2004 Conference on Diff. Eqns. and Appl. in Math. Biology, Nanaimo, BC, Canada.
Electronic Journal of Differential Equations, Conference 12, 2005, pp. 39–46.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

HOLOMORPHIC SOLUTIONS TO LINEAR FIRST-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study holomorphic solutions to linear first-order functional differential equations that have a nonlinear functional argument. We focus on the existence of local solutions at a fixed point of the functional argument and the holomorphic continuation of these solutions. We show that the Julia set for the functional argument dominates not only the conditions for holomorphic continuation, but also the existence of local solutions. In particular, nonconstant holomorphic solutions in a neighbourhood of a repelling or neutral fixed point are uncommon in that the functional argument must satisfy conditions that force it to have an exceptional point in the former case, and a Siegel fixed point in the latter case. In contrast, local holomorphic solutions always exist near attracting fixed points. In this case a subset of the Julia set forms a natural boundary for holomorphic continuation.

1. Introduction

We study initial-value problems of the form

$$y'(z) - by(z) - \lambda y(g(z)) = 0 \tag{1.1}$$

$$y(z_0) = y_0 \,, \tag{1.2}$$

where b, $\lambda \neq 0$, and $y_0 \neq 0$ are constants, and z_0 is a fixed point for the entire function g. The case $g(z) = \alpha z$, where α is a constant, corresponds to the well-known pantograph equation, which has been studied by numerous researchers including Kato and McLeod [12], Iserles [10], Iserles and Liu [11], Fredrickson [7], Morris *et al.* [15] and Derfel & Iserles [4]. A matrix version has been studied by Carr and Dyson [3], and a second order version has been studied by van Brunt *et al.* [21]. The pantograph equation has been used in models of current collection in an electric train [17], and cell and tumour growth [9, 1, 2] among other applications.

The case where g is a nonlinear entire function has received much less attention. Oberg [16] studied local solutions to a more general first order problem and made the connection of the existence of local solutions with attracting and neutral fixed points. Recently, Marshall *et al.* [13] and van Brunt *et al.* [22, 23, 24] looked at global properties of solutions to second order versions of the initial-value problem. Motivated by Oberg's work, they exploited certain results from complex dynamics

2000 *Mathematics Subject Classification.* 30D05, 34M05, 37F10, 37F50.
Key words and phrases. Complex functional differential equations; pantograph equation.

Published April 20, 2005.

to show that the Julia set of the functional argument determined where solutions can be holomorphically continued. This paper builds on this work, applying the results to the first order case and merging the results of these papers to provide a succinct summary.

The main results concerning holomorphic solutions to the initial-value problem (1.1), (1.2) are given in Section 3. Section 2 is devoted to a summary of some relevant terms and results from complex dynamics. The remainder of this section is devoted to a few simple examples that motivate and illustrate the analysis and results of Section 3.

Example 1.1. Suppose that $g(z) = \alpha z$, where α is a complex constant, and that $z_0 = 0$. If y is a solution to the initial-value problem that is holmorphic at the origin then y can be represented as a power series of the form $y(z) = \sum_{n=0}^{\infty} c_n z^n$. Substituting this expression into equation (1.1) gives

$$c_n = \frac{c_0}{n!} \prod_{k=0}^{n-1} (b + \lambda \alpha^k).$$

Since $y_0 \neq 0$, we have $c_0 \neq 0$. If $\lambda = -b/\alpha^j$ for some non-negative integer j, then it is clear that there is a polynomial solution to the initial-value problem. If $\lambda \neq -b/\alpha^j$ for any non-negative integer j. Then $c_k \neq 0$ and

$$\Big|\frac{c_{n+1}}{c_n}\Big| = \frac{|b + \lambda \alpha^n|}{n+1};$$

hence, the power series defines an entire function for $|\alpha| \leq 1$. If $|\alpha| > 1$, however, then

$$\Big|\frac{c_{n+1}}{c_n}\Big| \to \infty$$

as $n \to \infty$, so that the series converges only for $z = 0$. In this case there is no solution holomorphic at the origin. The initial-value problem always has a holomorphic solution if $|\alpha| \leq 1$, but if $|\alpha| > 1$, the initial-value problem has a holomorphic solution only for special values of λ.

Example 1.2. Consider the initial-value problem

$$y'(z) = -y(z^2) \tag{1.3}$$
$$y(0) = 1\,. \tag{1.4}$$

Substituting a power series of the form $\sum_{n=0}^{\infty} c_n z^n$ into equation (1.3) yields

$$y(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2^k - 1}}{\prod_{m=1}^{k} (2^m - 1)},$$

which defines a holomorphic function for $|z| < 1$. Note that the series defining y has large gaps: the coefficient c_n is nonzero only if $n = 2^k - 1$ for some $k = 0, 1, 2, \dots$. The sequence $\{2^k - 1\}$ meets the conditions of the Hadamard Gap Theorem (cf. [19]), *viz.*, there is a number $\nu > 0$ such that $n_{k+1} > (1 + \nu) n_k$ for all k. Since the radius of convergence for the power series is $R = 1$ we thus see that the circle $|z| = 1$ forms a natural boundary for the holomorphic continuation of y.

Example 1.3. The function $g(z) = z^2$ also has a fixed point at $z = 1$. In contrast with the previous example, this fixed point is repelling, i.e., $|g'(1)| > 1$. Consider the initial-value problem that consists of equation (1.3) and the initial condition

$$y(1) = 1.$$

Evidently, this problem has a solution

$$y(z) = \frac{1}{z},$$

which is holomorphic at $z = 1$.

2. Fixed Points and the Julia Set

The existence of a holomorphic local solution to the initial-value problem (1.1), (1.2) depends crucially on the nature of the fixed point. Let g be a function holomorphic at z_0 and let z_0 be a fixed point for g. The point z_0 is called *attracting* if there exists a neighbourhood U of z_0 such that

$$|g(z) - z_0| < |z - z_0|$$

for all $z \in U - \{z_0\}$. Similarly, z_0 is called *repelling* if there exists a neighbourhood U of z_0 such that

$$|g(z) - z_0| > |z - z_0|$$

for all $z \in U - \{z_0\}$. If z_0 is neither attracting nor repelling, then z_0 is called *neutral.* Since g is holomorphic at z_0, fixed points can be characterized as follows:

(1) z_0 is attracting if and only if $|g'(z_0)| < 1$;
(2) z_0 is repelling if and only if $|g'(z_0)| > 1$.

Let z_0 be an attracting fixed point of g. The *basin of attraction* is defined as

$$A_{z_0} = \{z : \lim_{n\to\infty} g_n(z) = z_0\}.$$

The set A_{z_0} need not be connected. The definition of an attracting fixed point, however, indicates that there is some neighbourhood U of z_0 that is in A_{z_0}, so that there is a non-empty connected open subset of A_{z_0} that contains z_0. We define the *immediate basin of attraction* $A^0_{z_0}$ as the connected component of A_{z_0} containing the point z_0. The boundary of $A^0_{z_0}$ is denoted by $\partial A^0_{z_0}$.

Let g_n denote the nth composition of g, e.g., $g_2(z) = g(g(z))$. A point $z_0 \in \mathbb{C}$ such that $g_n(z_0) = z_0$, and $g_k(z_0) \neq z_0$ for $k = 1, 2, \dots n-1$ is called a *periodic point of period n.* Evidently, a fixed point is a periodic point of period 1. Let z_0 be a periodic point of period n. We say that z_0 is *repelling* if there exists a neighbourhood U of z_0 such that

$$|g_n(z) - z_0| > |z - z_0|$$

for all $z \in U - \{z_0\}$.

We define the *Julia set* $J(g)$ of a holomorphic function g as the closure of the set of repelling periodic points of g. This is not the classical definition of the Julia set, which is in terms of normal families. Nonetheless, the definitions are equivalent. The next theorem gives an important property of Julia sets in connection with holomorphic continuation [5, 13].

Theorem 2.1. *Let $z \in J(g)$, U be a neighbourhood of z, and $g_n(U) = \{g_n(z) : z \in U\}$. Then the set G defined by*

$$G = \bigcup_{n=1}^{\infty} g_n(U)$$

omits at most one point in $\mathbb{C}$.

Theorem 2.1 shows that there is at most one point $\sigma \in \mathbb{C}$ such that $\sigma \notin G$. This point is called an *exceptional point*. The characterization of the exceptional point is particularly simple for polynomials (cf. [5]).

Theorem 2.2. *Let g be a polynomial. Suppose that there is a point $\sigma \in J(g)$ and a neighbourhood U of z such that*

$$\bigcup_{n=1}^{\infty} g_n(U) = \mathbb{C} - \{\sigma\}$$

for some $\sigma \in \mathbb{C}$. Then $g(z) = \sigma + \kappa(z-\sigma)^k$ for some $\kappa \in \mathbb{C} - \{0\}$ and some $k \in \mathbb{N}$.

It is clear from the above result that if there is an exceptional point σ for a nonlinear polynomial g it must be a fixed point for g. If $k \geq 2$, then $|g'(\sigma)| = 0$, so that the fixed point must be attracting. Suppose that z_0 is another fixed point for g. Then $g(z_0) = z_0$, so that

$$\kappa(z-\sigma)^{k-1} = 1;$$

hence, $g'(z_0) = k \geq 2$. We thus see that if g has an exceptional point, then any other fixed points must be repelling.

3. Holomorphic Solutions

The space of functions holomorphic in a set $\Omega \subseteq \mathbb{C}$ is denoted by $H(\Omega)$ and the disc of radius R centred at z_0 is denoted by $D(z_0; R)$. The closed disc is denoted by $\bar{D}(z_0; R)$.

The Julia set of g arises naturally in the holomorphic continuation of solutions to equation (1.1). Suppose, for example, that $y \in H(D(z_0;\delta))$ is a solution to equation (1.1) for some $\delta > 0$. Then equation (1.1) shows that $y \in H(g(D(z_0;\delta)))$, and since y is holomorphically continued into $g(D(z_0;\delta))$,

$$y'(g(z)) - by(g(z)) = \lambda y(g_2(z)),$$

for all $z \in D(z_0;\delta)$. The above argument can be applied any number of times so that

$$y'(g_n(z)) - by(g_n(z)) = \lambda y(g_{n+1}(z)),$$

for $n = 0, 1, 2, \dots$ and $z \in D(z_0;\delta)$. We thus have that y can be holomorphically continued to the set $G = \bigcup_{n=1}^{\infty} g_n(D(z_0;\delta))$. Now, it is possible that $G \subseteq D(z_0;\delta)$, so that the continuation process is trivial, but if $D(z_0;\delta)$ contains a point of the Julia set, then Theorem 2.1 shows that G covers the complex plane with at most one exception. We thus have the following result.

Theorem 3.1. *Suppose that y is a solution to equation* (1.1) *that is holomorphic at a point $z_1 \in J(g)$. Then y can be holomorphically continued to all points of the complex plane with at most one exception.*

Evidently, if g does not have an exceptional point and y is holomorphic at a point in $J(g)$ then y must be an entire function. If g is nonlinear, however, the maximum modulus theorem places severe restrictions on y.

Theorem 3.2. *Let $g \in H(\mathbb{C})$ be nonlinear and suppose that $y \in H(\mathbb{C})$ is a solution to equation* (1.1)*. Then y is a constant.*

Proof. For any $f \in H(\mathbb{C})$ let

$$M_f(R) = \sup_{|z|=R} |f(z)|.$$

Polya [18] showed that for any entire functions y and g such that $g(0) = 0$ there is a constant c, $0 < c < 1$, such that

$$M_{y\circ g}(R) \geq M_y(cM_g(\frac{R}{2})), \tag{3.1}$$

for all $R > 0$. We prove the theorem for the case $g(0) = 0$ and note the proof can be extended the more general case by use of a transformation [8].

Suppose that y is a nonconstant solution to equation (1.1) and let h be the function defined by

$$h(z) = \begin{cases} \frac{y(z)-y(0)}{z}, & \text{if } z \neq 0, \\ y'(0), & \text{if } z = 0\,. \end{cases}$$

Since y is entire and nonconstant h is an entire function. The above definition of h provides the inequalities

$$M_y \leq RM_h(R) + |y_0|\,, \tag{3.2}$$

$$RM_h(R) \leq M_y(R) + |y_0|, \tag{3.3}$$

for all $R > 0$.

The Cauchy integral formula gives

$$y'(z) = \frac{1}{2\pi i}\int_\gamma \frac{y(\xi)}{(\xi - z)^2} d\,\xi,$$

where γ is any simple closed contour oriented anticlockwise that encloses z. Since y is entire, we can use contours of the form $\gamma_\delta = \{\xi : |\xi - z| = \delta\}$, where δ is any positive number; thus, for $\delta > 0$,

$$|y'(z)| \leq M_{y'}(R) \leq \frac{1}{\delta} M_y(R + \delta).$$

Equation (1.1), the Maximum Modulus Theorem and the Polya inequality (3.1) imply

$$M_y(R+\delta)\big(\frac{1}{\delta} + |b|\big) \geq |\lambda| M_{y\circ g}(R) \geq |\lambda| M_y(cM_g(\frac{R}{2})). \tag{3.4}$$

Inequalities (3.2) - (3.4) thus yield

$$\big((R+\delta)M_h(R+\delta) + |y_0|\big)\big(\frac{1}{\delta} + |b|\big) \geq |\lambda|\big(cM_g(\frac{R}{2})M_h(cM_g(\frac{R}{2})) - |y_0|\big),$$

and since $M_h(R) \neq 0$ and $M_g(R/2) \neq 0$ for any $R > 0$, the above inequality can be recast as

$$M_h(R+\delta)L(R) \geq M_h(cM_g(\frac{R}{2})),$$

where

$$L(R) = \Big\{\frac{1}{|\lambda|}\big(R+\delta+\frac{|y_0|}{M_h(R+\delta)}\big)\big(\frac{1}{\delta}+|b|\big)+\frac{|y_0|}{M_h(R+\delta)}\Big\}\frac{1}{cM_g(\frac{R}{2})}.$$

Now, g is nonlinear and hence $R/M_g(R/2)\to 0$ as $R\to\infty$. Consequently there is an $\hat{R}>0$ such that

$$cM_g(\frac{R}{2}) > R+\delta \quad \text{and} \quad L(R)<1$$

for all $R\geq\hat{R}$. For such R we thus have

$$M_h(R+\delta) > M_h(cM_g(\frac{R}{2})),$$

which contradicts the Maximum Modulus Theorem. We thus conclude that y must be constant. ☐

Theorems 3.1 and 3.2 can be combined to give the following result.

Theorem 3.3. *Suppose that g is a nonlinear entire function without an exceptional point and that y is a solution to equation* (1.1) *that is holomorphic at some point in $J(g)$. Then $\lambda = b$, and y is a constant.*

Loosely speaking, the above theorem indicates that, generically, linear functional differential equations do not have solutions that are holomorphic at points in the Julia set of the functional argument. Nonconstant solutions that are holomorphic at points in the Julia set exist only if the nonlinear function g has an exceptional point, and Theorem 2.2 makes this a tractable condition for polynomials.

Since any repelling fixed points of g must be in the Julia set of g, Theorem 3.3 yields immediately the following corollary.

Corollary 3.4. *Let g be a nonlinear entire function with a repelling fixed point z_0. If there exists a nonconstant solution to initial-value problem* (1.1), (1.2) *that is holomorphic at z_0 then g must have an exceptional point.*

Example 1.3 illustrates that nonconstant holomorphic solutions can occur in the presence of an exceptional point, in this case $z=0$. Corollary 3.4, however, indicates that, in general, initial-value problems of the form (1.1), (1.2) do not have holomorphic solutions if z_0 is a repelling fixed point. In contrast, if z_0 is an attracting fixed point then the initial-value problem always has a solution holomorphic in a neighbourhood of z_0. Specifically, we have the following result, a proof of which can be found in [13, Theorem 2.2], and is a special case of a more general result (cf. [20]).

Theorem 3.5. *Let z_0 be an attracting fixed point of g. Then there exists a local solution to the initial-value problem* (1.1), (1.2) *that is unique among functions holomorphic at z_0.*

The Julia set, however, still plays a rôle in the holomorphic continuation of the local solution guaranteed by Theorem 3.5.

Theorem 3.6. *Let g be a nonlinear polynomial and z_0 be an attracting fixed point of g. Then the solution to initial-value problem (1.1), (1.2) has ∂A_{z_0} as a natural boundary, unless the solution is a constant.*

Proof. Let y be the local solution to the initial-value problem holomorphic at z_0. We show first that $y \in H(A_{z_0})$ and then that a nonconstant solution y cannot be holomorphic at any point on ∂A_{z_0}.

Suppose that y cannot be holomorphically continued to A_{z_0}. Then there is a $z \in A_{z_0}$ at which y is singular, and equation (1.1) shows that y must also be singular at $g(z)$. The same argument can be invoked any number of times and hence y has a sequence of singularities $\{g_n(z)\}$. The point z_0, however, is an attracting fixed point and $z \in A_{z_0}$; consequently, $g_n(z) \to z_0$ as $n \to \infty$. Hence we arrive at the contradiction that there is no neighbourhood of z_0 where y is holomorphic. We thus conclude that $y \in H(A_{z_0})$.

Suppose that y is a nonconstant solution. Now, $\partial A_{z_0} \subseteq J(g)$, so that Theorem 3.3 precludes a solution that is holomorphic at any point on ∂A_{z_0}, unless g has an exceptional point. But g is a nonlinear polynomial, so that if g has an exceptional point it must be the attracting fixed point z_0. We know that y is holomorphic at this point. We thus conclude that the solution cannot be holomorphically continued through any point of ∂A_{z_0}. □

A few remarks are in order for the case where z_0 is a neutral fixed point for the nonlinear polynomial g. The results for this case are considered in detail in van Brunt *et al.* [23]. Briefly, if $z_0 \in F(g) = \mathbb{C} \setminus J(g)$ (the Fatou set), then it can be shown that the initial-value problem (1.1), (1.2) has a local solution holomorphic at z_0. The condition that $z_0 \in F(g)$ corresponds to the condition that g be locally linearizable near z_0, and hence that z_0 is a Siegel point. The holomorphic solution can be continued into the Siegel disc corresponding to z_0, and the boundary of this disc forms a natural boundary for holomorphic continuation of nonconstant solutions.

Since z_0 is a neutral fixed point for the nonlinear polynomial g, we see that g cannot have an exceptional point. If $z_0 \in J(g)$ then Theorem 3.3 shows that holomorphic solutions are restricted to constant functions and the special case $\lambda = b$. If z_0 corresponds to a rational rotation, then $z_0 \in J(g)$; if z_0 corresponds to an irrational rotation, then Cremer's result (cf. Milnor [14]) indicates that generically $z_0 \in J(g)$, so that in this sense nonconstant holomorphic solutions are rare. Milnor (*op. cit.*), Chapter 11 describes the paucity of Siegel points in more detail.

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