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FUNCTIONAL DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. In this paper, we consider the third-order neutral functional differential equation with distributed deviating arguments. We give sufficient conditions for the oscillatory behavior of this functional differential equation.

1. Introduction

The aim of this paper is to develop some oscillation theorems for a third-order equations of the form

$$[a(t)[b(t)[x(t)+c(t)x(t-\tau)]']']' + \int_a^b p(t,\xi)x(\sigma(t,\xi))d\xi = 0, \tag{1.1}$$

where

- (a) $a(t), a'(t), b(t), c(t) \in C([t_0,\infty),(0,\infty))$, $0 < c(t) \le 1$, $a'(t) \ge 0$;
- (b) $\int^\infty \frac{dt}{b(t)} = \infty$ and $\int^\infty \frac{dt}{a(t)} = \infty$
- (c) $p(t,\xi) \in C([t_0,\infty)\times[a,b],[0,\infty))$, and $p(t,\xi)$ is not eventually zero on any half line $[t_m,\infty)\times[a,b]$, $t_m \ge t_0$
- (d) $\sigma(t,\xi) \in C([t_0,\infty)\times[a,b],R)$, $\sigma(t,\xi)+\tau \le t$, $\sigma(t,\xi)$ is nondecreasing with respect to t and ξ, respectively, and $\liminf_{t\to\infty\, \xi\in[a,b]} \sigma(t,\xi) = \infty$.

As is customary, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

Oscillatory solutions of third-order differential equations

$$(r_2(t)(r_1(t)x'(t))')' + q(t)f(x(g(t))) = h(t),$$

and

$$(b(t)(a(t)y'(t))')' + (q_1(t)y)' + q_2(t)y' = f(t)$$

were considered in [10] and [4], respectively. We refer to [1]-[3] and [7]-[9] for more studies. However, our results are more general with different proofs than those works.

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2. Main Results

Let $D_0 = \{(t,s)|t > s \geq t_0\}$, $D = \{(t,s)|t \geq s \geq t_0\}$.

Theorem 2.1. *Suppose that there exist* $\frac{d}{dt}\sigma(t,a)$ *and* $H(t,s) \in C'(D;R)$, $h(t,s) \in C(D_0;R)$ *and* $\rho(t) \in C'([t_0,\infty),(0,\infty))$ *such that*

(i) $H(t,t) = 0$, $H(t,s) > 0$

(ii) $H'_s(t,s) \leq 0$, *and* $-H'_s(t,s) - H(t,s)\frac{\rho'(s)}{\rho(s)} = h(t,s)\sqrt{H(t,s)}$.

If

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[& H(t,s)\rho(s) \int_a^b p(s,\xi)[1 - c(\sigma(s,\xi))]d\xi \\ & - \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a) - T]\sigma'(s,a)} \Big] ds = \infty \end{aligned} \tag{2.1}$$

and

$$\int_{t_3}^{\infty} \Big[\int_{t_3}^{r} \frac{du}{b(u)} \Big(\int_u^r \frac{dv}{a(v)} \Big) \Big] \int_a^b p(r,\xi)d\xi dr = \infty. \tag{2.2}$$

Then every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Assume, for the sake of contradiction, that equation (1.1) has an eventually positive solution $x(t)$. That is, there exists a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$. If we put

$$y(t) = x(t) + c(t)x(t - \tau) \tag{2.3}$$

then, from (1.1), we obtain

$$[a(t)[b(t)y'(t)]']' = -\int_a^b p(t,\xi)x(\sigma(t,\xi))d\xi. \tag{2.4}$$

Since $x(t)$ is an eventually positive solution of (1.1) and $\sigma(t,\xi) \to \infty$ as $t \to \infty$, $\xi \in [a,b]$, there exist a $t_1 \geq t_0$ such that $x(t-\tau) > 0$ and $x(\sigma(t,\xi)) > 0$ for $t \geq t_1$, $\xi \in [a,b]$. Thus in view of (2.3) and (2.4), we have $y(t) > 0$, $t \geq t_1$ and $[a(t)[b(t)y'(t)]']' \leq 0$, $t \geq t_1$. Thus, $y(t)$, $y'(t)$, $(b(t)y'(t))'$ are monotone and eventually one-signed. We want to show that there is a $t_2 \geq t_1$ such that

$$(b(t)y'(t))' > 0, \quad t \geq t_2. \tag{2.5}$$

Suppose on the contrary, $(b(t)y'(t))' \leq 0$. Since the right hand side of (2.4) is not identically zero and $a(t) > 0$, it is clear that there exists a $t_3 \geq t_2$ such that $a(t_3)(b(t_3)y'(t_3))' < 0$. Then we have

$$a(t)(b(t)y'(t))' \leq a(t_3)(b(t_3)y'(t_3))' < 0, \quad t \geq t_3. \tag{2.6}$$

Dividing (2.6) by $a(t)$ and integrating from t_3 to t, we obtain

$$b(t)y'(t) - b(t_3)y'(t_3) \leq a(t_3)(b(t_3)y'(t_3))' \int_{t_3}^{t} \frac{ds}{a(s)}. \tag{2.7}$$

Letting $t \to \infty$ in (2.7), and because of (b) we see that $b(t)y'(t) \to -\infty$ as $t \to \infty$. Thus, there is a $t_4 \geq t_3$ such that $b(t_4)y'(t_4) < 0$. By making use of $(b(t)y'(t))' \leq 0$, we obtain

$$b(t)y'(t) \leq b(t_4)y'(t_4) < 0, \quad t \geq t_4. \tag{2.8}$$

If we divide (2.8) by $b(t)$ and integrate from t_4 to t with $t \to \infty$, the right-hand side becomes negative. Thus, we have $y(t) \to -\infty$. But this is a contradiction, since

$y(t)$ is eventually positive, which therefore proves that (2.5) holds. Now we have two possibilities: (I) $y'(t) > 0$ for $t \geq t_2$, (II) $y'(t) < 0$ for $t \geq t_2$.

(I) Assume $y'(t) > 0$ for $t \geq t_2$. From (2.3), $y(t) \geq x(t)$ for $t \geq t_2$ and

$$y(\sigma(t,\xi)) \geq y(\sigma(t,\xi)-\tau) \geq x(\sigma(t,\xi)-\tau), \quad t \geq t_3 \geq t_2.$$

Thus from (2.3) and (2.4),

$$\begin{aligned}
[a(t)[b(t)y'(t)]']' &= -\int_a^b p(t,\xi)[y(\sigma(t,\xi)) - c(\sigma(t,\xi))x(\sigma(t,\xi)-\tau)]d\xi \\
&\leq -\int_a^b p(t,\xi)[y(\sigma(t,\xi)) - c(\sigma(t,\xi))y(\sigma(t,\xi))]d\xi \\
&= -\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]y(\sigma(t,\xi))d\xi \\
&\leq -y(\sigma(t,a))\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]d\xi.
\end{aligned}$$

Then, we have

$$\frac{[a(t)[b(t)y'(t)]']'}{y(\sigma(t,a))} \leq -\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]d\xi.$$

Now set

$$z(t) = \frac{a(t)(b(t)y'(t))'\rho(t)}{y(\sigma(t,a))}.$$

It is obvious that $z(t) > 0$ for $t \geq t_3$ and the derivative of $z(t)$ is

$$\begin{aligned}
z'(t) &= \frac{[a(t)(b(t)y'(t))']'\rho(t)}{y(\sigma(t,a))} + \frac{\rho'(t)}{\rho(t)}z(t) - \frac{[a(t)(b(t)y'(t))']\rho(t)y'(\sigma(t,a))\sigma'(t,a)}{y^2(\sigma(t,a))} \\
&\leq -\rho(t)\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]d\xi + \frac{\rho'(t)}{\rho(t)}z(t) - \frac{y'(\sigma(t,a))\sigma'(t,a)z(t)}{y(\sigma(t,a))}.
\end{aligned} \tag{2.9}$$

On the other hand, since $(a(t)(b(t)y'(t))')' \leq 0$ and $a'(t) \geq 0$, we find that

$$(b(t)y'(t))'' \leq 0. \tag{2.10}$$

Using the above inequality and

$$b(t)y'(t) = b(T)y'(T) + \int_T^t (b(s)y'(s))'ds,$$

we obtain

$$b(t)y'(t) \geq (t-T)(b(t)y'(t))', \quad t \geq T \geq t_3.$$

Since $(by')'$ is non-increasing, we have

$$b(\sigma(t,a))y'(\sigma(t,a)) \geq (\sigma(t,a)-T)(b(t)y'(t))', \quad t \geq t_4 \geq t_3.$$

Thus, we have

$$y'(\sigma(t,a)) \geq \frac{(\sigma(t,a)-T)(b(t)y'(t))'}{b(\sigma(t,a))}. \tag{2.11}$$

Then, substituting (2.11) in (2.9), it follows that

$$z'(t) \leq -\rho(t)\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]d\xi + \frac{\rho'(t)}{\rho(t)}z(t) - \frac{[\sigma(t,a)-T]\sigma'(t,a)z^2(t)}{a(t)\rho(t)b(\sigma(t,a))},$$

and

$$\rho(t)\int_a^b p(t,\xi)[1-c(\sigma(t,\xi))]d\xi \le -z'(t)+\frac{\rho'(t)}{\rho(t)}z(t)-\frac{[\sigma(t,a)-T]\sigma'(t,a)z^2(t)}{a(t)\rho(t)b(\sigma(t,a))}. \tag{2.12}$$

Multiplying both sides of equation (2.12) by $H(t,s)$, and integrating by parts from T^* to t, and using the properties (i) and (ii), we obtain

$$\begin{aligned}
&\int_{T^*}^t H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi ds\\
&\le -\int_{T^*}^t H(t,s)z'(s)ds+\int_{T^*}^t \frac{H(t,s)\rho'(s)z(s)}{\rho(s)}ds\\
&\quad -\int_{T^*}^t \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))}ds\\
&= H(t,T^*)z(T^*)+\int_{T^*}^t \Big[\frac{dH(t,s)}{ds}+H(t,s)\frac{\rho'(s)}{\rho(s)}\Big]z(s)ds\\
&\quad -\int_{T^*}^t \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))}ds\\
&= H(t,T^*)z(T^*)-\int_{T^*}^t h(t,s)\sqrt{H(t,s)}z(s)ds\\
&\quad -\int_{T^*}^t \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))}ds\\
&= H(t,T^*)z(T^*)-\int_{T^*}^t \Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s)\\
&\quad +\frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2 ds+\int_{T^*}^t \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}ds,
\end{aligned}$$

$t>T^*\ge t_4$. As a result of this, we get

$$\begin{aligned}
&\int_{T^*}^t \Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds\\
&= H(t,T^*)z(T^*)-\int_{T^*}^t \Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s)\\
&\quad +\frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2 ds\,.
\end{aligned}$$

From (ii) $H_s'(t,s)\le 0$, we have $H(t,t_4)\le H(t,t_0)$, $t_4\ge t_0$ and therefore

$$\begin{aligned}
&\int_{t_4}^t \Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds\\
&\le H(t,t_4)z(t_4)\le H(t,t_0)z(t_4)
\end{aligned}$$

which implies that

$$\frac{1}{H(t,t_0)}\int_{t_0}^t \Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds$$

$$
\begin{aligned}
&= \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t_4}+\int_{t_4}^{t}\Big]\Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi \\
&\quad - \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds \\
&\le z(t_4)+\int_{t_0}^{t_4}\frac{H(t,s)}{H(t,t_0)}\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi ds \\
&\le z(t_4)+\int_{t_0}^{t_4}\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi ds.
\end{aligned}
$$

Now taking upper limits as $t\to\infty$, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi \\
&- \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds \\
&\le z(t_4)+\int_{t_0}^{t_4}\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi ds = M<\infty,
\end{aligned}
$$

where M is a constant. Hence, this result leads to a contradiction to (2).

(II) Assume $y'(t)<0$ for $t\ge t_2$. We integrate (1.1) from t to ∞ and since

$$
a(t)(b(t)y'(t))'>0,\quad t\ge t_2,
$$

we have

$$
-a(t)(b(t)y'(t))'+\int_t^\infty\int_a^b p(r,\xi)x(\sigma(r,\xi))d\xi dr\le 0. \tag{2.13}
$$

Now integrating (2.13) from t to ∞ after dividing by $a(t)$ and using $b(t)y'(t)<0$, will lead to

$$
b(t)y'(t)+\int_t^\infty\Big(\int_t^r\frac{du}{a(u)}\Big)\int_a^b p(r,\xi)x(\sigma(r,\xi))d\xi dr\le 0. \tag{2.14}
$$

Dividing (2.14) by $b(t)$ and integrating again from t to ∞ gives

$$
\int_t^\infty\Big[\int_t^r\frac{du}{b(u)}\Big(\int_u^r\frac{dv}{a(v)}\Big)\Big]\int_a^b p(r,\xi)x(\sigma(r,\xi))d\xi dr\le y(t) \tag{2.15}
$$

for $t\ge t_3\ge t_2$. Replacing t by t_3 in (2.15), we get

$$
\int_{t_3}^\infty\Big[\int_{t_3}^r\frac{du}{b(u)}\Big(\int_u^r\frac{dv}{a(v)}\Big)\Big]\int_a^b p(r,\xi)x(\sigma(r,\xi))d\xi\, dr\le y(t_3). \tag{2.16}
$$

On the other hand, since y is monotonically decreasing function in the interval $[t_3,\infty]$, we get $\lim_{t\to\infty}y(t)=\lim_{t\to\infty}[x(t)+c(t)x(t-\tau)]=K\ge 0$. Suppose that $K>0$, then $[x(t)+c(t)x(t-\tau)]\ge\frac{K}{2}>0$ for $t\ge t_4\ge t_3$. From this we can observe that there exists K_1 such that $x(\sigma(r,\xi))\ge K_1>0$. Thus from (2.16)

$$
\int_{t_3}^\infty\Big[\int_{t_3}^r\frac{du}{b(u)}\Big(\int_u^r\frac{dv}{a(v)}\Big)\Big]\int_a^b p(r,\xi)K_1 d\xi dr\le y(t_3).
$$

From the previous equation, we have

$$
\int_{t_3}^\infty\Big[\int_{t_3}^r\frac{du}{b(u)}\Big(\int_u^r\frac{dv}{a(v)}\Big)\Big]\int_a^b p(r,\xi)d\xi dr<\infty.
$$

This is a contradiction to (2.2). Therefore, $\lim_{t\to\infty}[x(t)+c(t)x(t-\tau)]=0$. Then $\lim_{t\to\infty} x(t)=0$, so the proof is complete. $\square$

Example 2.2. Consider the functional differential equation

$$[x(t)+\frac{1}{2}x(t-\pi)]'''+\int_{1/5\pi}^{2/7\pi}\frac{(2-e^{-\pi})e^{1/\xi}}{\xi^2}x(t-\frac{1}{\xi})d\xi=0$$

so that $a(t)=b(t)=1$, $c(t)=\frac{1}{2}$, $\tau=\pi$, $p(t,\xi)=\frac{(2-e^{-\pi})e^{1/\xi}}{\xi^2}$, $\sigma(t,\xi)=t-\frac{1}{\xi}$, $\sigma(t)=\sigma(t,b)=t-\frac{7\pi}{2}$, $\rho(s)=s$, $H(t,s)=(t-s)^2$, $h(t,s)=[2-\frac{(t-s)}{s}]$.

We can see that the conditions of Theorem 2.1 are satisfied. It is easy to verify that $x(t)=e^t\sin t$ is a solution of this problem, which is oscillatory.

Theorem 2.3. *Suppose that the conditions of Theorem 2.1, and condition (2.2) holds, and*

$$0<\inf_{s\ge t_0}\Big[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\Big]\le\infty, \tag{2.17}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds<\infty. \tag{2.18}$$

If there exists a function $\phi(t)\in C([t_0,\infty),R)$ satisfying

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi\\&-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds\\&\ge\phi(u),\quad u\ge t_0,\end{aligned} \tag{2.19}$$

and

$$\begin{gathered}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{[\sigma(u,a)-T]\sigma'(u,a)\phi_+^2(u)du}{a(u)\rho(u)b(\sigma'(u,a))}=\infty,\\ \phi_+(u)=\max_{u\ge t_0}\{\phi(u),0\}.\end{gathered} \tag{2.20}$$

Then every solution of functional differential equation (1.1) is oscillatory or tends to zero as $t\to\infty$.

Proof. Assume, for the sake of contradiction, that equation (1.1) has positive solution, say $x(t)>0$, $t\ge t_0$. Then, proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}&\frac{1}{H(t,u)}\int_u^t\Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)}\Big]ds\\&\le z(u)-\frac{1}{H(t,u)}\int_u^t\Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s)\\&+\frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2ds,\end{aligned}$$

$t>u\ge t_1\ge t_0$. Thus taking upper limit as $t\to\infty$ and using (2.19), we have

$$z(u)\ge\phi(u)+\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s)$$

$$+\frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2 ds.$$

Then from the last inequality we see that $z(u) \geq \phi(u)$, and

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{H(t,u)} \int_u^t \Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s) \\
&+ \frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2 ds \\
&\leq z(u)-\phi(u) = M < \infty,
\end{aligned} \tag{2.21}$$

where M is a constant. On the other hand, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \Big[\sqrt{\frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}}z(s) \\
&+ \frac{\sqrt{a(s)b(\sigma(s,a))\rho(s)}h(t,s)}{2\sqrt{[\sigma(s,a)-T]\sigma'(s,a)}}\Big]^2 ds \\
&\geq \liminf_{t\to\infty} \Big[\frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)b(\sigma(s,a))\rho(s)} ds \\
&\quad + \frac{1}{H(t,t_1)} \int_{t_1}^t \sqrt{H(t,s)}h(t,s)z(s)ds\Big], \quad t > t_1.
\end{aligned} \tag{2.22}$$

Let

$$v_1(t) = \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)b(\sigma(s,a))\rho(s)} ds, \tag{2.23}$$

and

$$v_2(t) = \frac{1}{H(t,t_1)} \int_{t_1}^t \sqrt{H(t,s)}h(t,s)z(s)ds. \tag{2.24}$$

Thus, from (2.21) and (2.22), we see that

$$\liminf_{t\to\infty} [v_1(t)+v_2(t)] < \infty. \tag{2.25}$$

Now we want to show that

$$\int_{t_1}^{\infty} \frac{[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))} ds < \infty, \quad t > t_1. \tag{2.26}$$

Assume that

$$\int_{t_1}^{\infty} \frac{[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))} ds = \infty, \quad t > t_1. \tag{2.27}$$

Because of (2.17), there exists a constant $L > 0$ such that

$$\inf_{s\geq t_0} \Big[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\Big] > L > 0. \tag{2.28}$$

From (2.27) one can see that for any positive number $\lambda > 0$, there exists a $T > t_1$ such that

$$\int_{t_1}^{t} \frac{[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)\rho(s)b(\sigma(s,a))} ds \geq \frac{\lambda}{L}, \quad t > T. \tag{2.29}$$

On the other hand,

$$
\begin{aligned}
v_1(t) &= \frac{1}{H(t,t_1)}\int_{t_1}^{t} \frac{H(t,s)[\sigma(s,a)-T]\sigma'(s,a)z^2(s)}{a(s)b(\sigma(s,a))\rho(s)}ds\\
&= \frac{1}{H(t,t_1)}\int_{t_1}^{t} H(t,s)d\Big[\int_{t_1}^{s}\frac{[\sigma(u,a)-T]\sigma'(u,a)z^2(u)}{a(u)b(\sigma(u,a))\rho(u)}\Big]ds\\
&= \frac{1}{H(t,t_1)}\int_{t_1}^{t}\Big[\int_{t_1}^{s}\frac{[\sigma(u,a)-T]\sigma'(u,a)z^2(u)}{a(u)b(\sigma(u,a))\rho(u)}du\Big]\Big[-\frac{\partial H(t,s)}{\partial s}\Big]ds\\
&\geq \frac{1}{H(t,t_1)}\int_{T}^{t}\Big[\int_{t_1}^{s}\frac{[\sigma(u,a)-T]\sigma'(u,a)z^2(u)}{a(u)b(\sigma(u,a))\rho(u)}du\Big]\Big[-\frac{\partial H(t,s)}{\partial s}\Big]ds\\
&\geq \frac{\lambda}{LH(t,t_1)}\int_{T}^{t}\Big[-\frac{\partial H(t,s)}{\partial s}\Big]ds\\
&= \frac{\lambda}{L}\frac{H(t,T)}{H(t,t_1)} \geq \frac{\lambda}{L}\frac{H(t,T)}{H(t,t_0)}, \quad t \geq T > t_1.
\end{aligned}
\tag{2.30}
$$

Moreover, it follows from (2.28) that

$$
\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > L > 0, \quad s \geq t_0.
$$

Therefore, there exists a $t_2 > T$ such that

$$
\frac{H(t,T)}{H(t,t_0)} \geq L, \quad t \geq t_2. \tag{2.31}
$$

It follows from (2.30) and (2.31) that $v_1(t) \geq \lambda$, $t \geq t_2$. Since λ is arbitrary , we have

$$
\lim_{t\to\infty} v_1(t) = \infty. \tag{2.32}
$$

Moreover from (2.25), there exists a convergence subsequence $\{t_n\}_1^\infty$ on $[t_1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$
\lim_{n\to\infty}[v_1(t_n)+v_2(t_n)] = \liminf_{t\to\infty}[v_1(t)+v_2(t)] < \infty. \tag{2.33}
$$

As a result of (2.33) there exists a positive integer n_1 and constant k such that

$$
v_1(t_n)+v_2(t_n) < k, \quad n > n_1
$$

and from (2.32), we have

$$
\lim_{n\to\infty} v_1(t_n) = \infty. \tag{2.34}
$$

Thus (2.33) and (2.34) will give

$$
\lim_{n\to\infty} v_2(t_n) = -\infty. \tag{2.35}
$$

Moreover, for any $\epsilon \in (0,1)$, there exists a positive integer n_2 such that

$$
\frac{v_2(t_n)}{v_1(t_n)} + 1 < \epsilon, \quad n > n_2;
$$

then

$$
\frac{v_2(t_n)}{v_1(t_n)} < \epsilon - 1 < 0, \quad n > n_2. \tag{2.36}
$$

Thus (2.35) and (2.36) give

$$
\lim_{n\to\infty}\frac{v_2(t_n)}{v_1(t_n)}v_2(t_n) = \infty. \tag{2.37}
$$

By using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
0 \le v_2^2(t_n) &= \frac{1}{H^2(t_n,t_1)}\Big[\int_{t_1}^{t_n}\sqrt{H(t_n,s)}h(t_n,s)z(s)ds\Big]^2\\
&\le \Big[\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n}\frac{H(t_n,s)[\sigma(s,a)-T]\sigma'(s,a)}{a(s)b(\sigma(s,a))\rho(s)}z^2(s)ds\Big]\\
&\quad\Big[\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t_n,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds\Big]\\
&= v_1(t_n)\Big[\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t_n,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds\Big],\quad t\ge t_1
\end{aligned}$$

and

$$\frac{v_2^2(t_n)}{v_1(t_n)} \le \Big[\frac{1}{H(t_n,t_1)}\int_{t_1}^{t_n}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t_n,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds\Big]. \tag{2.38}$$

Using (2.31), we see that

$$\frac{1}{H(t_n,t_1)} \le \frac{1}{H(t_n,T)} = \frac{H(t_n,t_0)}{H(t_n,T)}\frac{1}{H(t_n,t_0)} \le \frac{1}{LH(t_n,t_0)},\quad T>t_1. \tag{2.39}$$

Therefore, we see from (2.38) and (2.39) that

$$\frac{v_2^2(t_n)}{v_1(t_n)} \le \frac{1}{LH(t_n,t_0)}\int_{t_1}^{t_n}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t_n,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds. \tag{2.40}$$

Then, it follows from (2.37) and (2.40) that

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t_n,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds=\infty,$$

and then

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{[\sigma(s,a)-T]\sigma'(s,a)}ds=\infty,$$

which contradicts (2.18). Thus, using $z(s)\ge\phi(s)$ with (2.26), we obtain

$$\int_{t_1}^{\infty}\frac{[\sigma(s,a)-T]\sigma'(s,a)\phi_+^2(s)ds}{a(s)\rho(s)b(\sigma'(s,a))} \le \int_{t_1}^{\infty}\frac{[\sigma(s,a)-T]\sigma'(s,a)z^2(s)ds}{a(s)\rho(s)b(\sigma'(s,a))}<\infty,$$

which leads to a contradiction to (2.20). This completes the proof. □

Theorem 2.4. *Suppose that the conditions of Theorem 2.1, and conditions* (2.2), (2.17), (2.20) *hold, in addition to*

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi ds<\infty. \tag{2.41}$$

If there exists a function $\phi(t)\in C([t_0,\infty),R)$ *satisfying*

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\Big[H(t,s)\rho(s)\int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi\\
&-\frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T)]\sigma'(s,a)}\Big]ds\\
&\ge\phi(u),\quad u\ge t_0,
\end{aligned}\tag{2.42}$$

then every solution of (1.1) is oscillatory or tends to zero as $t\to\infty$.

Proof. Assume, for the sake of contradiction, that equation (1.1) has positive solution, say $x(t) > 0$, $t \geq t_0$. It follows from (2.42) that

$$\begin{aligned}
\phi(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\rho(s) \int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi \\
- \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)} \Big] ds \\
\leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\rho(s) \int_a^b p(s,\xi)[1-c(\sigma(s,\xi))]d\xi \Big] ds \\
- \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)} ds.
\end{aligned} \tag{2.43}$$

Then, from (2.41) and (2.43)

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{a(s)\rho(s)b(\sigma(s,a))h^2(t,s)}{4[\sigma(s,a)-T]\sigma'(s,a)} ds < \infty.$$

Thus (2.18) holds in Theorem 2.3. Since the remaining part of the proof is similar to the proof of Theorem 2.3, it is omitted. □

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