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CONTINUOUS NEWTON METHOD FOR STAR-LIKE FUNCTIONS

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ABSTRACT. We study a continuous analogue of Newton method for solving the nonlinear equation

$$\varphi(z) = 0,$$

where $\varphi(z)$ holomorphic function and $0 \in \overline{\varphi(D)}$. It is proved that this method converges, to the solution for each initial data $z \in D$, if and only if $\varphi(z)$ is a star-like function with respect to either an interior or a boundary point. Our study is based on the theory of one parameter continuous semigroups. It enables us to consider convergence in the case of an interior as well as a boundary location of the solution by the same approach.

1. RESULTS

Let $D \subset \mathbb{C}$ be a domain (that is, an open connected subset of $\mathbb{C}$). The set of all holomorphic functions on D will be denoted by $\operatorname{Hol}(D, \mathbb{C})$. We consider a nonlinear equation

$$\varphi(z) = 0, \tag{1.1}$$

where $\varphi(z) \in \operatorname{Hol}(D, \mathbb{C})$ and $0 \in \overline{\varphi(D)}$.

In the known Newton's method [6, 7], the solution of (1.1) can be found as a limit of the sequence $\{z_n\}$, $n = 0, 1, 2 \dots$. The first term $z_0 \in D$ is given and other terms are constructed by the iterative process

$$z_{n+1} = z_n - \frac{\varphi(z_n)}{\varphi'(z_n)}. \tag{1.2}$$

It is well known that the convergence of process (1.2) depends on the choice of the initial approximation $z_0 \in D$. If z_0 is chosen arbitrarily, then sequence may diverge.

The Continuous Newton Method (CNM) and its modifications [2, 6] are an alternative approach to the solution of (1.1). The CNM has been considered as the solution of the following Cauchy problem (continuous analogue of process (1.2))

$$\begin{gathered} \frac{\partial u(t,z)}{\partial t} + \frac{\varphi(u(t,z))}{\varphi'(u(t,z))} = 0 \\ u(0,z) = z, \end{gathered} \tag{1.3}$$

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where the initial condition z is some point which belongs to the domain D. A solution of the (1.1) was obtained as the limit

$$\lim_{t\to\infty} u(t,z) = \tau \in \overline{D}, \tag{1.4}$$

where $u(t,z)$ is the solution of (1.3).

The continuous Newton Method has several advantages over the iterative method (1.2) because convergence theorems for CNM usually can be obtained easily. However, as well as in the iterative process (1.2), for its realization in a general case it is necessary to choose an initial condition by a special way.

This problem leads us to the following question: Are there functions $\varphi(z)$ for which the solution of (1.1) can be found by CNM under arbitrary initial condition $z \in D$ in the domain D?

In this article the question is answered. It is proved that star-like functions and only they satisfy this requirement. Our study of CNM is based on results of the theory of one-parameter continuous semigroups (see [10] and the references given there). It has permitted us to consider the convergence of CNM both to an interior point and to a boundary point by the same approach.

In addition the uniqueness of solution of (1.1) in $\overline{D}$ is proved. This solution is obtained as limit (1.4). More exact results are obtained when $D = \Delta$ is an open unit disk in $\mathbb{C}$. In particular, in this case the exponential convergence of CNM is established.

It is important to note one more problem in realization of CNM for the solution of (1.1). As it was mentioned above, $0 \in \overline{\varphi(D)}$. It means that the function φ may have no null point in D. Moreover, φ even may be undefined on the boundary ∂D. Therefore, we consider the solution of (1.1) at the boundary points of domain D in following generalized meaning.

Definition 1.1. A point $\tau \in \partial D$ is said to be a generalized solution of (1.1) on the set $\overline{D}$ if there is a Jordan curve $\gamma \in \overline{D}$ such that $\gamma \cap \partial D = \tau$ and

$$\lim_{z\to\tau,\ z\in\gamma} \varphi(z) = 0. \tag{1.5}$$

Here we give some definitions which will be used in the sequel.

Definition 1.2. We will say that CNM is well defined on the domain D if the Cauchy problem (1.3) has a unique solution for each $z \in D$ and

$$\{u(t,z),\ t \geq 0,\ z \in D\} \subset D.$$

Definition 1.3. We will say that the CNM converges globally in the domain D if the limit (1.4) exists for each solution $u(t,z)$ of Cauchy problem (1.3).

Remark 1.4. The CNM is well defined on the domain D if and only if the function $f(z) = \frac{\varphi(z)}{\varphi'(z)}$ is a generator of a one-parameter continuous semigroup of holomorphic self-mappings $S_f = \{F_t : D \to D,\ t \geq 0\}$, where

$$F_t = \varphi^{-1} \circ e^{-t} \circ \varphi\,, \tag{1.6}$$

$$u(t,z) = F_t(z) = \varphi^{-1}(e^{-t}\varphi(z)) \quad (t \geq 0,\ z \in D) \tag{1.7}$$

is the unique solution of the Cauchy problem (1.3), [9, 10]. In this case the set $\gamma_z(t) = \{u(t,z), t \geq 0\}$ is a Jordan curve for each $z \in D$.

The set of generators of one-parameter semigroups of holomorphic self-mappings in D will be denoted by $\mathcal{G}(D)$.

Definition 1.5. The set $\Omega \subset \mathbb{C}$ is called star-shaped if for any $\omega \in \Omega$, the point $t\omega$ belongs to Ω for every $t \in (0,1]$.

Definition 1.6. *A univalent holomorphic function $f : D \to \mathbb{C}$ is said to be star-like if the set $f(D)$ is star-shaped .*

Remark 1.7. It follows from [1], that univalent function $\varphi(z)$ ($\in \mathrm{Hol}(D,\mathbb{C})$) is a star-like function, if and only if the mapping $f(z) = \frac{\varphi(z)}{\varphi'(z)} \in \mathcal{G}(D)$.

Now we will formulate the main result of this paper.

Theorem 1.8. *Let $\varphi(z)$ ($\in \mathrm{Hol}\,\Delta,\mathbb{C})$) be a univalent function such that $\overline{\varphi(\Delta)} \ni 0$. Then continuous Newton Methos is well defined in Δ if and only if the following inequality holds:*

$$\mathrm{Re}\{\overline{z}\frac{\varphi(z)}{\varphi'(z)}\} \geq (1-|z|^2)\cdot \mathrm{Re}\{\overline{z}\frac{\varphi(0)}{\varphi'(0)}\}, z \in \Delta. \tag{1.8}$$

Moreover, in this case CNM converges globally to a unique point $\tau \in \overline{\Delta}$. In addition,

(i) *If $\tau \in \Delta$ is a solution of (1.1) and $|\tau| \leq \rho < 1$, then*

$$|\tau - u(t,z)| \leq \delta^{-1}\exp\{-\frac{1-|z|}{1+|z|}\delta t\}|\tau - z|, \quad z \in \Delta,\ t \geq 0, \tag{1.9}$$

where $\delta = \frac{1-\rho}{1+\rho}$ and $u(t,z)$ is a solution of Cauchy problem (1.3).

(ii) *If $\tau \in \partial\Delta$ is a generalized solution of (1.1), then the limit*

$$\beta = \lim_{r\to 1^-} \frac{\varphi(r\tau)}{\varphi'(r\tau)(r-1)\tau} > 0 \tag{1.10}$$

exist and

$$|\tau - u(t,z)| \leq \frac{\sqrt{2}e^{-\frac{\beta}{2}t}}{\sqrt{1-z^2}}|\tau - z|, \quad z \in \Delta,\ t \geq 0, \tag{1.11}$$

where $u(t,z)$ is a solution of the Cauchy problem (1.3).

Remark 1.9. As a matter of fact [8], if $\tau \in \partial\Delta$ is a generalized solution of (1.1), then

$$\lim_{z\to\tau,\tau\in\gamma} \varphi(z) = 0$$

along each non-tangential curve γ (i.e. there exists a non-tangential limit at point τ).

The proof of Theorem 1.8 is based on the following result.

Theorem 1.10. *Let D be a bounded domain with Jordan boundary ∂D and $\varphi(z)$ ($\in \mathrm{Hol}(D,\mathbb{C})$) be a univalent function such that $\overline{\varphi(D)} \ni 0$. Then following two conditions are equivalent:*

(i) *$\varphi(z)$ is a star-like function.*
(ii) *The continuous Newton method is well defined in the domain D.*

Moreover, if it is this case, CNM globally converges to a unique point $\tau \in \overline{\Delta}$.

Proof. The equivalence $(i) \Longleftrightarrow (ii)$ follows from Remarks 1.4 and 1.7. Therefore, it is sufficient to prove the latter assertion of this theorem on the global convergence.

Let $\Omega = \varphi(D)$. We will consider following two cases separately: $0 \in \Omega$ and $0 \in \partial\Omega$. If $0 \in \Omega$, then $\tau = \varphi^{-1}(0) \in D$ is a unique solution of (1.1). Since $\varphi^{-1}(z)$ is a continuous function at point 0, we obtain by (1.7) that

$$\lim_{t\to\infty} u(t,z) = \lim_{t\to\infty} \varphi^{-1}(e^{-t}\varphi(z)) = \varphi^{-1}(\lim_{t\to\infty} e^{-t}\varphi(z)) = \varphi^{-1}(0) = \tau$$

for each $z \in D$. So in this case CNM converges globally.

Suppose now, that $0 \in \partial\Omega$. Let $h : D \to \Delta$ be any conformal mapping of D onto unit open disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Then the linear invertible operator $T : \mathrm{Hol}(\Delta,\mathbb{C}) \to \mathrm{Hol}(D,\mathbb{C})$, defined by

$$T(f) = [(h^{-1})']^{-1} f \circ h^{-1} \tag{1.12}$$

is invertible and maps $\mathcal{G}(\Delta)$ onto $\mathcal{G}(D)$ (see [3, 10]). Moreover, if

$$\{F_t : D \to D,\ t \geq 0\} \quad \text{and} \quad \{\Psi_t : \Delta \to \Delta,\ t \geq 0\}$$

are semigroups of holomorphic self-mappings, generated by f and $\psi = T(f)$, respectively (see [10]), then

$$F_t = h^{-1} \circ \Psi_t \circ h\,. \tag{1.13}$$

In the considered case $f(z) = \frac{\varphi(z)}{\varphi'(z)} \in \mathcal{G}(D)$ has no null point in D. It follows from (1.12), that the function $\psi(z)$ has no null point in Δ. Therefore, for each point $z \in \Delta$ there exists a unique limit

$$e = \lim_{t\to\infty} \Psi_t\ (z) \in \partial\Delta\,.$$

In supposition of the theorem the boundary ∂D is a Jordan curve, thus, applying Caratheodory Theorem, we conclude, that the function $h(z)$ has a continuous extension to $D \cup \partial D$, [8]. Therefore $\tau = h^{-1}(e) \in \partial D$ and for any $z \in D$ by (1.13), we have

$$\begin{aligned}
\lim_{t\to\infty}\ u(t,z) &= \lim_{t\to\infty}\ F_t(z) \\
&= \lim_{t\to\infty}\ h^{-1}(\Psi_t(h(z))) \\
&= h^{-1}(\lim_{t\to\infty} \Psi_t(h(z))) \\
&= h^{-1}(e) = \tau\,.
\end{aligned}$$

Further, it follows from (1.7), that for each point $z_0 \in D$ along the curve $\gamma_{z_0}(t) = \{u(t,z_0), t \geq 0\}$,

$$\lim_{z\to\tau} \varphi(z) = \lim_{t\to\infty} \varphi(F_t(z_0)) = \lim_{t\to\infty} \varphi(\varphi^{-1}(e^{-t}(z_0))) = 0\,. \tag{1.14}$$

Thus τ is a generalized solution of equation (1.1) in the set $\overline{D}$. Therefore, to complete our proof in the case $0 \in \partial\Omega$, we need to show the uniqueness of generalized solution τ. Assume, that there exist another generalized solution $\tau_1 \in \partial D$ of (1.1). Then there is a Jordan curve $\gamma_1 \subset \overline{D}$ which begins at some point $z_0 \in D$ such that $\gamma_1 \cap \partial D = \tau_1$ and

$$\lim_{z\to\tau_1,\, z\in\gamma_1} \varphi(z) = 0. \tag{1.15}$$

Since $\gamma_{z_0}(t) = \{u(t,z_0), t \geq 0\}$ is a Jordan curve, then the curve $\gamma = \gamma_1 \cup \gamma_{z_0}(t)$ is Jordan too (see Fig .1).

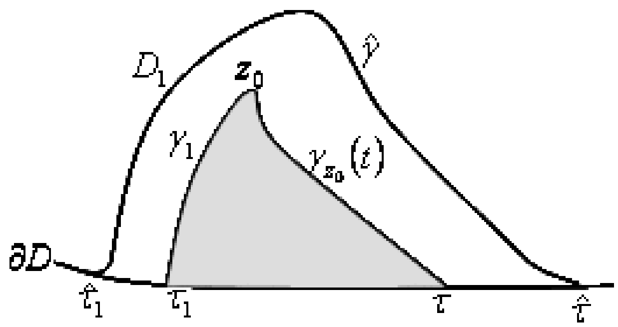

Fig. 1 : The domain D_1

Let the points $\widehat{\tau}$, $\widehat{\tau}_1 \in \partial D$ be different from τ, τ_1. We will use the following notation:

$\widehat{\gamma}$ is some curve which connects the points $\widehat{\tau}$ and $\widehat{\tau}_1$, such that $\widehat{\gamma} \in \overline{D}$, $\gamma \cap \widehat{\gamma} = \emptyset$ and $\widehat{\gamma} \cap \partial D = \{\widehat{\tau}, \widehat{\tau}_1\}$;

λ (respectively λ_1) is the part of boundary ∂D which connects the points τ and $\widehat{\tau}$ (respectively τ_1 and $\widehat{\tau}_1$).

Let D_1 be a domain which boundary is $\partial D_1 = \gamma \cup \lambda \cup \widehat{\gamma} \cup \lambda_1$ and $\Omega_1 = \varphi(D_1)$. Then it follows from (1.14) and (1.15), that the curve $\varphi(\gamma)$ ($\subset \partial\Omega_1$) is closed and $0 \in \varphi(\gamma)$ (see Fig. 2).

Moreover, the domain Ω_1 is placed in the external part of the complex plane $\mathbb{C}$ with respect to the curve $\varphi(\gamma)$. Therefore, there are $\omega \in \Omega_1$ and $t \in (0, 1]$ such that $t\omega \notin \Omega_1$. It means, that Ω_1 is not star-shaped. This contradicts the star-likeness of the function $\varphi(z)$. Thus uniqueness of the generalized solution τ is proved. The Theorem 1.10 is proved. □

Proof of Theorem 1.8. It is proved in [10] that the function $f(z) \in \mathcal{G}(\Delta)$ if and only if the following inequality holds.

$$\mathrm{Re}\{\overline{z}f(z)\} \le (1 - |z|^2) \cdot \mathrm{Re}\{\overline{z}f(0)\}, \quad z \in \Delta.$$

Hence, by Theorem 1.10 and Remark 1.7 we obtain, that inequality (1.8) is equivalent to the assertions (i), (ii) of the Theorem 1.10. Thus the CNM converges globally to a unique solution $\tau \in \overline{\Delta}$ of (1.1). Now we will show that the estimate (1.9) holds.

Since $\tau \in \Delta$ is a solution of (1.1) and $(\frac{\varphi}{\varphi'})'(\tau) = 1$, we obtain by [10] that

$$\Big|\frac{\tau - u(t,z)}{1 - u(t,z)\tau}\Big| \le |M_\tau(z)| \cdot \exp\{-\frac{1 - M_\tau(z)}{1 + M_\tau(z)}t\}, \quad z \in \Delta \tag{1.16}$$

where

$$M_\tau(z) = \frac{\tau - z}{1 - \tau\overline{z}} \tag{1.17}$$

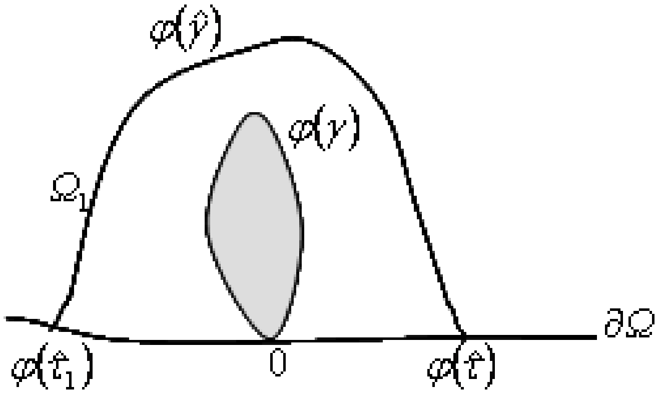

Fig.2 : The domain $\Omega_1 = \varphi(D_1)$

is the Möbius transform of the unit open disk and all values of $M_\tau(z)$ are found in the open disk centered at

$$c = -\frac{1-\rho^2}{1-\rho^2|z|^2} \cdot z \quad \text{with radius} \quad r = \frac{1-\rho^2}{1-\rho^2|z|^2} \cdot \rho$$

Therefore,

$$|M_\tau(z)| \le |c| + \rho \le \frac{|z|+\rho}{1+\rho|z|},$$

and from (1.17), we obtain

$$\frac{1-|M_\tau(z)|}{1+|M_\tau(z)|} \ge [1 - \frac{|z|+\rho}{1+\rho|z|}] \cdot [1 + \frac{|z|+\rho}{1+\rho|z|}]^{-1} = \frac{(1-\rho)(1-|z|)}{(1+\rho)(1+|z|)}.$$

Now, it follows by (1.16) , that

$$|\tau - u(t,z)| \le \frac{|1-u(t,z)\overline{\tau}|}{|1-\tau\overline{z}|} \cdot |\tau - z| \exp\{-\frac{(1-\rho)(1-|z|)}{(1+\rho)(1+|z|)} t\}.$$

Since

$$\frac{|1-u(t,z)\overline{\tau}|}{|1-\tau\overline{z}|} \le \frac{(1+\rho)}{(1-\rho)},$$

then we obtain that estimate (1.9) holds.

Now we will prove assertion (ii). It is known that $\varphi(z)$ is a star-like function, therefore the function $\varphi_\tau(z) = \varphi(\tau z)$ is star-like too and

$$f_\tau(z) = \frac{\varphi_\tau(z)}{(\varphi_\tau(z))'} = \frac{\varphi(\tau z)}{\tau\varphi'(\tau z)} \in \mathcal{G}(\Delta).$$

Then, it follows by [5], that

$$\lim_{r \to 1^-} \frac{f_\tau(z)}{r-1} = \beta > 0,$$

(i.e. (1.10) holds), and there exist following representation of function $\varphi_\tau(z)$:

$$\varphi_\tau(z) = \frac{(1-z)^2}{z} \cdot q_\tau(z),$$

where $q_\tau(z)$ is a star-like function, such that $q_\tau(0) = 0$. Now, by using (1.15) and the dynamical extension of the Julia-Wolf-Caratheodory Theorem given in [5], we obtain (1.11). Then Theorem 1.8 is proved. □

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