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STABILITY OF THE PRINCIPAL EIGENVALUE OF A NONLINEAR ELLIPTIC SYSTEM

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ABSTRACT. This paper concerns some special properties of the principal eigenvalue of nonlinear elliptic systems with Dirichlet boundary conditions. We study the stability with respect to the exponents p and q; and the dependence on the domain variations.

1. INTRODUCTION

In this work, we survey two results concerning the nonlinear elliptic system

$$\begin{aligned} -\Delta_p u &= \lambda |u|^\alpha |v|^\beta v \quad \text{in } \Omega \\ -\Delta_q v &= \lambda |u|^\alpha |v|^\beta u \quad \text{in } \Omega \\ (u,v) &\in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, ($N \geq 3$), is a bounded domain not necessary regular; α, β, p, q are reel numbers such that $p > 1$, $q > 1$, $\alpha \geq 0$, $\beta \geq 0$ satisfying

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1. \tag{1.2}$$

The principal eigenvalue $\lambda_1(p,q)$ of (1.1) is defined as

$$\lambda_1 = \inf \big\{ A(u,v) : (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega),\, B(u,v) = 1 \big\}, \tag{1.3}$$

where

$$A(u,v) = \frac{\alpha+1}{p} \int_\Omega |\nabla u|^p \,dx + \frac{\beta+1}{q} \int_\Omega |\nabla v|^q \,dx,$$
$$B(u,v) = \int_\Omega |u|^\alpha |v|^\beta uv \,dx.$$

It is known (see [2]) that this eigenvalue is simple, isolated, and can be expressed as (1.3). It is also known that the associated principal eigenfunction (u,v) can be chosen strictly positive in Ω. This eigenfunction is a regular pair of functions where

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the infimum of the energy functional A is attained on the C^1-manifold $\{(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega): B(u,v)=1\}$.

The operator $-\Delta_p$ arises in problems from pure mathematics, such as the theory of quasiregular and quasiconformal mapping, as well as in problems from a variety of applications, e.g. non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, petroleum extraction, astronomy, etc.

The main purpose of this work is to prove that the mapping $(p,q)\mapsto\lambda_1(p,q)$ is continuous (stable) on the set

$$I_{\alpha,\beta}=\{(p,q)\in(]1,+\infty[)^2 \text{ such that (1.2) is satisfied}\},$$

for any bounded domain Ω having the segment property. Recall that an open subset Ω of $\mathbb{R}^N$ is said to have the segment property if, given any $x\in\partial\Omega$, there exist an open set G_x in $\mathbb{R}^N$ with $x\in G_x$ and y_x of $\mathbb{R}^N\setminus\{0\}$ such that, if $z\in\overline{\Omega}\cap G_x$ and $t\in]0,1[$, then $z+ty_x\in\Omega$.

This property allows us by a translation to push the support of a function u in Ω. This class of domains for which the boundary is sufficiently regular to guarantee that

$$W^{1,t}(\Omega)\cap_{1<r<t}W_0^{1,r}(\Omega)=W_0^{1,t}(\Omega),$$

for any $t>1$. See [3] for more details about this property.

The difficulty is that as the exponents (p,q) change, the appropriate spaces $L^p(\Omega)$ and $L^q(\Omega)$ also change. In the case of equation $\Delta_p u+\lambda|u|^{p-2}u=0$, the same result is studied by Lindqvist [5] but in a regular bounded domain. El Khalil et al. [3] studied the equation $\Delta_p u+\lambda g|u|^{p-2}u=0$ with $g\in L^\infty_{\rm loc}(\Omega)\cap L^r(\Omega)$ and Ω having the segment property. Note that the stability was partially studied in [4].
For the dependence of λ_1 upon variations of the domain, instead of the exponents p and q, we prove that $\lambda_1^{(p,q)}(\Omega_j)\to\lambda_1^{(p,q)}(\Omega)$ as $j\to+\infty$, where $\Omega=\cup_{j\in\mathbb{N}^*}\Omega_j$. In the case of only one equation, see [1] for contracting domain, and see [6] for variations of the domain.

2. Stability

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the segment property. Then, the function $(p,q)\mapsto\lambda_1(p,q)$ is continuous from $I_{\alpha,\beta}$ to $\mathbb{R}^+$. where*

$$I_{\alpha,\beta}=\{(p,q)\in(]1,+\infty[)^2 \textit{ such that } (1.2) \textit{ is satisfied }\}.$$

Proof. Let $(t_n)_{n\geq1}$, $t_n=(p_n,q_n)$ be a sequence in $I_{\alpha,\beta}$ converging at $t=(p,q)\in I_{\alpha,\beta}$. We will prove that

$$\lim_{n\to+\infty}\lambda_1(p_n,q_n)=\lambda_1(p,q).$$

Indeed, let $(\phi,\psi)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$ such that $B(\phi,\psi)>0$; hence,

$$\lambda_1(p_n,q_n)\leq\frac{\frac{\alpha+1}{p_n}\|\nabla\phi\|_{p_n}^{p_n}+\frac{\beta+1}{q_n}\|\nabla\psi\|_{q_n}^{q_n}}{B(\phi,\psi)},$$

since $\lambda_1(p_n,q_n)$ being the infimum. Letting n tend to infinity, we deduce from Lebesgue's theorem

$$\limsup_{n\to+\infty}\lambda_1(p_n,q_n)\leq\frac{\frac{\alpha+1}{p}\|\nabla\phi\|_p^p+\frac{\beta+1}{q}\|\nabla\psi\|_q^q}{B(\phi,\psi)}. \tag{2.1}$$

Then,

$$\limsup_{n\to+\infty} \lambda_1(p_n,q_n) \leq \lambda_1(p,q). \tag{2.2}$$

On the other hand, let $\{(p_{n_k},q_{n_k})\}_{k\geq 1}$ be a subsequence of $(t_n)_n$ such that

$$\lim_{k\to+\infty} \lambda_1(p_{n_k},q_{n_k}) = \liminf_{n\to+\infty} \lambda_1(p_n,q_n).$$

Let us fix $\varepsilon_0>0$ small enough, so that for all $\varepsilon\in(0,\varepsilon_0)$, we have

$$1<\min(p-\varepsilon,q-\varepsilon), \tag{2.3}$$
$$\max(p+\varepsilon,q+\varepsilon)<\min((p-\varepsilon)^*,(q-\varepsilon)^*). \tag{2.4}$$

For each $k\in\mathbb{N}^*$, let $(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})})\in W_0^{1,p_{n_k}}(\Omega)\times W_0^{1,q_{n_k}}(\Omega)$ be a principal eigenfunction of $(S_{p_{n_k},q_{n_k}})$ related with $\lambda_1(p_{n_k},q_{n_k})$. Then, by Hölder's inequality, for $\varepsilon\in(0,\varepsilon_0)$, the following inequalities hold

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\varepsilon} \leq \|\nabla u_{(p_{n_k},q_{n_k})}\|_{p_{n_k}} |\Omega|^{\frac{p_{n_k}-p+\varepsilon}{p_{n_k}(p-\varepsilon)}}, \tag{2.5}$$
$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\varepsilon} \leq \|\nabla v_{(p_{n_k},q_{n_k})}\|_{q_{n_k}} |\Omega|^{\frac{q_{n_k}-q+\varepsilon}{q_{n_k}(q-\varepsilon)}}. \tag{2.6}$$

Combining these two inequalities and using the variational characterization of λ_1, we have

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\varepsilon} \leq \Big\{\frac{p_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\alpha+1}\Big\}^{\frac{1}{p_{n_k}}} |\Omega|^{\frac{p_{n_k}-p+\varepsilon}{p_{n_k}(p-\varepsilon)}} \tag{2.7}$$
$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\varepsilon} \leq \Big\{\frac{q_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\beta+1}\Big\}^{\frac{1}{q_{n_k}}} |\Omega|^{\frac{q_{n_k}-q+\varepsilon}{q_{n_k}(q-\varepsilon)}}. \tag{2.8}$$

Therefore, via (2.3) and (2.4), for a subsequence

$$(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})}) \rightharpoonup (u,v) \quad \text{weakly in } W_0^{1,p-\varepsilon}(\Omega)\times W_0^{1,q-\varepsilon}(\Omega),$$
$$(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})}) \to (u,v) \quad \text{strongly in } L^{p+\varepsilon}(\Omega)\times L^{q+\varepsilon}(\Omega).$$

Passing to the limit in (2.7) and (2.8), respectively as $k\to\infty$ and as $\epsilon\to0^+$, we have

$$\|\nabla u\|_p^p \leq \frac{p}{\alpha+1}\lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k})<+\infty,$$
$$\|\nabla v\|_q^q \leq \frac{q}{\beta+1}\lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k})<+\infty.$$

Then,

$$u\in W_0^{1,p-\varepsilon}(\Omega)\cap W^{1,p}(\Omega)=W_0^{1,p}(\Omega),\quad v\in W_0^{1,q-\varepsilon}(\Omega)\cap W^{1,q}(\Omega)=W_0^{1,q}(\Omega),$$

because Ω satisfies the segment property.

On the other hand, from the variational characterization of $\lambda_1(p_{n_k},q_{n_k})$, (2.5), (2.6), and using weak lower semi continuity of the norm; it follows that

$$\frac{1}{|\Omega|^{\frac{\varepsilon}{p-\varepsilon}}}\frac{\alpha+1}{p}\|\nabla u\|_{p-\varepsilon}^p + \frac{1}{|\Omega|^{\frac{\varepsilon}{q-\varepsilon}}}\frac{\beta+1}{q}\|\nabla v\|_{q-\varepsilon}^q \leq \lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k}).$$

Letting $\varepsilon\to0^+$ in (2.10), the Fatou lemma yields,

$$\frac{\alpha+1}{p}\|\nabla u\|_p^p + \frac{\beta+1}{q}\|\nabla v\|_q^q \leq \lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k}).$$

Since $B(u_{(p_{n_k},q_{n_k})}, v_{(p_{n_k},q_{n_k})}) = 1$ via compactness of B, (u,v) is admissible in the variational characterization of $\lambda_1(p,q)$; hence

$$\lambda_1(p,q) \le \lim_{k\to+\infty} \lambda_1(p_{n_k}, q_{n_k}) = \liminf_{n\to+\infty} \lambda_1(p_n, q_n).$$

This and (2.2) complete the proof. $\square$

Remark 2.2. Observe that the segment property is used only for to prove

$$\lambda_1(p,q) \le \liminf_{n\to+\infty} \lambda_1(p_n, q_n).$$

3. Dependence with variations domain

Theorem 3.1. *Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ be an exhaustion of Ω such that*

$$\Omega = \bigcup_{j\in\mathbb{N}^*} \Omega_j.$$

Then

$$\lim_{j\to+\infty} \lambda_1^{(p,q)}(\Omega_j) = \lambda_1^{(p,q)}(\Omega), \tag{3.1}$$

$$\lim_{j\to+\infty} \int_\Omega |\nabla\widetilde{u_j} - \nabla u|^p\,dx = \lim_{j\to+\infty} \int_\Omega |\nabla\widetilde{v_j} - \nabla v|^q\,dx = 0, \tag{3.2}$$

where (u,v) is a positive principal eigenfunction of (1.1) related to $\lambda_1^{(p,q)}(\Omega)$; (u_j,v_j) is the principal eigenfunction related to $\lambda_1^{(p,q)}(\Omega_j)$ and $(\widetilde{u_j},\widetilde{v_j})$ is the extended by 0 in $\Omega\backslash\Omega_j$ of (u_j,v_j).

Proof. To prove (3.1), we note that

$$\lambda_1^{(p,q)}(\Omega_1) \ge \lambda_1^{(p,q)}(\Omega_2) \ge \dots \ge \lambda_1^{(p,q)}(\Omega).$$

For each $\varepsilon > 0$, there is a pair of function $(\varphi_\varepsilon, \psi_\varepsilon) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ such that

$$\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon\,dx > 0$$

$$\lambda_1^{(p,q)}(\Omega) > \frac{\frac{\alpha+1}{p}\int_\Omega |\nabla\varphi_\varepsilon|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\nabla\psi_\varepsilon|^q\,dx}{\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon\,dx} - \varepsilon,$$

since $\lambda_1^{(p,q)}(\Omega)$ is the infimum.

The support of φ_ε and ψ_ε being compact sets, are covered a finite number of the sets Ω_j; hence, there exist $j_0 \in \mathbb{N}^*$ such that

$$\operatorname{supp}\varphi_\varepsilon \subset \Omega_j \quad\text{and}\quad \operatorname{supp}\psi_\varepsilon \subset \Omega_j \quad \forall j \ge j_0.$$

Then,

$$\begin{aligned}
\lambda_1^{(p,q)}(\Omega_j) &\le \frac{\frac{\alpha+1}{p}\int_{\Omega_j} |\nabla\varphi_\varepsilon|^p\,dx + \frac{\beta+1}{q}\int_{\Omega_j} |\nabla\psi_\varepsilon|^q\,dx}{\int_{\Omega_j} |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon\,dx} \\
&= \frac{\frac{\alpha+1}{p}\int_\Omega |\nabla\varphi_\varepsilon|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\nabla\psi_\varepsilon|^q\,dx}{\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon\,dx} \\
&\le \lambda_1^{(p,q)}(\Omega) + \epsilon
\end{aligned}$$

for all large j. It is clear that $\lambda_1^{(p,q)}(\Omega) \ge \lim_{j\to+\infty} \lambda_1^{(p,q)}(\Omega_j)$. which proves the desired result.

For the strong convergence (3.2), we proceed as follows: First, we have $(\widetilde{u_j},\widetilde{v_j})\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$; $\widetilde{u_j}\geq 0$, $\widetilde{v_j}\geq 0$ a.e in Ω;

$$\int_\Omega |\widetilde{u_j}|^\alpha|\widetilde{v_j}|^\beta\widetilde{u}_j\widetilde{v}_j\,dx=1,$$

because

$$\int_\Omega |\widetilde{u_j}|^\alpha|\widetilde{v_j}|^\beta\widetilde{u}_j\widetilde{v}_j\,dx=\int_{\Omega_j}|u_j|^\alpha|v_j|^\beta u_jv_j\,dx=1;$$

and $\nabla\widetilde{u_j}=\widetilde{\nabla u_j}$ and $\nabla\widetilde{v_j}=\widetilde{\nabla v_j}$ a.e. in Ω. Hence, $(\widetilde{u_j},\widetilde{v_j})$ is admissible function in definition of $\lambda_1^{(p,q)}(\Omega)$. Which implies

$$\lambda_1^{(p,q)}(\Omega)\leq\frac{\alpha+1}{p}\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\widetilde{v_j}|^q\,dx=\lambda_1^{(p,q)}(\Omega_j).$$

Therefore, via (3.1) and for a subsequence,

$$\begin{gathered}(\widetilde{u_j},\widetilde{v_j})\rightharpoonup(\varphi,\psi)\quad\text{weakly in } W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega),\\ (\widetilde{u_j},\widetilde{v_j})\to(u,v)\quad\text{strongly in } L^p(\Omega)\times L^q(\Omega),\\ \varphi\geq 0,\quad \psi\geq 0\quad\text{a.e in }\Omega\end{gathered}$$

Since B is compact, we have $\int_\Omega\varphi^{\alpha+1}\psi^{\beta+1}dx=1$. Then (φ,ψ) is admissible function in definition of $\lambda_1^{(p,q)}(\Omega)$. We deduce that

$$\begin{aligned}\lambda_1^{(p,q)}(\Omega)&\leq\frac{\alpha+1}{p}\int_\Omega|\nabla\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\psi|^q\,dx\\ &\leq\liminf_{j\to\infty}\Big(\frac{\alpha+1}{p}\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\widetilde{v_j}|^q\,dx\Big)\\ &=\liminf_{j\to\infty}\lambda_1^{(p,q)}(\Omega_j).\end{aligned}$$

Then

$$\lambda_1^{(p,q)}(\Omega)=\frac{\alpha+1}{p}\int_\Omega|\nabla\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\psi|^q\,dx$$

which implies (φ,ψ) is positive eigenfunction related to $\lambda_1^{(p,q)}(\Omega)$ that it is simple (see [2]) and $\int_\Omega|\varphi|^\alpha|\psi|^\beta\varphi\psi dx=1$, hence $\phi\equiv u$ and $\psi\equiv v$. Also, (u,v) is independent of the subsequence. Then $\{(\widetilde{u_j},\widetilde{v_j})\}_j$ converge to (u,v) least in $L^p(\Omega)\times L^q(\Omega)$.

For the strong convergence (2.3), we use Clarckson inequality and distinguish three possible cases for $(p,q)\in I_{\alpha,\beta}$.

case 1: $p\geq 2$ and $q\geq 2$. For all $j\geq j_0$, we have

$$\begin{aligned}&\frac{\alpha+1}{p}\int_\Omega\Big|\frac{\nabla\widetilde{u_j}-\nabla u}{2}\Big|^p\,dx+\frac{\beta+1}{q}\int_\Omega\Big|\frac{\nabla\widetilde{v_j}-\nabla v}{2}\Big|^q\,dx\\ &\leq\frac{\alpha+1}{p}\Big[-\int_\Omega\Big|\frac{\nabla\widetilde{u_j}+\nabla u}{2}\Big|^p\,dx+\frac12\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac12\int_\Omega|\nabla u|^p\,dx\Big]\\ &\quad+\frac{\beta+1}{q}\Big[-\int_\Omega\Big|\frac{\nabla\widetilde{v_j}+\nabla v}{2}\Big|^p\,dx+\frac12\int_\Omega|\nabla\widetilde{v_j}|^q\,dx+\frac12\int_\Omega|\nabla v|^q\,dx\Big].\end{aligned}$$

Also, we have

$$\lambda_1^{(p,q)}(\Omega)\int_\Omega \big(\frac{\widetilde{u_j}+u}{2}\big)^{\alpha+1}\big(\frac{\widetilde{v_j}+v}{2}\big)^{\beta+1}dx \\ \leq \frac{\alpha+1}{p}\int_\Omega \big|\frac{\nabla\widetilde{u_j}+\nabla u}{2}\big|^p\,dx + \frac{\beta+1}{q}\int_\Omega \big|\frac{\nabla\widetilde{v_j}+\nabla v}{2}\big|^q\,dx.$$

Combining the last inequality and using Lebegue's Theorem; we obtain

$$\limsup_{j\to+\infty}\Big\{\frac{\alpha+1}{p}\int_\Omega \big|\frac{\nabla\widetilde{u_j}-\nabla u}{2}\big|^p\,dx + \frac{\beta+1}{q}\int_\Omega \big|\frac{\nabla\widetilde{v_j}-\nabla v}{2}\big|^q\,dx\Big\}\leq 0,$$

Case 2: $1<p<2$ and $2<q$ or $1<q<2$ and $2<p$. Using the second Clarckson inequality, we prove that

$$\lim_{j\to+\infty}\int_\Omega |\nabla\widetilde{u}_j-\nabla u|^p\,dx = 0.$$

Using first Clarckson inequality, we prove that

$$\lim_{j\to+\infty}\int_\Omega |\nabla\widetilde{v}_j-\nabla v|^q\,dx = 0.$$

Which completes the proof. □

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