes are considered, i.e.

$$s = S/N, \quad i = I/N, \quad r = R/N\,. \tag{4.4}$$

With these variables system (4.1)-(4.3) becomes

$$s' = b(1 - s) + \varrho r + \epsilon si + \delta sr - i\Phi(s, i) \tag{4.5}$$
$$i' = -(b + \epsilon + \gamma)i + \epsilon i^2 + \delta ir + i\Phi(s, i) + i\Psi(r, i) \tag{4.6}$$
$$r' = \gamma i - (b + \varrho + \delta)r + \epsilon ri + \delta r^2 - i\Psi(r, i) \tag{4.7}$$

where $\Phi(s, i) = \Phi(s, i, 1) = \Phi(S/N, I/N, N/N) = \Phi(S, I, N)$ and $\Psi(r, i) = \Psi(r, i, 1) = \Psi(R/N, I/N, N/N) = \Psi(R, I, N)$.

In [11] a theorem was presented and proved in order to establish under which conditions system (4.5)-(4.7) does not possess periodic solutions in the feasibility region

$$\mathcal{D} = \{s \geq 0, i \geq 0, r \geq 0 : s + i + r = 1\} \tag{4.8}$$

An example of the nonexistence of periodic solutions was then introduced, namely a special SIRI case of the general model (4.5)-(4.7) with $\varrho = \delta = 0$, $\Phi(s, i) = \phi s$, and $\Psi(r, i) = \psi r$. Since $s + i + r = 1$ it is possible to eliminate r and finally obtain the following system:

$$s' = b(1-s) - (\phi - \epsilon)si \tag{4.9}$$

$$i' = i[(\phi - \psi)s + (\epsilon - \psi)i - (\epsilon + b + \gamma - \psi)] \tag{4.10}$$

In [51] Lie group analysis was applied to system (4.9)-(4.10), namely to either the second-order equation in the unknown s that one obtains by deriving i from (4.9) or the second-order equation in the unknown i that one obtains by deriving s from (4.10). Several cases were found, even instances of hidden linearity. Here we show that when

$$b = 0, \quad \phi = 2\epsilon - \psi, \quad \gamma = \psi - \epsilon$$

then a two-dimensional Lie symmetry algebra is admitted by equation

$$\begin{aligned} i'' = & -((\psi i^2 - i')i' + \gamma\phi i^3 + b^2 i^2 + (i-1)\epsilon^2 i^3 + ((i-1)\psi i + i')\phi i^2 \\ & - (\gamma i + 2i' + (\phi + \psi)(i-1)i)\epsilon i^2 + (\psi i^2 + i' + \gamma i \\ & + (i-1)\phi i - (2i-1)\epsilon i)bi)/i \end{aligned} \tag{4.11}$$

which is obtained from system (4.9)-(4.10) by deriving s from equation (4.10), i.e.

$$s = \frac{[b + \gamma - (\epsilon - \psi)(i-1)]i + i'}{(\phi - \psi)i} \tag{4.12}$$

and substituting it into equation (4.9). The Lie symmetry algebra is generated by the operators

$$\Gamma_1 = t\partial_t - i\partial_i, \quad \Gamma_2 = \partial_t\,. \tag{4.13}$$

This means that equation (4.11) can be easily integrated by quadrature as was shown in the previous section. Its general solution is

$$i = \frac{a_1}{\sin\left(\frac{a_1\, a_2 - a_1\, t}{\epsilon - \psi}\right)(\epsilon^2 - 2\,\epsilon\,\psi + \psi^2)} \tag{4.14}$$

and from (4.12) one obtains:

$$s = \frac{1}{2}\,\frac{a_1\,\left(\cos\left(\frac{a_1\, a_2 - a_1\, t}{\epsilon - \psi}\right) - 1\right)}{\sin\left(\frac{a_1\, a_2 - a_1\, t}{\epsilon - \psi}\right)(\epsilon^2 - 2\,\epsilon\,\psi + \psi^2)} \tag{4.15}$$

This general solution of system (4.9)-(4.10) is clearly periodic in apparent contrast with the findings in [11]. Note that the functions (4.14)-(4.15) are neither bounded nor positive nor continuous, and do not belong to the feasibility region (4.8). In fact b must be positive for nonexistence of periodic solutions. However in [11] the condition $b = 0$ was allowed in order to show that system (4.5)-(4.7) has periodic solutions if $\Phi(s,i) = \phi si$, and $\Psi(r,i) = 0$.

5. Final remarks

In the Introduction to his Principia, Newton stated [37, 9]:

> I wish we could derive the rest of the phenomena of nature by the same kind of reasoning from mechanical principles.

However, Pulte [50] has reminded us that in his lectures on analytical mechanics Jacobi wrote [25]:

> Wherever Mathematics is mixed up with anything, which is outside its field, you will find attempts to demonstrate these merely propositions *a priori*, and it will be your task to find out the false deduction in each case ... Mathematics cannot invent how the relations of system of points depend on each other.

In 1964 Arscott in the Preface to his book on periodic differential equations wrote [4]:

> Only rarely does one find mention, at post-graduate level, of any problem in connection with the process of actually solving such equations. The electronic computer may perhaps be partly to blame for this, since the impression prevails in many quarters that almost any differential equation problem can be merely "put on the machine", so that finding an analytical solution is largely a waste of time. This, however, is only a small part of the truth, for at higher levels there are generally so many parameters or boundary conditions involved that numerical solutions, even if practicable, give no real idea of the properties of the equation. Moreover, any analyst of sensibility will feel that to fall back on numerical techniques savours somewhat of breaking a door with a hammer when one could, with a little trouble, find the key.

In conclusion, Lie group analysis should be considered an essential tool for anyone who wants to "comprehend" differential equations of relevance in physics and other scientific fields. As brilliantly stated by Ibragimov [20]

cherchez le groupe!

References

[1] W. F. Ames; *Nonlinear Partial Differential Equations in Engineering*, Vol. 2, Academic Press, New York, 1972.

[2] R. M. Anderson; *The role of mathematical models in the study of HIV transmission and the epidemiology of AIDS*, J. of AIDS **1** (1988), 240-255.

[3] R. M. Anderson and R. M. May; *Infectious Diseases of Humans: Dynamics and Control,* Oxford University Press, Oxford, 1991.

[4] F. M. Arscott; *Periodic Differential Equations*, Pergamon Press, Oxford, 1964.

[5] G. W. Bluman and S. Anco; *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, Berlin, 2002.

[6] G. W. Bluman and J. D. Cole; *Similarity Methods for Differential Equations*, Springer-Verlag, Berlin, 1974.

[7] G. W. Bluman and S. Kumei; *Symmetries and Differential Equations*, Springer-Verlag, Berlin, 1989.

[8] F. Brauer; *Basic ideas of mathematical epidemiology*, in: C. Castillo-Chavez, S. Blower Sally, P. van den Driessche, D. Kirschner, A.A. Yakubu, (Eds.), Mathematical Approaches for Emerging and Reemerging Infectious Diseases, Springer-Verlag, New York, 2002, pp. 31–65.

[9] F. Cajori; *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World.* Translated into English by Andrew Motte in 1729, University of California Press, Berkeley, 1946.

[10] F. Calogero; *The neatest many-body problem amenable to exact treatments (a "goldfish"?)*, Physica D **152-153** (2001), 78-84.

[11] W. R. Derrick and P. van den Driessche; *A disease transmission model in a nonconstant population*, J. Math. Bio. **31** (1993), 495-512.

[12] M. Edwards and M. C. Nucci; *Application of Lie group analysis to a core group model for sexually transmitted diseases*, (2005) submitted.

[13] F. González-Gascón and A. González-López; *Symmetries of differential equations.* IV, J. Math. Phys. **24** (1983), 2006-2021.

[14] R. M. Grant, J. A. Wiley, and W. Winkelstein; *Infectivity of the Human Immuno-deficiency Virus: estimates from a prospective study of homosexual men*, J. Infect. Dis. **156** (1987), 189-193.

[15] K. P. Hadeler and C. Castillo-Chavez; *A core group model for disease transmission*, Math. Biosci. **128** (1995), 41-55.

[16] T. Hawkins; *The Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics 1869-1926*, Springer-Verlag, Berlin, 2000.

[17] T. Hawkins; *Jacobi and the birth of Lie's theory of groups*, Arch. Hist. Exact Sciences **42** (1991), 187-278.

[18] J. M. Hill; *Differential Equations and Group Methods for Scientists and Engineers*, CRC Press, Boca Raton, 1992.

[19] P. E. Hydon; *Symmetry Methods for Differential Equations: A Beginner's Guide*, Cambridge University Press, Cambridge, 2000.

[20] N. H. Ibragimov; *Opyt Gruppovogo Analiza Obyknovennykh Differentsial'nykh Uravnenii*, Znanie, Moscow, 1991.

[21] N. H. Ibragimov, (ed.), *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. I: Symmetries, Exact Solutions, and Conservation Laws*, CRC Press, Boca Raton, 1994.

[22] N. H. Ibragimov, (ed.), *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. II: Applications in Engineering and Physical Sciences*, CRC Press, Boca Raton, 1995.

[23] N. H. Ibragimov; *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, New York, 1999.

[24] C. G. J. Jacobi, *Problema trium corporum mutuis attractionibus cubis distantiarum inverse proportionalibus recta linea se moventium*, in Gesammelte Werke, t. 4, Reimer, Berlin, 1866, 533-539.

[25] C. G. J. Jacobi, *Vorlesungen über Analytische Mechanik (1847-1848)*, ed. H. Pulte, Braunschweig/Wiesbaden, Vieweg, 1996.

[26] S. Kowalevski; *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Math. **12** (1889), 177-232.

[27] J. Krause; *On the complete symmetry group of the classical Kepler system*, J. Math Phys **35** (1994), 5734-5748.

[28] P. G. L. Leach and M. C. Nucci; *Reduction of the classical MICZ-Kepler problem to a two-dimensional linear isotropic harmonic oscillator*, J. Math. Phys. **45** (2004), 3590-3604.

[29] S. Lie; *Theorie der Transformationsgruppen. Erster Abschnitt*, Teubner, Leipzig, 1888.

[30] S. Lie; *Theorie der Transformationsgruppen. Zweiter Abschnitt*, Teubner, Leipzig, 1890.

[31] S. Lie; *Theorie der Transformationsgruppen. Dritter und Letzter Abschnitt*, Teubner, Leipzig, 1893.

[32] S. Lie; *Geometrie der Berührungstransformationen*, Teubner, Leipzig, 1896.

[33] S. Lie; *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Teubner, Leipzig, 1912.

[34] E. N. Lorenz; *Deterministic nonperiodic flow*, J. Atmos. Sci. **20** (1963), 130-141.

[35] M. Marcelli and M. C. Nucci; *Lie point symmetries and first integrals: the Kowalevsky top*, J. Math. Phys. **44** (2003), 2111-2132.

[36] J. D. Murray; *Mathematical Biology,* Springer-Verlag, Berlin, 1989.

[37] I. Newton; *Philosophiæ Naturalis Principia Mathematica*, J. Streater, London, 1687.

[38] E. Noether; *Invariante Variationsprobleme*, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse (1918), 235-257 (English translation: E. Noether and M.A. Tavel, Invariant variation problems, Transport Theory and Statistical Mechanics **1** (1971), 183-207).

[39] M. C. Nucci; *The complete Kepler group can be derived by Lie group analysis*, J. Math. Phys. **37** (1996), 1772-1775.

[40] M. C. Nucci; *Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: manual and floppy disk*, in CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends in Theoretical Developments and Computational Methods, ed. N. H. Ibragimov, CRC Press, Boca Raton, 1996, 415-481.

[41] M. C. Nucci; *Lorenz integrable system moves à la Poinsot*, J. Math. Phys. **44** (2003), 4107-4118.

[42] M. C. Nucci; *Calogero's "goldfish" is indeed a school of free particles*, J. Phys. A: Math. Gen. **37** (2004), 11391-11400.

[43] M. C. Nucci; *Jacobi's three-body system moves like a free particle*, J. Nonlinear Math. Phys. **12-s1** (2005), 499-506.

[44] M. C. Nucci; *Jacobi last multiplier and Lie symmetries: a novel application of an old relationship*, J. Nonlinear Math. Phys. **12** (2005), 141-161.

[45] M. C. Nucci and P. G. L. Leach; *The harmony in the Kepler and related problems*, J. Math. Phys. **42** (2001), 746-764.

[46] M. C. Nucci and P. G. L. Leach; *An integrable S-I-S model*, J. Math. Anal. Appl. **290** (2004), 506-518.

[47] M. C. Nucci and P. G. L. Leach; *Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story*, J. Phys. A: Math. Gen. **37** (2004), 7743-7753.

[48] P. J. Olver; *Applications of Lie Groups to Differential Equations*, Springer-Verlag, Berlin, 1986.

[49] L. V. Ovsjannikov; *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[50] H. Pulte; *Jacobi's criticism of Lagrange: the changing role of mathematics in the foundations of classical mechanics*, Historia Math. **25** (1998), 154-184.

[51] M. Rellini and M.C. Nucci; *Application of Lie group analysis to a SIRI disease transmission model*, (2005) in preparation.

[52] C. Rogers and W. F. Ames; *Nonlinear Boundary Value Problems in Science and Engineering*, Academic Press, New York, 1989.

[53] H. Stephani, *Differential Equations. Their Solution Using Symmetries*, Cambridge University Press, Cambridge, 1989.

[54] *V. Torrisi and M. C. Nucci*, Application of Lie group analysis to a mathematical model which describes HIV transmission, in The Geometrical Study of Differential Equations, eds. J. A. Leslie and T. P. Hobart, AMS, Providence, 2001, 31-40.

Dipartimento di Matematica e Informatica, Università di Perugia, 06123 Perugia, Italy

E-mail address: nucci@unipg.it