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DECAY ESTIMATES FOR SOLUTIONS OF SOME SYSTEMS FOR ELASTICITY WITH NONLINEAR BOUNDARY FEEDBACK

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ABSTRACT. We study the energy decay rate for the Euler-Bernoulli and Kirchhoff plate equations. Under suitable growth assumptions on the nonlinear dissipative boundary feedback functions, we obtain some new results.

1. Introduction and main results

We consider the following Euler-Bernoulli beam equation with nonlinear boundary feedback controls:

$$\begin{gathered} y_{tt} + y_{xxxx} = 0 \quad \text{in } (0,1),\ t>0, \\ y(0,t) = y_x(0,t) = 0 \quad (\text{clamped at } x=0),\ t>0, \\ -y_{xx}(1,t) = h(y_{xt}(1,t)) \quad (\text{moment}), t>0, \\ y_{xxx}(1,t) = g(y_t(1,t)) \quad (\text{force}), t>0, \\ y(.,0) = y_0 \in H_E^2, \quad y_t(.,0) = y_1 \in L^2(0,1), \end{gathered} \tag{1.1}$$

where $H_E^2 = \{u \in H^2(0,1);\ u(0) = u'(0) = 0\}$, x stands for the position and t the time. The flexural rigidity of the beam and the mass density are assumed to be equal to one. One end is controlled by a point force and point bending moment which are assumed to be nonlinear functions of the observation. By observation we mean velocity and angular velocity of the transversal deflexion at the end.

Throughout this paper, g and h are assumed to satisfy the following hypothesis

$$g \in C^0(\mathbb{R}),\ \text{nondecreasing},\ g(0)=0,\ g(s)s>0\ \forall s \neq 0\,, \tag{1.2}$$

$$h \in C^0(\mathbb{R}),\ \text{nondecreasing},\ h(0)=0,\ h(s)s>0\ \forall s \neq 0\,. \tag{1.3}$$

The total energy of system (1.1) is

$$\widetilde{E}(t) = \frac{1}{2}\int_0^1 (y_t^2(x,t) + y_{xx}^2(x,t))dx. \tag{1.4}$$

Formally,

$$\frac{d}{dt}\widetilde{E}(t) = -y_t(1,t)\ g(y_t(1,t)) - y_{xt}(1,t)\ h(y_{xt}(1,t)) \tag{1.5}$$

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Assumptions (1.2) and (1.3) imply that the energy $\widetilde{E}(t)$ is non-increasing and is a Lyapounov function. It is well-known that for any initial data $(y_0, y_1) \in \mathcal{H} = H^2_E \times L^2(0,1)$, Problem (1.1) admits a unique weak solution y such that $(y(t), y_t(t)) \in \mathcal{H}$ and $y \in C^0(\mathbb{R}^+; H^4(0,1)) \cap C^1(\mathbb{R}^+; L^2(0,1))$ (see Conrad and Pierre [2]). The study of the strong asymptotic stability of the solution of (1.1) in $\mathcal{H}$, using the invariance principle of Lasalle has been proved by Conrad and Pierre [2] in the case where g and h are multivalued maximal monotone graphs. Our aim in this study is to estimate the rate of decay of the energy $\widetilde{E}(t)$ when the nonlinear feedback functions g and h satisfy suitable growth conditions. However, the extension of this method to the case where g and h are maximal monotone graphs seems difficult. Our two main results are as follows.

Theorem 1.1. *Assume* (1.2) *and* (1.3) *hold. Then for every solution y of system* (1.1)*, we have*
(i) If there exist positive constants C_1, C_2, C_3, C_4 such that for all $x \in \mathbb{R}$,

$$\begin{gathered} C_1|x| \le |g(x)| \le C_2|x|, \\ C_3|x| \le |h(x)| \le C_4|x|, \end{gathered} \tag{1.6}$$

then given any $M > 1$, there exists a constant $\omega > 0$ such that

$$\widetilde{E}(t) \le M\widetilde{E}(0)e^{-\omega t} \quad \forall t > 0.$$

(ii) If there exist positive constants C_1, C_2, C_3, C_4, and p, q in $[1, +\infty[$ such that $\max(p,q) = p \vee q > 1$ such that for all $x \in \mathbb{R}$,

$$\begin{gathered} C_1 \min(|x|, |x|^p) \le |g(x)| \le C_2|x|, \\ C_3 \min(|x|, |x|^q) \le |h(x)| \le C_4|x|, \end{gathered} \tag{1.7}$$

then, given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $\widetilde{E}(0)$ such that

$$\widetilde{E}(t) \le M\widetilde{E}(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}} \quad \forall t \ge 0.$$

Theorem 1.2. *Assume* (1.2) *and* (1.3) *hold. Then for every solution y of system* (1.1) *we have*
(i)If there exist positive constants C_1, C_2, C_3, C_4 and $p, q \in]0,1]$ with $\min(p,q) = p \wedge q < 1$ such that for all $x \in \mathbb{R}$,

$$\begin{gathered} C_1|x| \le |g(x)| \le C_2 \max(|x|, |x|^p), \\ C_3|x| \le |h(x)| \le C_4 \max(|x|, |x|^q), \end{gathered} \tag{1.8}$$

then, given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $\widetilde{E}(0)$ such that

$$\widetilde{E}(t) \le M\widetilde{E}(0)(1+\omega t)^{-\frac{2(p\wedge q)}{1-(p\wedge q)}}, \quad \forall t \ge 0.$$

(ii) If there exists positive constants C_1, C_2, C_3, C_4 and $(p,q) \in]0,1] \times [1, +\infty[$ with $\frac{1}{p} \vee q > 1$ such that for all $x \in \mathbb{R}$,

$$\begin{gathered} C_1|x| \le |g(x)| \le C_2 \max(|x|, |x|^p), \\ C_3 \min(|x|, |x|^q) \le |h(x)| \le C_4|x|, \end{gathered} \tag{1.9}$$

then given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $\widetilde{E}(0)$ such that

$$\widetilde{E}(t) \le M\widetilde{E}(0)(1+\omega t)^{-\frac{2}{(\frac{1}{p}\vee q)-1}} \quad \forall t \ge 0.$$

Theorems 1.1 and 1.2 will also be valid for the following model of a Kirchhoff plate equation (see Ciarlet [1] and Lagnese [5]), in star-shaped domain by nonlinear boundary feedback:

$$\begin{gathered} y_{tt} + \Delta^2 y = 0 \quad \text{in } \Omega\times]0,+\infty[, \\ \Delta y + (1-\mu)B_1 y = v_1 \quad \text{on } \Gamma\times]0,+\infty[, \\ \frac{\partial}{\partial\nu}\Delta y + (1-\mu)\frac{\partial}{\partial\tau}B_2 y = v_2 \quad \text{on } \Gamma\times]0,+\infty[, \\ y(0) = y_0 \in H^2(\Omega), \quad y_t(0) = y_1 \in L^2(\Omega). \end{gathered} \tag{1.10}$$

We assume that Ω is a bounded strongly star-shaped domain of $\mathbb{R}^2$ with respect to $x_0 \in \Omega$ and having smooth boundary $\Gamma = \partial\Omega$ of class C^2, which means there exists a positive constant δ such that

$$m(x).\nu(x) \geq \delta^{-1} \quad \forall x \in \Gamma, \tag{1.11}$$

where $\nu(x) = (\nu_1(x), \nu_2(x))$ is the unit outer normal vector to Γ, $m(x) = x - x_0$, and the dot "." denotes the scalar product in $\mathbb{R}^2$. $\tau(x) = (-\nu_2(x), \nu_1(x))$ is a unit tangent vector. We denote by $\frac{\partial}{\partial\nu}$ (resp. $\frac{\partial}{\partial\tau}$,) the normal derivative (resp. tangent derivative). The constant $0 < \mu < 1/2$ is the Poisson coefficient and the boundary operators B_1, B_2 are defined by

$$B_1 y = 2\nu_1\nu_2\frac{\partial^2 y}{\partial x_1\partial x_2} - \nu_1^2\frac{\partial^2 y}{\partial x_2^2} - \nu_2^2\frac{\partial^2 y}{\partial x_1^2}, \quad B_2 y = (\nu_1^2 - \nu_2^2)\frac{\partial^2 y}{\partial x_1\partial x_2} + \nu_1\nu_2(\frac{\partial^2 y}{\partial x_1^2} - \frac{\partial^2 y}{\partial x_2^2}),$$

where $y = y(x_1, x_2, t)$ is the vertical displacement of the point $x = (x_1, x_2) \in \Omega$ at the time t of the plate.

The aim of this paper is to study the uniform energy decay rate of the system (1.10) by the nonlinear feedback laws v_1 and v_2 given as follows

$$v_1(t) = -\beta\frac{\partial y}{\partial\nu} - h(\frac{\partial y_t}{\partial\nu}), \quad v_2(t) = \alpha y + g(y_t) \quad \text{on } \Gamma\times]0,+\infty[, \tag{1.12}$$

$$(\alpha,\beta) \in (L^\infty(\Gamma))^2, \quad 0 < \alpha_0 \leq \alpha(x) \leq \alpha_1, \quad 0 < \beta_0 \leq \beta(x) \leq \beta_1 \quad \forall x \in \Gamma. \tag{1.13}$$

It is well-known that for any initial data $(y_0, y_1) \in V = H^2(\Omega) \times L^2(\Omega)$, the system (1.10) has a unique weak solution y such that $(y(t), y_t(t)) \in V$ and $y \in C^0(\mathbb{R}^+, H^2(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega))$ (see Rao [7]).

We introduce the energy associated with the system (1.10) as follows

$$E(t) = \frac{1}{2}(\int_\Omega |y_t|^2 dx + a(y,y) + \int_\Gamma (\alpha|y|^2 + \beta|\frac{\partial y}{\partial\nu}|^2) d\Gamma), \tag{1.14}$$

where

$$a(y,y) = \int_\Omega ((\frac{\partial^2 y}{\partial x_1^2})^2 + (\frac{\partial^2 y}{\partial x_2^2})^2 + 2\mu\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} + 2(1-\mu)(\frac{\partial^2 y}{\partial x_1\partial x_2})^2) dx.$$

Using Green's formula, the derivative of the energy $E(t)$ is

$$\frac{dE}{dt} = -\int_\Gamma (g(y_t)y_t + h(\frac{\partial y_t}{\partial\nu})\frac{\partial y_t}{\partial\nu}) d\Gamma. \tag{1.15}$$

Assumptions (1.2) and (1.3) imply that the energy is non-increasing and is a Lyapounov function. It is easy to prove the strong stabilization by applying Holmgren's theorem (see Lions [6]) and Lasalle's invariance principle (see [8]). The problem of estimating the rate of decay of the energy $E(t)$ has been studied extensively by many authors, among which we can mention Lagnese [5], Komornik - Zuazua [4] and Zuazua [9]. In the nonlinear cases, under suitable growth conditions on the functions g and h, Rao [7] has established exponential or rational rates of decay of the energy for any positive functions α, β given by (1.13). We obtain here an improvement of these results in the sense that we prove the estimates when the nonlinear feedback functions g and h satisfy more general growth conditions. Now, we state our main results.

Theorem 1.3. *Assume* (1.2), (1.3), (1.11), (1.13). *Then for every solution y of system* (1.10) *with feedback laws* (1.12), *we have*
(i) In addition, if we assume that g and h such that (1.7) *holds. Then, given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $E(0)$ such that*

$$E(t) \leq ME(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}}, \quad \forall t \geq 0.$$

(ii) In addition, if we assume that g and h such that (1.8) *holds. Then, given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $E(0)$ such that*

$$E(t) \leq ME(0)(1+\omega t)^{-\frac{2(p\wedge q)}{1-(p\wedge q)}}, \quad \forall t \geq 0.$$

(iii) In addition, if we assume that g and h such that (1.9) *holds. Then, given any $M > 1$, there exists a constant $\omega > 0$ depending continuously on $E(0)$ such that*

$$E(t) \leq ME(0)(1+\omega t)^{-\frac{2}{(\frac{1}{p}\vee q)-1}} \quad \forall t \geq 0.$$

2. Proof of Theorem 1.1

Let y be a smooth solution of (1.1). We define the functional

$$\rho(t) = \rho_1(t) + \rho_2(t) + \rho_3(t),$$

where

$$\rho_1 = 4\int_0^1 xy_t y_x dx, \quad \rho_2 = \frac{1}{4}y(1,t)\int_0^1 x^2(3-2x)y_t dx,$$
$$\rho_3 = \frac{1}{4}y_x(1,t)\int_0^1 x^2(x-1)y_t dx.$$

We can show that there exist positive constants K_0, K_1 and K_2 such that for any $t \geq 0$, the following estimates hold

$$|\rho(t)| \leq K_0\widetilde{E}(t), \tag{2.1}$$

$$\frac{d\rho}{dt}(t) \leq -\widetilde{E}(t) + K_1(y_{xt}^2(1,t) + y_t^2(1,t)) + K_2(g^2(y_t(1,t)) + h^2(y_{xt}(1,t))). \tag{2.2}$$

Given $\varepsilon > 0$, we introduce the perturbed energy by

$$\widetilde{E}_\varepsilon(t) = \widetilde{E}(t) + \varepsilon\rho(t)\big[\widetilde{E}(t)\big]^{\frac{(p\vee q)-1}{2}}. \tag{2.3}$$

This together with the non-increasing of the energy $\widetilde{E}(t)$ implies that for any $M > 1$

$$M^{-\frac{1}{2}}\big[\widetilde{E}_\varepsilon(t)\big]^{\frac{(p\vee q)+1}{2}} \leq \big[\widetilde{E}(t)\big]^{\frac{(p\vee q)+1}{2}} \leq M^{1/2}\big[\widetilde{E}_\varepsilon(t)\big]^{\frac{(p\vee q)+1}{2}} \tag{2.4}$$

with

$$\varepsilon \leq K_0^{-1}\big[\widetilde{E}(0)\big]^{\frac{1-(p\vee q)}{2}}(1 - M^{\frac{-1}{(p\vee q)+1}}). \tag{2.5}$$

Now, we calculate the derivative of the perturbed energy $\widetilde{E}_\varepsilon(t)$

$$\widetilde{E}'_\varepsilon(t) = \widetilde{E}'(t) + \varepsilon\frac{(p\vee q)-1}{2}\rho(t)\widetilde{E}'(t)[\widetilde{E}(t)]^{\frac{(p\vee q)-3}{2}} + \varepsilon\rho'(t)[\widetilde{E}(t)]^{\frac{(p\vee q)-1}{2}}, \tag{2.6}$$

on the other hand, from (1.6), (1.7) and (2.2) one obtains

$$\rho'(t) \le -\widetilde{E}(t) + K_3 y_t^2(1,t) + K_4 y_{xt}^2(1,t) \tag{2.7}$$

where $K_3 = K_1 + K_2C_2^2$ and $K_4 = K_1 + K_2C_4^2$. Plugging (2.1) and (2.7) into equation (2.6), one obtains

$$\widetilde{E}'_\varepsilon(t) \le (-1+\varepsilon\frac{(p\vee q)-1}{2}K_0[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}})(-\widetilde{E}'(t))+F_1+F_2-\varepsilon[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}} \tag{2.8}$$

where $F_1 = \varepsilon K_3[\widetilde{E}(t)]^{\frac{(p\vee q)-1}{2}}y_t^2(1,t)$ and $F_2 = \varepsilon K_4[\widetilde{E}(t)]^{\frac{(p\vee q)-1}{2}}y_{xt}^2(1,t))$. Now we distinguish the case $p\vee q = 1$ and $p\vee q > 1$.

(i) Case $p\vee q = 1$. In this case (2.8) yields

$$\widetilde{E}'_\epsilon(t) \le (-1+\varepsilon K_3/C_1)y_t(1,t)g(y_t(1,t)+(-1+\varepsilon K_4/C_3)y_{xt}(1,t)h(y_{xt}(1,t))-\varepsilon\widetilde{E}(t).$$

If we choose $\varepsilon \le \min(C_1/K_3, C_3/K_4, K_0^{-1}[\widetilde{E}(0)]^{\frac{1-(p\vee q)}{2}}(1-M^{\frac{-1}{(p\vee q)+1}}))$, this implies

$$\widetilde{E}'_\varepsilon(t) \le -\varepsilon\widetilde{E}(t) \le -\varepsilon M^{\frac{-1}{2}}\widetilde{E}_\varepsilon(t),$$

so we obtain

$$\widetilde{E}(t) \le M\widetilde{E}(0)e^{-\varepsilon M^{\frac{-1}{2}}t}, \quad \forall t > 0.$$

(ii) Case $p\vee q > 1$. If $y_{xt}^2(1,t) > 1$, it follows from hypothesis (1.2), (1.3) and (1.7) that

$$F_2 \le \varepsilon(K_4/C_3)[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}y_{xt}(1,t)h(y_{xt}(1,t)). \tag{2.9}$$

However, while $y_{xt}^2(1,t) \le 1$, we have $\min(|y_{xt}(1,t)|, |y_{xt}(1,t)|^q) = |y_{xt}(1,t)|^q$ and

$$|y_{xt}(1,t)|^{(p\vee q)+1} \le |y_{xt}(1,t)|^{q+1} \le (1/C_3)y_{xt}(1,t)h(y_{xt}(1,t)),$$

by Young's inequality, we have for any $\delta > 0$,

$$\begin{aligned} F_2 \le{}& \varepsilon\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}}[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}} \\ &+ \varepsilon\frac{2}{C_3(p\vee q+1)}(K_4\delta)^{\frac{(p\vee q)+1}{2}}y_{xt}(1,t)h(y_{xt}(1,t)). \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), one has

$$F_2 \le \varepsilon\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}}[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}} + \varepsilon K_5 y_{xt}(1,t)h(y_{xt}(1,t)), \tag{2.11}$$

where

$$K_5 = \frac{2}{C_3(p\vee q+1)}(K_4\delta)^{\frac{(p\vee q)+1}{2}} + (K_4/C_3)[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}.$$

Similarly, we can show that

$$F_1 \le \varepsilon\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}}[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}} + \varepsilon K_6 y_t(1,t)g(y_t(1,t)), \tag{2.12}$$

with

$$K_6 = \frac{2}{C_1(p\vee q+1)}(K_4\delta)^{\frac{(p\vee q)+1}{2}} + (K_3/C_1)[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}.$$

Inserting (1.5),(2.11) and (2.12) into (2.8), we obtain

$$\begin{aligned}\widetilde{E}'_\varepsilon(t) &\le \varepsilon\Big(-1+2\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}}\Big)[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}} + (-1+\varepsilon\lambda_0)y_t(1,t)g(y_t(1,t))\\ &\quad + (-1+\varepsilon\lambda_1)y_{xt}(1,t)h(y_{xt}(1,t)),\end{aligned}$$

where $\lambda_i = K_{6-i} + K_0\frac{(p\vee q)-1}{2}[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}$ for $i=0,1$. This implies that

$$\widetilde{E}'_\varepsilon(t) \le -\mu\varepsilon[\widetilde{E}(t)]^{\frac{(p\vee q)+1}{2}}, \tag{2.13}$$

provided δ is chosen such that for some $\mu>0$, $2\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} \le 1-\mu$ and ε is chosen as follows $-1+\varepsilon(K_{6-i}+K_0\frac{(p\vee q)-1}{2}[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}) \le 0$ for $i=0,1$. Combining (2.4) and (2.13), we get

$$\widetilde{E}'_\varepsilon(t) \le -\mu\varepsilon M^{\frac{-1}{2}}[\widetilde{E}_\varepsilon(t)]^{\frac{(p\vee q)+1}{2}}. \tag{2.14}$$

Finally, solving the differential inequality (2.14) and using (2.4) we obtain

$$\widetilde{E}(t) \le M\widetilde{E}(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}},$$

with $\omega = \frac{(p\vee q)-1}{2}\mu\varepsilon M^{\frac{-(p\vee q)}{(p\vee q)+1}}[\widetilde{E}(0)]^{\frac{(p\vee q)-1}{2}}$. This completes the proof of theorem 1.1.

3. Proof of Theorem 1.2

(i) First, by the conditions (1.8) and (2.2), we can deduce the following estimate

$$\rho'(t) \le -\widetilde{E}(t) + K(g^2(y_t(1,t)) + h^2(y_{xt}(1,t))), \tag{3.1}$$

with $K = K_2 + K_1(1/C_1^2 + 1/C_2^2)$. Next, given $\varepsilon>0$, we introduce the perturbed energy by

$$\widetilde{E}_\varepsilon(t) = \widetilde{E}(t) + \varepsilon\rho(t)[\widetilde{E}(t)]^{\frac{1-(p\wedge q)}{2(p\wedge q)}}. \tag{3.2}$$

Then, for any $M>1$, we have (2.4) and (2.5) with $1/p$ instead of p. We have also (2.6) with $1/p$ instead of p. Plugging (1.5), (2.1) and (3.1) into equation (2.6) one obtains

$$\widetilde{E}'_\varepsilon(t) \le \Big(-1+\varepsilon K_0\frac{1-(p\wedge q)}{2(p\wedge q)}[\widetilde{E}(0)]^{\frac{1-(p\wedge q)}{2(p\wedge q)}}\Big)(-\widetilde{E}'(t)) + D_1 + D_2 - \varepsilon[\widetilde{E}(t)]^{\frac{1+(p\wedge q)}{2(p\wedge q)}}. \tag{3.3}$$

where $D_1 = \varepsilon K(\widetilde{E}(t))^{\frac{1+(p\wedge q)}{2(p\wedge q)}} g^2(y_t(1,t)$ and $D_2 = \varepsilon K(\widetilde{E}(t))^{\frac{1+(p\wedge q)}{2(p\wedge q)}} h^2(y_{xt}(1,t)$.

If $y_{xt}^2(1,t) \ge 1$, it follows from hypothesis (1.2), (1.3) and (1.8), we have

$$D_2 \le \varepsilon C_4 K[\widetilde{E}(0)]^{\frac{1-(p\wedge q)}{2(p\wedge q)}} h(y_{xt}(1,t))y_{xt}(1,t). \tag{3.4}$$

However, while $y_{xt}^2(1,t) < 1$, we have $\max(|y_{xt}(1,t)|, |y_{xt}(1,t)|^q) = |y_{xt}(1,t)|^q$, so

$$|h(y_{xt}(1,t))|^{\frac{1+(p\wedge q)}{(p\wedge q)}} \le C_4 y_{xt}(1,t)h(y_{xt}(1,t),$$

by Young's inequality we obtain

$$D_2 \le (\varepsilon/4)[\widetilde{E}(t)]^{\frac{1+(p\wedge q)}{2(p\wedge q)}} + \varepsilon C_4(4K)^{\frac{1+(p\wedge q)}{2(p\wedge q)}} y_{xt}(1,t)h(y_{xt}(1,t)). \tag{3.5}$$

Setting $\theta_i = C_{4-i}((4K)^{\frac{1+(p\wedge q)}{2(p\wedge q)}} + K\,(\widetilde{E}(0))^{\frac{1-(p\wedge q)}{2(p\wedge q)}})$ for $i=0$ and $i=2$ and combining (3.4) and (3.5), one obtains

$$D_2 \le (\varepsilon/4)[\widetilde{E}(t)]^{\frac{1+(p\wedge q)}{2(p\wedge q)}} + \varepsilon\theta_0 y_{xt}(1,t)h(y_{xt}(1,t)). \tag{3.6}$$

Similarly we can show that

$$D_1 \le (\varepsilon/4)[\widetilde{E}(t)]^{\frac{1+(p\wedge q)}{2(p\wedge q)}} + \varepsilon\theta_2 y_t(1,t)g(y_t(1,t)) \tag{3.7}$$

Inserting (1.5), (3.6) and (3.7) into (3.3), it follows

$$\widetilde{E}'_\varepsilon(t) \le -\frac{\varepsilon}{2}[\widetilde{E}(t)]^{\frac{1+(p\wedge q)}{2(p\wedge q)}} + C(\varepsilon)(y_t(1,t)g(y_t(1,t)) + y_{xt}(1,t)h(y_{xt}(1,t))),$$

where $C(\varepsilon) = \varepsilon(\theta_0+\theta_2) - 1 + \varepsilon K_0 \frac{1-(p\wedge q)}{2(p\wedge q)}(\widetilde{E}(0))^{\frac{1-(p\wedge q)}{2(p\wedge q)}}$. Using (2.4)–(2.5) with $1/p$ instead of p, we can show that

$$\widetilde{E}(t) \le M\widetilde{E}(0)(1+\omega t)^{-\frac{2(p\wedge q)}{1-(p\wedge q)}}, \tag{3.8}$$

where $\omega = \varepsilon\frac{1-(p\wedge q)}{4(p\wedge q)} M^{-\frac{1}{1+(p\wedge q)}}(\widetilde{E}(0))^{\frac{1-(p\wedge q)}{2(p\wedge q)}}$ and ε is chosen such that

$$\begin{aligned}\varepsilon \le \min\Big(&((\theta_0+\theta_2) + K_0\frac{1-(p\wedge q)}{2(p\wedge q)}(\widetilde{E}(0))^{\frac{1-(p\wedge q)}{2(p\wedge q)}})^{-1},\\ &K_0^{-1}(\widetilde{E}(0))^{\frac{(p\wedge q)-1}{2(p\wedge q)}}(1-M^{-\frac{(p\wedge q)}{(p\wedge q+1)}})\Big).\end{aligned}$$

(ii) First, by the conditions (1.9) and (2.2), we have

$$\rho'(t) \le -\widetilde{E}(t) + Ay_{xt}^2(1,t) + Bg^2(y_t(1,t)), \tag{3.9}$$

with $A = K_1 + K_2C_4^2$ and $B = (K_1/C_1^2) + K_2$. Next, given $\varepsilon > 0$, we introduce the perturbed energy by

$$\widetilde{E}_\varepsilon(t) = \widetilde{E}(t) + \varepsilon\rho(t)[\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)-1}{2}}. \tag{3.10}$$

Then, for any $M > 1$, we have (2.4) and (2.5) with $1/p$ instead of p. We have also (2.6) with $1/p$ instead of p. Plugging (2.1) and (3.9) into (2.4) one obtains

$$\widetilde{E}'_\varepsilon(t) \le (-1+\varepsilon K_0\frac{(\frac{1}{p}\vee q)-1}{2}[\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}})(-\widetilde{E}'(t)) + E_1 + E_2 - \varepsilon[\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}} \tag{3.11}$$

where

$$E_1 = \varepsilon A(\widetilde{E}(t))^{\frac{(\frac{1}{p}\vee q)-1}{2}} y_{xt}^2(1,t), \quad E_2 = \varepsilon B(\widetilde{E}(t))^{\frac{(\frac{1}{p}\vee q)-1}{2}} g^2(y_t(1,t))).$$

If $y_{xt}^2(1,t) > 1$, it follows from hypothesis (1.2), (1.3) and (1.9), that

$$E_1 \le (A/C_3)\varepsilon[\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}} y_{xt}(1,t)h(y_{xt}(1,t)) \tag{3.12}$$

However, while $y_{xt}^2(1,t) \le 1$, we have $min(|y_{xt}(1,t)|, |y_{xt}(1,t)|^q) = |y_{xt}(1,t)|^q$,

$$|y_{xt}(1,t)|^{1+(\frac{1}{p}\vee q)} \le |y_{xt}(1,t)|^{q+1} \le (1/C_3)y_{xt}(1,t)h(y_{xt}(1,t)),$$

and by Young's inequality, for any $\delta > 0$, we have

$$E_1 \le \varepsilon/\beta' C_3)(A\delta)^{\beta'} y_{xt}(1,t)h(y_{xt}(1,t)) + (\varepsilon/\beta)\delta^{-\beta}[\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}}. \tag{3.13}$$

Combining (3.11) and (3.12), one has

$$E_1 \le (\varepsilon/\beta)\delta^{-\beta}[\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}} + \varepsilon A(\delta)\ y_{xt}(1,t)h(y_{xt}(1,t)), \tag{3.14}$$

with $A(\delta) = (A/C_3)[\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}} + (1/\beta' C_3)(A\delta)^{\beta'}$. If $y_t^2(1,t) \geq 1$, it follows from hypothesis (1.2), (1.3) and (1.9), that

$$E_2 \leq \varepsilon B C_2 [\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}} y_t(1,t) g(y_t(1,t)). \tag{3.15}$$

However, while $y_t^2(1,t) < 1$, we have $\max(|y_t(1,t)|, |y_t(1,t)|^p) = |y_t(1,t)|^p$,

$$|g(y_t(1,t))|^{1+(\frac{1}{p}\vee q)} \leq (C_2)^{(\frac{1}{p}\vee q)} y_t(1,t) g(y_t(1,t))$$

and by Young's inequality, we can deduce that

$$E_2 \leq \varepsilon (4B)^{\frac{(\frac{1}{p}\vee q)-1}{2}} (C_2)^{\frac{1}{p}\vee q} y_t(1,t) g(y_t(1,t)) + \varepsilon 4^{-\beta} [\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}}. \tag{3.16}$$

Combining (3.14) and (3.15), we obtain

$$E_2 \leq \varepsilon (C_2 B [\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}} + (4B)^{\frac{(\frac{1}{p}\vee q)-1}{2}} (C_2)^{\frac{1}{p}\vee q}) y_t(1,t) g(y_t(1,t))) + \frac{\varepsilon}{4} [\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}}. \tag{3.17}$$

Inserting (3.13) and (3.16) into (2.6), one obtains

$$\begin{aligned} \widetilde{E}'_\varepsilon(t) \leq & -\varepsilon (1 - \frac{1}{4} - (\delta^{-\beta}/\beta)) [\widetilde{E}(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}} + (-1 + \varepsilon \sigma_1) y_t(1,t) g(y_t(1,t)) \\ & + (-1 + \varepsilon \sigma_2) y_{xt}(1,t) h(y_{xt}(1,t)) \end{aligned} \tag{3.18}$$

where

$$\sigma_1 = (C_2 B + K_0 \frac{(\frac{1}{p}\vee q) - 1}{2}) [\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}} + (4B)^{\beta'} (C_2)^{\frac{1}{p}\vee q})$$

and

$$\sigma_2 = A(\delta) + K_0 \frac{(\frac{1}{p}\vee q) - 1}{2} [\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}}.$$

By choosing δ so that $\frac{1}{4} + (\delta^{-\beta}/\beta) = \frac{1}{2}$, and

$$\varepsilon \leq \min(\sigma_1^{-1}, \sigma_2^{-1}, K_0^{-1} [\widetilde{E}(0)]^{\frac{1-(\frac{1}{p}\vee q)}{2}} (1 - M^{\frac{-1}{(\frac{1}{p}\vee q)+1}})).$$

This with (3.17) and using (2.4) with $1/p$ instead of p, we obtain

$$\widetilde{E}(t) \leq M \widetilde{E}(0) (1 + \omega t)^{-\frac{2}{(\frac{1}{p}\vee q)-1}},$$

with $\omega = \frac{\varepsilon}{4}((\frac{1}{p}\vee q) - 1) M^{-\frac{\frac{1}{p}\vee q}{(\frac{1}{p}\vee q)+1}} [\widetilde{E}(0)]^{\frac{(\frac{1}{p}\vee q)-1}{2}}$. The proof of theorem 1.2 is thus complete.

4. Proof of Theorem 1.3

(i) It follows from (1.7) that (1.7) is also true when p is replaced by $p \vee q$ and q is replaced by $p \vee q$. So we get the desired estimate, by applying Rao's theorem 1.1 (ii) (see [7]).

(ii) (1.8) implies that (1.8) is also true when p is replaced by $p \wedge q$ and q is replaced by $p \wedge q$. This is exactly the condition (1.28)-(1.29) with exponent $p \wedge q < 1$ of Rao's theorem 1.2 (see [7]), consequently, the proof of this part is thus complete.

(iii) We adopt the method used by Rao [7] but in our case we have two nonlinear feedback functions g and h satisfying more general growth conditions. We introduce the same functional defined in [7],

$$\rho(t)=\int_\Omega y_t(m.\nabla y)dx+C_0\int_\Omega y_t\varphi dx, \tag{4.1}$$

where C_0 is a positive constant and φ is the solution of the problem

$$\begin{gathered}\Delta^2\varphi=0 \quad \text{in } \Omega,\\ \varphi=y,\quad \frac{\partial\varphi}{\partial\nu}=\frac{\partial y}{\partial\nu}\quad \text{on } \Gamma,\end{gathered}$$

we verify that the following estimates hold

$$\int_\Omega \varphi^2dx\le\gamma^2\int_\Gamma\big((y)^2+(\frac{\partial y}{\partial\nu})^2\big)d\Gamma,\quad a(\varphi,y)=a(\varphi,\varphi)\ge 0,$$

where γ is a constant depending only on the domain Ω. We can show that there exist positive constants K_0,K_1,K_2 are such that for any $t\ge 0$,

$$|\rho(t)|\le K_0E(t),\quad \forall t\ge 0, \tag{4.2}$$

$$\rho'(t)\le -E(t)+K_1\int_\Gamma((y_t)^2+(\frac{\partial y_t}{\partial\nu})^2)d\Gamma+K_2\int_\Gamma(g^2(y_t)+h^2(\frac{\partial y_t}{\partial\nu}))d\Gamma. \tag{4.3}$$

This with (1.9) implies

$$\rho'(t)\le -E(t)+A\int_\Gamma|\frac{\partial y_t}{\partial\nu}|^2d\Gamma+B\int_\Gamma g^2(y_t)d\Gamma, \tag{4.4}$$

where $A=K_1+KC_4^2$ and $B=(K_1/C_1^2)+K_2$.

Then, for any $M>1$, Plugging (4.2) and (4.4) into (2.6) with p replaced by $1/p$, one obtains

$$E_\varepsilon'(t)\le(-1+\varepsilon K_0\frac{(\frac1p\vee q)-1}{2}[E(0)]^{\frac{(\frac1p\vee q)-1}{2}})(-E'(t))-\varepsilon[E(t)]^{\frac{(\frac1p\vee q)+1}{2}}+L, \tag{4.5}$$

where

$$L=\varepsilon[E(t)]^{\frac{(\frac1p\vee q)-1}{2}}(A\int_\Gamma|\frac{\partial y_t}{\partial\nu}|^2d\Gamma+B\int_\Gamma g^2(y_t)d\Gamma).$$

¿From (1.2), (1.3) and (1.9) we have

$$(g(s))^2\le C_2g(s)s\ \ \forall|s|\ge 1;\quad |g(s)|^{(\frac1p\vee q)+1}\le(C_2)^{\frac1p\vee q}g(s)s\ \ \forall|s|\le 1; \tag{4.6}$$

$$|s|^2\le(1/C_3)h(s)s\ \ \forall|s|\ge 1;\quad |s|^{(\frac1p\vee q)+1}\le(1/C_3)h(s)s\ \ \forall|s|\le 1. \tag{4.7}$$

Using (4.6) and (4.7), we have

$$L\le Q+\varepsilon C[E(0)]^{\frac{(\frac1p\vee q)-1}{2}}\Big(\int_{\{|\frac{\partial y_t}{\partial\nu}|\ge 1\}}h(\frac{\partial y_t}{\partial\nu})\frac{\partial y_t}{\partial\nu}d\Gamma+\int_{\{|y_t|\ge 1\}}g(y_t)y_td\Gamma\Big), \tag{4.8}$$

where

$$Q=\varepsilon[E(t)]^{\frac{(\frac1p\vee q)-1}{2}}\Big(A\int_{\{|\frac{\partial y_t}{\partial\nu}|\le 1\}}(\frac{\partial y_t}{\partial\nu})^2d\Gamma+B\int_{\{|y_t|\le 1\}}g^2(y_t)d\Gamma\Big)$$

and $C = (A+B)(C_2 + 1/C_3)$, With the exponents $\beta = \frac{(\frac{1}{p}\vee q)+1}{(\frac{1}{p}\vee q)-1}$, $\beta' = \frac{(\frac{1}{p}\vee q)+1}{2}$, applying Young's inequality to Q, it follows from Hôlder's inequality and (4.6), (4.7) that for any parameter $\delta > 0$,

$$\begin{aligned} Q &\leq 2\varepsilon\delta^{-\beta}/\beta[E(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}} + (\varepsilon/\beta')|\Gamma|^{\beta'/\beta} \\ &\times \Big[(A\delta)^{\beta'}/C_3 \int_{\{|\frac{\partial y_t}{\partial\nu}|\leq 1\}} h(\frac{\partial y_t}{\partial\nu})\frac{\partial y_t}{\partial\nu} d\Gamma + (B\delta)^{\beta'} C_2{}^{(\frac{1}{p}\vee q)} \int_{\{|y_t|\leq 1\}} g(y_t)y_t d\Gamma\Big]. \end{aligned} \tag{4.9}$$

Inserting (4.6) into (4.8) gives

$$L \leq \varepsilon\eta(-E'(t)) + 2\varepsilon\delta^{-\beta}/\beta[E(t)]^{\frac{(\frac{1}{p}\vee q)+1}{2}} \tag{4.10}$$

where $\eta = C(E(0))^{\frac{(\frac{1}{p}\vee q)-1}{2}} + (1/\beta')|\Gamma|^{\beta'/\beta}((A\delta)^{\beta'}/C_3 + (B\delta)^{\beta'}(C_2)^{(\frac{1}{p}\vee q)})$. Inserting (4.10) into (4.5), it follows

$$\begin{aligned} E'_\varepsilon(t) &\leq \Big(-1 + \varepsilon(\eta + K_0\frac{(\frac{1}{p}\vee q)-1}{2}(E(0))^{\frac{(\frac{1}{p}\vee q)-1}{2}})\Big)(-E'(t)) \\ &- \varepsilon(1-2\delta^{-\beta}/\beta)(E(t))^{\frac{(\frac{1}{p}\vee q)+1}{2}}. \end{aligned} \tag{4.11}$$

By choosing δ so that $\delta = (4/\beta)^{(1/\beta)}$, and ε as

$$\varepsilon \leq \min((\eta + K_0\frac{(\frac{1}{p}\vee q)-1}{2}(E(0))^{\frac{(\frac{1}{p}\vee q)-1}{2}})^{-1}, K_0^{-1}(E(0))^{\frac{1-(\frac{1}{p}\vee q)}{2}}(1-M^{\frac{-1}{((\frac{1}{p}\vee q)+1)}})).$$

This with (4.11) and using (2.4) with p replaced by $1/p$, we obtain

$$E(t) \leq ME(0)(1+\omega t)^{-\frac{2}{(\frac{1}{p}\vee q)-1}},$$

with

$$\omega = \frac{\varepsilon}{4}((\frac{1}{p}\vee q)-1)M^{-\frac{\frac{1}{p}\vee q}{(\frac{1}{p}\vee q)+1}}(E(0))^{\frac{(\frac{1}{p}\vee q)-1}{2}}.$$

The proof of theorem 1.3 is thus complete.

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