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ON THE ISOSPECTRAL BEAMS

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ABSTRACT. The free undamped infinitesimal transverse vibrations of a thin straight beam are modelled by a forth-order differential equation. This paper investigates the families of fourth-order systems which have one spectrum in common, and correspond to four different sets of end-conditions. The analysis is based on the transformation of the beam operator into a fourth-order self-adjoint linear differential operator. This operator is factorized as a product $L = H^*H$, where H is a second-order differential operator of the form $H = D^2 + rD + s$, and H^* is its adjoint operator.

1. BEAM EQUATION

The free undamped infinitesimal transverse vibrations, of frequency ω, of a thin straight beam of length ℓ are governed by the Euler-Bernoulli equation

$$(EI(x)u''(x))'' = \omega^2 \rho A(x)u(x), \quad 0 \le x \le \ell. \tag{1.1}$$

Here E is the Young's modulus, ρ is the density, both assumed constant, $A(x)$ is the cross-sectional area at x, $I(x)$ is the second moment of this area about the axis through the centroid at right angle to the plane of vibration (the natural axis). Euler-Bernoulli equation (1.1) can be written even in simpler form by the following substitutions. Put

$$x = \ell s, \quad y(s) = u(x), \quad r(s) = \frac{I(x)}{I(x_0)},$$
$$a(s) = \frac{A(x)}{A(x_0)}, \quad \lambda = \frac{\rho \ell^4 \omega^2 A(x_0)}{EI(x_0)},$$

where x_0 is a fixed point in $[0, 1]$. Then (1.1) becomes

$$(r(s)y''(s))'' = \lambda a(s)y(s), \quad 0 \le s \le 1, \tag{1.2}$$

where $r(s), a(s) > 0$ for all $s \in [0, 1]$. For the beam equation (1.2) the most commonly used end-conditions are

(F) $y'' = 0 = (ry'')'$ (Free)
(C) $y = 0 = y'$ (Clamped)
(P) $y = 0 = y''$ (Pinned)

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(S) $y' = 0 = (ry'')'$ (Sliding).

By using a transformation, we can obtain another form of equation (1.2) which is useful in practice. This transformation is as follows:

$$z = \int_s^1 \big[\frac{a(t)}{r(t)}\big]^{1/4} dt, \quad b(z) = \big[\frac{a(s)}{r(s)}\big]^{1/4}, \quad c^2(z) = [r^3(s)a(s)]^{1/4}. \tag{1.3}$$

We have

$$\frac{d}{ds} = \frac{d}{dz}\frac{dz}{ds} = -\big[\frac{a(s)}{r(s)}\big]^{1/4}\frac{d}{dz} = -b(z)\frac{d}{dz},$$

and

$$r(s)\frac{d^2}{ds^2} = r(s)b(z)\frac{d}{dz}\big[b(z)\frac{d}{dz}\big] = c^2(z)\frac{d}{dz}\big[b(z)\frac{d}{dz}\big].$$

Therefore, the (1.2) becomes

$$b\frac{d}{dz}\big\{b\frac{d}{dz}\big[c^2\frac{d}{dz}\big(b\frac{dy}{dz}\big)\big]\big\} = \lambda b^3c^2y,$$

which can be written as

$$\big(b\big(c^2(by'(z))'\big)'\big)' = \lambda b^2c^2y(z), \quad 0 \le z \le L, \tag{1.4}$$

where $L = \int_0^1 [\frac{a(t)}{r(t)}]^{1/4}dt$. For more details see [1] or [3]. Equation (1.2) can be written in the form $v^{(4)} + (Av')' + Bv = \lambda v$ as follows. Put $y = \frac{v}{bc}$, then

$$by' = b\big[\frac{v'}{bc} - \frac{v}{b^2c^2}(bc' + b'c)\big].$$

If we put $\frac{b'}{b} = \beta$ and $\frac{c'}{c} = \gamma$, then after some algebraic calculations we find

$$\begin{aligned}
by' &= \frac{1}{c}[v' - (\beta+\gamma)v] \\
(by')' &= \frac{1}{c}[v'' - (\beta+\gamma)v' - (\beta'+\gamma')v - \gamma v' + \gamma(\beta+\gamma)v] \\
(c^2(by')')' &= c\{v''' - (\beta+\gamma)v'' - [2\beta' + 3\gamma' + \gamma^2]v' - (\beta''+\gamma'')v \\
&\quad + [\gamma(\beta+\gamma)' + \gamma'(\beta+\gamma) - \gamma(\beta+\gamma)' + \gamma^2(\beta+\gamma)]v\} \\
\big(b\big(c^2(by')'\big)'\big)' &= bc\{v^{(4)} - [(\beta+\gamma)' + 2\beta' + 3\gamma' + \gamma^2 + (\beta+\gamma)^2]v'' \\
&\quad - [2\gamma\gamma' + \beta'' + \gamma'' - \gamma'(\beta+\gamma) - \gamma^2(\beta+\gamma)]v' \\
&\quad - [2\beta'' + 3\gamma'' + 2\beta'(\beta+\gamma) + 3\gamma'(\beta+\gamma) + \gamma^2(\beta+\gamma)]v' \\
&\quad + [\gamma'(\beta+\gamma)' + \gamma''(\beta+\gamma) + [\gamma^2(\beta+\gamma)]']v \\
&\quad - [\beta''' + \gamma''' + (\beta+\gamma)(\beta''+\gamma'')]v \\
&\quad + [\gamma'(\beta+\gamma)^2 + \gamma^2(\beta+\gamma)^2]v\}.
\end{aligned} \tag{1.5}$$

It is clear that clamped is the only end-condition that remains invariant under the transformation (3), i.e., $v = 0 = v'$. By using (5) we can write the equation (4) in v as follows:

$$v^{(4)} + (Av')' + Bv = \lambda v, \tag{1.6}$$

where

$$A = -[3\beta' + 4\gamma' + \gamma^2 + (\beta+\gamma)^2], \tag{1.7}$$

$$\begin{aligned} B = &-(\beta+\gamma)''' - \beta''(\beta+\gamma) + \gamma'(\beta+\gamma)' + \gamma'(\beta+\gamma)^2 \\ &+ [\gamma^2(\beta+\gamma)]' + \gamma^2(\beta+\gamma)^2. \end{aligned} \tag{1.8}$$

Put $\theta = \beta + \gamma$ and $\phi = \gamma' + \gamma^2$, then A and B can be written as follows

$$\begin{aligned} A &= -3\theta' - \theta^2 - \phi \\ B &= -\theta''' - (\theta'' - \gamma'')\theta + \gamma'\theta' + \gamma'\theta^2 + (\gamma^2\theta)' + \gamma^2\theta^2 \\ &= -\theta''' - \theta''\theta + (\gamma'\theta)' + \gamma'\theta^2 + (\gamma^2\theta)' + \gamma^2\theta^2 \\ &= -\theta''' - \theta''\theta + [(\gamma' + \gamma^2)\theta]' + (\gamma' + \gamma^2)\theta^2 \\ &= -\theta''' - \theta''\theta + (\theta\phi)' + \theta^2\phi \\ &= (-\theta'' + \theta\phi)' + (-\theta'' + \theta\phi)\theta. \end{aligned} \tag{1.9}$$

If we substitute $\beta = \frac{b'}{b}$ and $\gamma = \frac{c'}{c}$ back in (1.7) and (1.8) then A and B can be written in terms of b, c and their derivatives as follows:

$$A = 2c'^2/c^2 - 4c''/c - 3b''/b + 2b'^2/b^2 - 2b'c'/bc \tag{1.10}$$

$$\begin{aligned} B = &\, 4b'^4/b^4 - c^{(4)}/c + 4c'^4/c^4 + 4c''^2/c^2 - b^{(4)}/b \\ &+ 3b''^2/b^2 - 10c'^2c''/c^3 + 4c'c'''/c^2 + 3b'b'''/b^2 \\ &- 9b''b'^2/b^3 - 2b'c'^3/bc^3 + 3b''b'c'/b^2c - c'b'''/bc \\ &- 2b'^3c'/b^3c + 4b'c'c''/bc^2 + b''c''/bc. \end{aligned} \tag{1.11}$$

2. Isospectral rods

The linear, free, undamped longitudinal vibrations of a thin elastic rod with variable cross section $A = A(x)$ are governed by the equation

$$(Au')' + \lambda Au = 0. \tag{2.1}$$

Gladwell and Morassi [2] found isospectral rods, rods with identical spectrum, by using the Darboux lemma. In this section we apply a different method, namely factorizing the rod operator, to find the the isospectral rods. Then we develop this method to find isospectral beams in the following section. This method first was proposed by Pöchel J and Trubowitz E [7] to Sturm-Liouville operator. The rod equation (2.1) can be written in the Sturm-Liouville form. The cross section function A is positive; write $A = a^2$, $y = au$. Then

$$Au' = a^2u' = ay' - a'y$$

so that (2.1) reduces to the Sturm-Liouville form

$$-y'' + qy = \lambda y, \quad \text{where } q = \frac{a''}{a}. \tag{2.2}$$

Define the rod operator

$$L = -D^2 + q. \tag{2.3}$$

We want to write the rod operator L as a product of two first order differential operators. Every linear differential operator of the form

$$H(v) = \sum_{k=0}^{n} p_k(x) D^k(v) \tag{2.4}$$

has the adjoint operator H^* of the form

$$H^*(u) = \sum_{k=0}^{n} D^k \left[(-1)^k p_k(x) u\right]. \tag{2.5}$$

See for example Lanczos [6]. Therefore the adjoint operator of $H = D + \alpha$ is $H^* = -D + \alpha$, where α is a real function. Now consider the rod operator L given by (2.3). We want to write L as a product of the form $L = H^*H$, where $H = D+\alpha$. Let

$$L = -D^2 + q = -D^2 + \frac{a''}{a} = H^*H = (-D+\alpha)(D+\alpha).$$

Thus

$$q = \alpha^2 - \alpha' = \frac{a''}{a}. \tag{2.6}$$

The operator $L = H^*H$ and $\hat{L} = HH^*$ are isospectral. Now let

$$\hat{L} = HH^* = (D+\alpha)(-D+\alpha) = -D^2 + \hat{q} = -D^2 + \frac{\hat{a}''}{\hat{a}}.$$

Hence

$$\hat{q} = \alpha^2 + \alpha'. \tag{2.7}$$

Substituting the solution $\alpha = -\frac{a'}{a}$ in (2.7) yields

$$\hat{q} = 2\frac{a'^2}{a^2} - \frac{a''}{a}. \tag{2.8}$$

On the other hand

$$\hat{q} = \frac{\hat{a}''}{\hat{a}}.$$

In order to determine the isospectral rods we need the general solution of the following differential equation

$$\frac{\hat{a}''}{\hat{a}} = 2\frac{a'^2}{a^2} - \frac{a''}{a}. \tag{2.9}$$

The general solution of the equation (2.9) is

$$\hat{a} = \frac{1}{a}\{1 + K\int_0^x a^2(s)ds\}, \quad \text{where } K \in \mathbb{R}.$$

This family of isospectral rods coincide with those obtained by Gladwell and Morassi [2].

Remark 2.1. In general if we consider the Sturm-Liouville operator L given by (2.3), using a similar argument we can factor $L - \mu$ as follows

$$L - \mu = H^*H = (-D - \frac{g'}{g})(D - \frac{g'}{g}), \tag{2.10}$$

where g is a nonzero solution of the Sturm-Liouville equation, i.e.,

$$-g'' + (q - \mu)g = 0.$$

If we reverse the factors in (2.10) then we obtain the isospectral Sturm-Liouville operator

$$\hat{L} - \mu = HH^* = -D^2 + (\hat{q} - \mu), \tag{2.11}$$

where $\hat{q} = q - 2\frac{d^2}{dx^2}(\ln g)$. For more details see [7].

3. Isospectral beams

According to the beam equation (1.6) we define the beam operator as follows

$$L = H^*H = D^4 + D(AD) + B, \tag{3.1}$$

where A and B are given by (1.7) and (1.8). Using (2.4) and (2.5) it can be easily checked that the adjoint operator of a general operator of the form $H = aD^2 + bD + c$ is $H^* = aD^2 + (2a' - b)D + (a'' - b' + c)$. The idea is to find the isospectral beams by factorizing the beam operator L and then reversing the factors to find the isospectral beams. We therefore suppose that L can be factorized as follows

$$L = H^*H = [D^2 - rD + (s - r')][D^2 + rD + s]$$

that is equivalent to the nonlinear system:

$$\{2s + r' - r^2 = A,\ s^2 + s'' - (rs)' = B\}.$$

Define $\hat{L} = HH^*$. Then L and $\hat{L}$ are isospectral and we have

$$L = H^*H = D^4 + D(AD) + B,\ \hat{L} = HH^* = D^4 + D(\hat{A}D) + \hat{B}, \tag{3.2}$$

where

$$\begin{aligned} A &= 2s + r' - r^2 \\ B &= s^2 + s'' - (rs)'\,, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \hat{A} &= 2s - 3r' - r^2 \\ \hat{B} &= s^2 + s'' - r''' - rr'' + rs' - sr'. \end{aligned} \tag{3.4}$$

Let us call the system (3.3) the *principal* system because the solutions of (3.3) will produce isospectral beams. It is easy to verify that $r = -\beta - 2\gamma$, and $s = \gamma^2 + \beta\gamma - \beta' - \gamma'$ is a solution to the nonlinear system (3.3). Comparing (3.3) and (3.4) we obtain

$$A - \hat{A} = 4r', B - \hat{B} = r''' + rr'' - 2rs'. \tag{3.5}$$

Substituting $r = -\beta - 2\gamma$, and $s = \gamma^2 + \beta\gamma - \beta' - \gamma'$ in (3.5) we find

$$\hat{A} = \beta' + 4\gamma' - \beta^2 - 2\gamma^2 - 2\beta\gamma, \tag{3.6}$$

and

$$\begin{aligned} \hat{B} = {}& \gamma''' + \beta''\gamma + \gamma'(\beta + \gamma)' - 4\beta\gamma\gamma' - \beta^2\gamma' - 4\gamma'\gamma^2 \\ & - 3\beta'\gamma^2 - 2\beta\beta'\gamma + \gamma^2(\beta + \gamma)^2. \end{aligned} \tag{3.7}$$

It is clear that

$$\begin{aligned} \hat{A}(\beta, \gamma) &= A(\beta, -\beta - \gamma), \\ \hat{B}(\beta, \gamma) &= B(\beta, -\beta - \gamma). \end{aligned}$$

This implies that $\hat{\beta} = \beta$ and $\hat{\gamma} = -\beta - \gamma$. Therefore, $\hat{b} = b$ and $\hat{c} = \frac{1}{bc}$.

Remark 3.1. We can verify this isospectral family as follows. Consider the original beam equation

$$\big(b\big(c^2(by')'\big)'\big)' = \lambda b^2c^2y.$$

Put $H_1 = \frac{1}{b^2c^2}(DbD)$ and $H_2 = c^2(DbD)$. Then

$$\big(b\big(c^2(by')'\big)'\big)' = \lambda b^2c^2y \Longleftrightarrow H_1H_2(y) = \lambda y.$$

Therefore $H_2H_1(H_2y) = \lambda(H_2y)$, which can be written as $H_2H_1(v) = \lambda v$, where $v = H_2y$.

Remark 3.2. Similar to Remark 2.1, if we factorize the operator $L-\mu$ instead the beam operator L given by (3.1) then the corresponding principal system will be

$$\begin{gathered} 2s + r' - r^2 = A \\ s^2 + s'' - (rs)' = B - \mu = -\frac{g^{(4)}}{g} - \frac{(Ag')'}{g} \end{gathered} \tag{3.8}$$

where g is a nonzero solution of the beam equation, i.e.,

$$g^{(4)} + (Ag')' + (B-\mu)g = 0.$$

In contrast to the principal system (3.3), the system (3.8) does not seem to be easy to solve. The operator $\hat{L}$ coincide with the beam operator given by Gottlieb [5]. There he finds seven classes of beams isospectral to the standard unit coefficient beam equation

$$D^4v = \lambda v. \tag{3.9}$$

Since L and $\hat{L}$ are isospectral, by using the operator L we can find another seven class of beams isospectral to the standard unit-coefficient beam. Referring to Gottlieb's notations we define

$$b = \eta(\delta z + \epsilon)^{\mu}, \quad c = \zeta(\delta z + \epsilon)^{\nu}, \tag{3.10}$$

where $\delta, \epsilon, \eta, \zeta$ are constants, and μ, ν are parameters to be determined from the expressions of A, B in terms of b, c and their derivatives given by (1.10) and (1.11). After some algebraic calculations we find

$$A = -\delta^2(\delta z + \epsilon)^{-2}[\mu^2 + 2\mu\nu + 2\nu^2 - 4\nu - 3\mu] \tag{3.11}$$

$$\begin{aligned} B = \delta^4(\delta z + \epsilon)^{-4}&[\nu^4 - 4\nu^3 + \mu^2\nu^2 + \nu^2] \\ &\times \big[2\mu\nu^3 - 2\mu^2 - \nu\mu^2 - \mu\nu - 5\mu\nu^2 + 6(\mu+\nu)\big]. \end{aligned} \tag{3.12}$$

Using the transformation given by (1.3), we get

$$\begin{aligned} 1 - s &= \int_0^z \frac{dz}{b(z)} \\ &= \int_0^z \frac{1}{\eta}(\delta z + \epsilon)^{-\mu} dz \\ &= \frac{1}{\eta(1-\mu)}(\delta z + \epsilon)^{1-\mu} - \frac{1}{\eta(1-\mu)}\epsilon^{1-\mu}. \end{aligned} \tag{3.13}$$

Now put

$$Z = \delta z + \epsilon, \quad \text{and} \quad \xi(s) = \epsilon^{1-\mu} + \eta(1-\mu)(1-s). \tag{3.14}$$

Then $Z = [\xi(s)]^{\chi}$, where $\chi = \frac{1}{1-\mu}$, and hence

$$z(s) = \delta^{-1}\left([\xi(s)]^{\chi} - \epsilon\right). \tag{3.15}$$

Thus

$$b(z(s)) = \eta[\xi(s)]^{\lambda}, \quad \text{with } \lambda = \frac{\mu}{1-\mu} \tag{3.16}$$

$$c(z(s)) = \zeta[\xi(s)]^{\kappa}, \quad \text{with } \kappa = \frac{\nu}{1-\mu}. \tag{3.17}$$

Therefore, the seven class of beams isospectral to the standard unit-coefficient beam is corresponding to $A = B = 0$; so that we can determine these classes by solving the algebraic equations

$$\begin{gathered} \mu^2 + 2\mu\nu + 2\nu^2 - 4\nu - 3\mu = 0 \\ \nu^4 - 4\nu^3 + \mu^2\nu^2 + \nu^2 + 2\mu\nu^3 - 2\mu^2 - \nu\mu^2 - \mu\nu - 5\mu\nu^2 + 6(\mu+\nu) = 0 \end{gathered} \tag{3.18}$$

which gives the solutions cited in the following table:

class	μ	ν	χ	λ	κ
(0)	0	0	1	0	0
(1)	0	2	1	0	2
(2)	-1	2	$1/2$	$-1/2$	1
(3)	-1	1	$1/2$	$-1/2$	$1/2$
(4)	2	1	-1	-2	-1
(5)	2	-1	-1	-2	1
(6)	3	-1	$-1/2$	$-3/2$	$1/2$
(7)	3	0	$-1/2$	$-3/2$	0

More isospectral beams. For this purpose, we put $\theta = \beta + \gamma$ and $\phi = \gamma' + \gamma^2$. Then (1.9) implies

$$\begin{gathered} A = A(\theta, \phi) = -3\theta' - \theta^2 + \phi \\ B = B(\theta, \phi) = (-\theta'' + \theta\phi)' + (-\theta'' + \theta\phi)\theta. \end{gathered} \tag{3.19}$$

Let $\tilde{A} = A(\tilde{\theta}, \tilde{\phi})$, and $\tilde{B} = B(\tilde{\theta}, \tilde{\phi})$. If $A = \tilde{A}$ and $B = \tilde{B}$ then L and $\tilde{L}$ are isospectral, where L is given by (3.1) and

$$\tilde{L} = D^4 + D(\tilde{A}D) + \tilde{B}. \tag{3.20}$$

Let $\theta = \tilde{\theta}$, $\phi = \tilde{\phi}$, then $A = \tilde{A}$ and $B = \tilde{B}$. Thus we obtain isospectral class as follows

$$\theta = \tilde{\theta} \Longleftrightarrow bc = \tilde{b}\tilde{c}. \tag{3.21}$$

Now the equation $\phi = \tilde{\phi}$ implies

$$\frac{c''}{c} = \frac{\tilde{c}''}{\tilde{c}} \Longleftrightarrow c''\tilde{c} - \tilde{c}''c = 0 \Longleftrightarrow c'\tilde{c} - \tilde{c}'c = \text{const}. \tag{3.22}$$

This equation implies that for arbitrary real constants K_1, K_2 we find

$$\tilde{c} = c \int \frac{K_1}{c^2} ds + K_2 c.$$

Now with a similar way we can find another family isospectral to $\hat{L}$ which is isospectral to L. Put $\phi = \theta^2 - \theta'$. After some algebraic computation we can write $\hat{A}$ and $\hat{B}$ as follows

$$\begin{gathered} \hat{A} = 3\gamma' - \gamma^2 - \phi \\ \hat{B} = (\gamma'' - \gamma\phi)' - \gamma(\gamma'' - \gamma\phi). \end{gathered} \tag{3.23}$$

If $\gamma = \tilde{\gamma}$ and $\phi = \tilde{\phi}$ then $\hat{L}$ and $\tilde{\hat{L}}$ are isospectreal. The equation $\gamma = \tilde{\gamma}$ implies $c = \tilde{c}$. Now we find $\tilde{b}$ by solving the equation $\phi = \tilde{\phi}$. We have

$$\phi = \tilde{\phi} \Longleftrightarrow (\tilde{\theta} - \theta)' = \tilde{\theta}^2 - \theta^2 \Longleftrightarrow \frac{(\tilde{\theta} - \theta)'}{\tilde{\theta} - \theta} = \tilde{\theta} + \theta.$$

This equation implies that $\tilde{\theta} - \theta = bc\tilde{b}\tilde{c} = b\tilde{b}c^2$, which is equivalent to

$$\frac{\tilde{b}'}{\tilde{b}} - \frac{b'}{b} = b\tilde{b}c^2 \Longleftrightarrow \tilde{b}'b - b'\tilde{b} = (b\tilde{b})^2 c^2 \Longleftrightarrow \Big(\frac{\tilde{b}}{b}\Big)' = \tilde{b}^2 c^2.$$

This equation can be easily solved. In fact, putting $u = \frac{\tilde{b}}{b}$, the above equation becomes

$$u' = u^2 b^2 c^2.$$

By solving this differential equation, we find

$$\tilde{b} = \frac{b}{K_1 - \int b^2 c^2 ds}, \quad \text{where } K_1 \in \mathbb{R}.$$

Suppose

$$b = c^2 \Big(\int \frac{K_1}{c^2} ds + K_2 \Big).$$

Then we can define another isospectral class as follows: Define $\tilde{b} = c^2$ and $\tilde{c} = \sqrt{b}$. Then $bc^2 = \tilde{b}\tilde{c}^2$. Hence $r = \tilde{r}$ where $r = \beta + 2\gamma$. It is clear that $s = \tilde{s}$, thus we have $A = \tilde{A}$ and $B = \tilde{B}$ according to the principal system (3.3).

Conclusion. Factoring the beam operator $L = D^4 + D(AD) + B$ into the product of the form $L = HH^*$, where H is a second order differential operator of the form $H = D^2 + rD + s$ and H^* is the adjoint operator of H, and reversing the order of this product we managed to find some 12 classes of isospectral beams. However, determining all classes is linked to nonlinear system (3.3) for which we found particular solution.

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