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ASYMPTOTIC BEHAVIOR OF DISCRETE SOLUTIONS TO IMPULSIVE LOGISTIC EQUATIONS

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Abstract. We study the stability characteristics of a time-dependent system of impulsive logistic equations by using discrete modelling.

1. Introduction

One of the most used models for a single species dynamics has been derived by many researchers in the form of a differential equation

$$\dot{x}(t) = x(t)f(t, x(t)) - g(t, x(t)), \tag{1.1}$$

where the solution $x(t)$ is the size or biomass of the resource population at time $t > 0$, the function $f(t, x(t))$ characterizes the population change at time t, and the function $g(t, x(t))$ defines the continuous effects — influences of external factors. When $g(t, x(t)) = 0$, then the model is called classical logistic equation, which represents a population isolated from external factors.

As an illustrative example, a special case of (1.1), consider the well known elementary nonlinear ordinary differential equation

$$\frac{dy(t)}{dt} = y(t)[1 - y(t)], \quad t > 0\,. \tag{1.2}$$

The autonomous system has two equilibrium states: $y(t) = 0$ and $y(t) = 1$. It is not difficult to see that the trivial solution is unstable while the positive equilibrium of (1.2) is asymptotically stable and all solutions of (1.2) satisfy $y(0) > 0 \Rightarrow y(t) > 0$ for $t > 0$ and $y(t) \to 1$ as $t \to \infty$ and the convergence is monotonic. One of the discrete versions of (1.2) is given by the quadratic logistic equation (an Euler-type scheme)

$$y(n+1) = y(n)[1 + h - hy(n)], \quad h > 0,\ n \in \mathbb{Z}_0^+. \tag{1.3}$$

The trivial equilibrium of (1.3) is unstable for $h > 0$. The positive equilibrium of (1.3) is asymptotically stable for $0 < h \leq 2$, for $h > 2$ it becomes unstable, and oscillatory and chaotic behavior of solutions of (1.3) is possible. See more details

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in [11, 13, 14] and references therein. Equation (1.3) can be reformulated so as to obtain the following popular logistic equation:

$$\begin{gathered} x(n+1) = rx(n)[1-x(n)], \\ r = 1+h, \quad x(n) = \frac{h}{1+h}y(n)\,. \end{gathered} \tag{1.4}$$

It has infinitely many, a one-parameter family of nonzero equilibria for positive r. The dynamical characteristics of (1.4) are well known in the literature on chaotic systems [8, 14].

Let us consider a second order Runge-Kutta algorithm for (1.2); such an algorithm leads to a discrete-time system defined by the following expressions:

$$\begin{gathered} x(n+1) = x(n) + \frac{1}{2}(k_1+k_2), \\ k_1 = hx(n)(1-x(n)), \\ k_2 = h(x(n)+k_1)(1-x(n)-k_1), \end{gathered} \tag{1.5}$$

where $n \in \mathbb{Z}_0^+$ and $h > 0$ is a fixed constant. An algebraic simplification of (1.5) leads to

$$\begin{aligned} x(n+1) = \Big(1 + \frac{h(2+h)}{2}\Big)x(n) - \frac{h(2+3h+h^2)}{2}x^2(n) \\ + \Big((1+h)h^2\Big)x^3(n) - \frac{h^3}{2}x^4(n), \quad n \in \mathbb{Z}_0^+. \end{aligned} \tag{1.6}$$

The equilibria of (1.6) are given by x_i^*, $i = 1,2,3,4$, where

$$\begin{gathered} x_1^* = 0, \quad x_2^* = 1, \\ x_3^* = \frac{1}{2h}\big(2+h-\sqrt{h^2-4}\big), \quad x_4^* = \frac{1}{2h}\big(2+h+\sqrt{h^2-4}\big). \end{gathered}$$

Thus it follows that for $h > 2$, (1.6) has four equilibrium points while its mother version (1.2) has only two equilibria. Also for $h > 2$, x_1^* and x_2^* are unstable while x_3^* and x_4^* are stable. The equilibria x_3^* and x_4^* are born through the process of discretization of (1.2) and these are parasitic, spurious or ghost solutions of the system.

Higher order Runge-Kutta numerical algorithms can give rise to more parasitic solutions including periodic, quasiperiodic and chaotic behaviour. For more details see [7, 9] and [15, 16, 17].

2. Main result

Consider the non-autonomous logistic equation

$$\frac{dx}{dt} = r(t)x(t)\big(1 - \frac{x(t)}{K(t)}\big), \quad t > 0,\ t \neq \tau_k, \tag{2.1}$$

$$\Delta x(t) = I_k(x(t)), \quad t = \tau_k,\ k = 1,2,\dots, \tag{2.2}$$

in which $r(t)$ is nonnegative and $K(t)$ is a strictly positive continuous function, and I_k are bounded operators. In the real evolutionary processes of the population, the perturbation or the influence from outside occurs “instantly” as impulses, and not continuously. The duration of these perturbations is negligible compared to the duration of the whole process, for more details about the theory of impulsive differential equations and applications see [10, 18] and references therein. Also impulsive perturbations (harvest, taking out, hunting, fishing, etc.) are more practical and realistic compared to any kind of continuous harvest. For instance, a fisherman can not fish 24 hours a day and furthermore, the seasons also determine the fishing period. Similar considerations are applicable for hunting and taking away a huge part of any biomass.

In system (2.1), (2.2) the instants of impulse effect τ_k, $k = 1, 2, \dots$, form a strictly increasing sequence such that $\lim_{k\to\infty} \tau_k = +\infty$. By a solution of system (2.1), (2.2) we mean a piecewise continuous function on $[0, +\infty)$ which satisfies the equation (2.1) for $t > 0$, $t \neq \tau_k$, with discontinuities of the first kind at τ_k, $k = 1, 2, \dots$, at which it is continuous from the left and satisfies the impulsive conditions

$$\Delta x(\tau_k) \equiv x(\tau_k + 0) - x(\tau_k) = I_k(x(\tau_k)).$$

The logistic equation (2.1) has been intensively studied by various researchers [1, 2, 4] and [5, 6], considering existence of the solutions, asymptotic properties of the solutions, sufficient conditions for the oscillation of the solutions and so on. In this paper we will use a suitable differential equation with piecewise constant argument to approximate the solution of the system (2.1), (2.2) and introduce a sufficient condition for the existence of the solution of the impulsive systems. The approximation is given by

$$\begin{gathered} \frac{dx}{dt} = x(t)\Big(r\big(\big[\frac{t}{h}\big]h\big) - \frac{r([\frac{t}{h}]h)x(t)}{K([\frac{t}{h}]h)}\Big), \\ t \in [nh, (n+1)h),\ n \in \mathbb{Z}_0^+,\ n \neq \big[\frac{\tau_k}{h}\big], \\ x\Big(\big(\big[\frac{\tau_k}{h}\big] + 1\big)h\Big) = x\big(\big[\frac{\tau_k}{h}\big]h\big) + I_k x\big(\big[\frac{\tau_k}{h}\big]h\big), \quad k = 1, 2, \dots, \end{gathered} \tag{2.3}$$

where $h > 0$ denotes a uniform discretization step size and $[\mu]$ denotes the integer part of $\mu \in \mathbb{R}$. Then the system (2.1), (2.2) becomes

$$\begin{gathered} \frac{dx}{dt} = r(n)x(t) - \frac{r(n)}{K(n)}x^2(t), \quad t \in [nh, (n+1)h),\ n \neq m_k, \\ x(m_k + 1) = x(m_k) + I_k(x(m_k)), \quad k = 1, 2, \dots, \end{gathered} \tag{2.4}$$

where $\big[\frac{t}{h}\big] = n$, $\big[\frac{\tau_k}{h}\big] = m_k$, and we use the notation $f(n) = f(nh)$.

An integration of the differential equation in (2.4) over $[nh, t)$, where $t < (n+1)h$, leads to

$$\frac{1}{x(t)}e^{r(n)t} - \frac{1}{x(n)}e^{r(n)nh} = \frac{e^{r(n)t}}{K(n)} - \frac{e^{r(n)nh}}{K(n)}, \quad \tau_k \notin [nh, t),$$

and by allowing $t \to (n+1)h$, we obtain after some simplification

$$\begin{gathered} x(n+1) = \frac{e^{r(n)h}x(n)}{1 + \big(\frac{e^{r(n)h}-1}{K(n)}\big)x(n)}, \quad n \neq m_k, \\ x(m_k + 1) = x(m_k) + I_k(x(m_k)), \quad k = 1, 2, \dots \end{gathered} \tag{2.5}$$

Theorem 1. *Let the following conditions hold:*

$$\begin{gathered} r(n) \geq 0,\ n \in \mathbb{Z}_0^+, \quad R = \sup_{n\in\mathbb{Z}_0^+} r(n) < \infty, \quad 0 < K_* \leq \inf_{n\in\mathbb{Z}_0^+} K(n), \\ \sup_{n\in\mathbb{Z}_0^+} K(n) < \infty, \quad I_k(x(m_k)) = cx(m_k), \end{gathered}$$

where $c > 0$. Then for $h > 0$ satisfying the inequality $h \le \ln(1+c)/R$ a solution $x(n)$ of (2.5) corresponding to $x(0) > 0$ satisfies the inequality

$$\frac{1}{x(n)} \le \frac{1}{x(0)} \exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) + \sum_{j=1}^{n} \frac{1-e^{-r(n-j)h}}{K(n-j)} \exp\Big(-\sum_{\ell=1}^{j-1} r(n-\ell)h\Big). \tag{2.6}$$

Proof. Since $x(n) > 0$ for all $n \in \mathbb{Z}_0^+$, we let $y(n) = \dfrac{1}{x(n)}$ in (2.5), and obtain

$$\begin{gathered} y(n+1) = e^{-r(n)h} y(n) + \frac{1-e^{-r(n)h}}{K(n)}, \quad n \in \mathbb{Z}_0^+,\ n \ne m_k, \\ y(m_k+1) = \frac{1}{1+c} y(m_k), \quad k = 1, 2, \dots \end{gathered} \tag{2.7}$$

We will show by induction that (2.7) leads to

$$\begin{aligned} y(n) &\le y(0) \exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) \\ &\quad + \sum_{j=0}^{n-1} \frac{1}{K(j)} \Big\{ \exp\Big(-\sum_{\ell=j+1}^{n-1} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{n-1} r(\ell)h\Big)\Big\}. \end{aligned} \tag{2.8}$$

In fact, for $n = 0$ (2.8) is obviously true. Suppose that it is true for some $n \ne m_k$. Then from (2.7) we find

$$\begin{aligned} &y(n+1) \\ &= e^{-r(n)h} y(n) + \frac{1-e^{-r(n)h}}{K(n)} \\ &\le e^{-r(n)h} y(0) \exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) \\ &\quad + e^{-r(n)h} \sum_{j=0}^{n-1} \frac{1}{K(j)} \Big\{ \exp\Big(-\sum_{\ell=j+1}^{n-1} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{n-1} r(\ell)h\Big)\Big\} + \frac{1-e^{-r(n)h}}{K(n)} \\ &= y(0) \exp\Big(-\sum_{i=0}^{n} r(i)h\Big) + \sum_{j=0}^{n-1} \frac{1}{K(j)} \Big\{ \exp\Big(-\sum_{\ell=j+1}^{n} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{n} r(\ell)h\Big)\Big\} \\ &\quad + \frac{1}{K(n)} \Big\{ \exp\Big(-\sum_{\ell=n+1}^{n} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=n}^{n} r(\ell)h\Big)\Big\} \\ &= y(0) \exp\Big(-\sum_{i=0}^{n} r(i)h\Big) + \sum_{j=0}^{n} \frac{1}{K(j)} \Big\{ \exp\Big(-\sum_{\ell=j+1}^{n} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{n} r(\ell)h\Big)\Big\}. \end{aligned}$$

Next suppose that inequality (2.8) is true for $n = m_k$. Then, using the inequality $\exp\big(r(m_k)h\big) \le 1 + c$ which follows from our assumptions, we obtain

$$\begin{aligned}
&y(m_k+1)\\
&= \frac{1}{1+c} y(m_k)\\
&\le e^{-r(m_k)h} y(m_k)\\
&\le y(0)\exp\Big(-\sum_{i=0}^{m_k} r(i)h\Big) + \sum_{j=0}^{m_k-1} \frac{1}{K(j)}\Big\{\exp\Big(-\sum_{\ell=j+1}^{m_k} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{m_k} r(\ell)h\Big)\Big\}\\
&\le y(0)\exp\Big(-\sum_{i=0}^{m_k} r(i)h\Big) + \sum_{j=0}^{m_k} \frac{1}{K(j)}\Big\{\exp\Big(-\sum_{\ell=j+1}^{m_k} r(\ell)h\Big) - \exp\Big(-\sum_{\ell=j}^{m_k} r(\ell)h\Big)\Big\}.
\end{aligned}$$

Thus (2.8) is proved. By changing the summation variables in (2.8), we get

$$y(n) \le y(0)\exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) + \sum_{m=1}^{n} \frac{1-e^{-r(n-m)h}}{K(n-m)} \exp\Big(-\sum_{\ell=1}^{m-1} r(n-\ell)h\Big). \tag{2.9}$$

Since $y(n) > 0$ for all $n \in \mathbb{Z}_0^+$, we can substitute $x(n) = \frac{1}{y(n)}$, $n \in \mathbb{Z}_0^+$, and this completes the proof. □

Now we will study the asymptotic behavior of the solutions of (2.5) where $r(\cdot)$ and $K(\cdot)$ are time dependent.

Theorem 2. *Let all assumptions of Theorem 1 hold and suppose further that there exists a number $\hat{r} > 0$ such that*

$$\lim_{m\to\infty} \frac{1}{m}\Big\{\sum_{j=1}^{m} r(n-j)\Big\} = \hat{r},\ m \in \mathbb{Z}^+,\ \text{uniformly on } n \in \mathbb{Z}. \tag{2.10}$$

Then the solution $x(n)$ of the system (2.5) tends to $x^*(n)$ as $n \to \infty$, where $x^*(n)$ is given by

$$x^*(n) = \Big[\sum_{j=1}^{\infty} \frac{1-e^{-r(n-j)h}}{K(n-j)} \exp\Big(-\sum_{\ell=1}^{j-1} r(n-\ell)h\Big)\Big]^{-1}$$

in the sense that $x(n) - x^*(n) \to 0$ as $n \to \infty$.

Proof. Since $x(n)$ is positive, we can use $y(n) = 1/x(n)$, $n \in \mathbb{Z}_0^+$, and from Theorem 1 we have

$$y(n) \le y(0)\exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) + \sum_{j=1}^{n} \frac{1-e^{-r(n-j)h}}{K(n-j)} \exp\Big(-\sum_{\ell=1}^{j-1} r(n-\ell)h\Big),$$

then

$$\begin{aligned}
&y(n) - \sum_{j=1}^{\infty} \frac{1-e^{-r(n-j)h}}{K(n-j)} \exp\Big(-\sum_{\ell=1}^{j-1} r(n-\ell)h\Big)\\
&\le y(0)\exp\Big(-\sum_{i=0}^{n-1} r(i)h\Big) - \sum_{j=n+1}^{\infty} \frac{1-e^{-r(n-j)h}}{K(n-j)} \exp\Big(-\sum_{\ell=1}^{j-1} r(n-\ell)h\Big)
\end{aligned}$$

and hence

$$\begin{aligned}&\Big|y(n)-\sum_{j=1}^{\infty}\frac{1-e^{-r(n-j)h}}{K(n-j)}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)\Big|\\&\le y(0)\exp\Big(-\sum_{i=0}^{n-1}r(i)h\Big)+\sum_{j=n+1}^{\infty}\frac{1-e^{-r(n-j)h}}{K(n-j)}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big).\end{aligned}\tag{2.11}$$

Let us choose a number ξ satisfying $0<\xi<\hat{r}$. From the assumption (2.10), there corresponds a positive integer $N=N(\xi)$ such that

$$n(\hat{r}-\xi)h\le\sum_{i=0}^{n-1}r(i)h\le n(\hat{r}+\xi)h\quad\forall\, n\ge N.$$

By substituting into (2.11), for $n\ge N$ we obtain

$$\begin{aligned}&\Big|y(n)-\sum_{j=1}^{\infty}\frac{1-e^{-r(n-j)h}}{K(n-j)}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)\Big|\\&\le y(0)e^{-n(\hat{r}-\xi)h}+\frac{1}{K_*}\sum_{j=n+1}^{\infty}e^{-(j-1)(\hat{r}-\xi)h}\\&=y(0)e^{-n(\hat{r}-\xi)h}+\frac{1}{K_*}\frac{e^{-n(\hat{r}-\xi)h}}{1-e^{-(\hat{r}-\xi)h}}\to0\quad\text{as } n\to\infty.\end{aligned}$$

Let us denote

$$y^*(n)=\sum_{j=1}^{\infty}\frac{1-e^{-r(n-j)h}}{K(n-j)}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big).$$

Then $y(n)-y^*(n)\to0$ as $n\to\infty$, $y^*(n)>0$ for all $n\in\mathbb{Z}$. In the following, we will show that $y^*(n)<\infty$, *i.e.*, the series $y^*(n)$ is convergent with respect to $n\in\mathbb{Z}$. As above, for the number ξ there corresponds a positive integer $N=N(\xi)$ such that

$$m(\hat{r}-\xi)h\le\sum_{j=1}^{m}r(n-j)h\le m(\hat{r}+\xi)h \text{ for } m\ge N.\tag{2.12}$$

Now, let us consider the series $\sum_{j=1}^{\infty}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)$ and rewrite it as

$$\sum_{j=1}^{\infty}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)=\sum_{j=1}^{N}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)+\sum_{j=N+1}^{\infty}\exp\Big(\sum_{\ell=1}^{j-1}r(n-\ell)h\Big).$$

We note that the integer N is positive and finite and the first sequence in the above equality is bounded and non-negative for all $n\in\mathbb{Z}$. Thus there exists a finite positive real number A for which $\sum_{j=1}^{N}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)\le A$ for all $n\in\mathbb{Z}$.

Furthermore, we have that

$$\sum_{j=N+1}^{\infty}\exp\Big(-\sum_{\ell=1}^{j-1}r(n-\ell)h\Big)\le\frac{e^{-N(\hat{r}-\xi)h}}{1-e^{-(\hat{r}-\xi)h}}.$$

Thus we have

$$y^*(n)\le\frac{1}{K_*}\Big\{A+\frac{e^{-N(\hat{r}-\xi)h}}{1-e^{-(\hat{r}-\xi)h}}\Big\}.$$

As above, using the right-hand side in (2.12) and the condition $\sup_{n\in\mathbb{Z}_0^+} K(n) < \infty$, we find that $\liminf_{n\to\infty} y^*(n) > 0$. Since $y(n) > 0$ for all n and $y(n) - y^*(n) \to 0$ as $n \to \infty$, we have $\liminf_{n\to\infty} y(n) > 0$ and $\sup_{n\in\mathbb{Z}_0^+} y(n) < \infty$. Thus $x(n)$ satisfies

$$\lim_{n\to\infty}(x(n) - x^*(n)) = \lim_{n\to\infty}\big(\frac{1}{y(n)} - \frac{1}{y^*(n)}\big) = 0$$

and the proof is complete. □

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