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# THERMISTOR PROBLEM: A NONLOCAL PARABOLIC PROBLEM

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Abstract. In this paper, we study a nonlocal parabolic problem arising in Ohmic heating. Firstly, some existence and uniqueness results for the continuous problem are proposed. secondly, a time discretization technique by Euler forward scheme is proposed and a study of the discrete associated dynamical system is presented.

## 1. Introduction

In this work, we shall deal with the following nonlocal parabolic problem

$$\begin{gathered} \frac{\partial u}{\partial t} - \triangle u = \lambda \frac{f(u)}{(\int_\Omega f(u)\,dx)^2}, \quad \text{in } \Omega \times ]0;T[, \\ u = 0 \quad \text{on } \partial\Omega \times ]0;T[, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded regular domain, $\lambda$ is a positive parameter and $f$ is a function with prescribed conditions. Let us recall first that (1.1) arises by reducing the following system of two equations which model a thermistor problem

$$\begin{gathered} u_t = \nabla.(k(u)\nabla u) + \sigma(u)|\nabla\varphi|^2, \\ \nabla(\sigma(u)\nabla\varphi) = 0, \end{gathered} \tag{1.2}$$

where, $u$ represents the temperature generated by the electric current flowing through a conductor, $\varphi$ the electric potential, $\sigma(u)$ and $k(u)$ are respectively the electric and thermal conductivities. For more information, we refer the reader to [7, 9, 10, 15].

In section 2, our gaol concerns the existence and uniqueness of weak solutions to (1.1). Some results have been obtained by many authors in the case where $N = 1$ and $f$ taking particular forms: Montesinos and Gallego [12] proved the existence of weak solution under

$$0 < \sigma_1 \leq \sigma(s) \leq \sigma_2, \forall s \in \mathbb{R}. \tag{1.3}$$

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In [9, 10, 15], major emphasis is placed on cases where the spatial dimension $N$ is 1 or 2 and $f$ is of the form $f(u) = \exp(u) or \exp(-u)$. In these works, additional regularity assumptions are made on $u_0$ and a combination of usual Lyapounov functional and a comparison method is the main ingredient. Our purpose is to extend some of the results therein to problem (1.1), where here, the condition (1.3) is weakened to (H2) below.

We recall also that the Euler forward method has been used by several authors in the semi-discretization of non linear parabolic problems, see for example [5, 6]. Concerning the existence and uniqueness of solutions to (1.1) under particular forms of $f$, we refer the reader to [2] and the references therein. On the other hand, little is known about the solutions to the following discrete problem:

$$\begin{gathered} U^n - \tau \triangle U^n = U^{n-1} + \lambda\tau \frac{f(U^n)}{\big(\int_\Omega f(U^n)\,dx\big)^2}, \text{ in } \Omega, \\ U^n = 0 \quad \text{on } \partial\Omega, \\ U^0 = u_0 \quad \text{in } \Omega. \end{gathered} \tag{1.4}$$

Whereas, semi-discretization has been used for equations of the thermistor problem in [13, 1]. Our aim here is to continue the study of problem (1.1) initiated in section 2, where an a priori $L^\infty-$estimate is derived. In addition to the usual existence and uniqueness questions concerning the solutions of (1.3), we shall prove some results of stability and proceed to error estimates analysis. In [1], the authors derived an $L^2$ and $H^1$ norm error by requiring regularity on the solution $u$, for instance $u, u_t$ in $H^2(\Omega)\cap W^{1,\infty}(\Omega)$. Unfortunately, such smoothness is not always possible since the function $f$ is non linear. We end this paper by studying the asymptotic behaviour of the solutions to the discrete dynamical system associated with (1.3).

## 2. Existence and uniqueness for the continuous problem

We assume the following hypotheses:

- (H1) $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function.
- (H2) There exist positive constants $\sigma, c_1, c_2$ and $\alpha$ such that $\alpha < \frac{4}{d-2}$ and for all $\xi \in \mathbb{R}$
$$\sigma \le f(\xi) \le c_1|\xi|^{\alpha+1} + c_2.$$

We adopt the following weak formulation for (1.1): $u$ is a solution of (1.1) if and only if

$$\begin{gathered} u \in L^\infty(\tau, +\infty, H_0^1(\Omega) \cap L^\infty(\Omega)) \text{ with} \frac{\partial u}{\partial t} \in L^2(\tau, +\infty, L^2(\Omega)) \\ \text{for any } \tau > 0, \text{ and satisfying} \\ \int_0^T \int_\Omega u\frac{\partial}{\partial t}\phi - \nabla u \nabla \phi \,dx\,dt = \int_0^T (\frac{\lambda}{\big(\int_\Omega f(u)\,dx\big)^2} \int_\Omega f(u)\phi dx)dt, \\ \text{for any } \phi \in C^\infty((0,\infty), \Omega). \end{gathered}$$

Now, we state our main result.

**Theorem 2.1.** *Let hypotheses (H1)-(H2) be satisfied. Assume that $u_0 \in L^{k_0+2}(\Omega)$ with $k_0$ such that*

$$k_0 \ge \max\big(0, \frac{\alpha N}{2} - 2\big). \tag{2.1}$$

*Then, there exists $d_0 > 0$ such that if $\|u_0\|_{k_0+2} < d_0$, the problem* (1.1) *admits a solution $u$ verifying for all $\tau > 0$*

$$u \in L^\infty(\tau, +\infty, L^{k_0+2}(\Omega)), \quad |u|^\gamma u \in L^\infty(\tau, +\infty, H_0^1(\Omega)), \text{ with } \gamma = \frac{k_0}{2}.$$

*Moreover, if $u_0 \in L^\infty(\Omega)$, then $u \in L^\infty(\tau, +\infty, L^\infty(\Omega))$ and is unique.*

**Remark.** The value of $d_0$ will be given in the course of the proof.

*Proof.* We use a Faedo-Galerkin method see [11]. Let $u_m \subseteq D(\Omega)$ be such that $u_{0m} \to u_0$ in $H_0^1(\Omega)$ and let $(w_j)_j \subseteq H_0^1(\Omega)$ a special basis. We seek $u$ to be the limit of a sequence $(u_m)_m$ such that

$$u_m(t) = \Sigma_{j=1}^m g_{jm}(t) w_j,$$

where $g_{jm}$ is the solution of the following ordinary differential system

$$\begin{gathered} \langle u_m', w_j\rangle + (u_m, w_j) = \frac{\lambda}{\left(\int_\Omega f(u_m)\,dx\right)^2} \langle f(u_m), w_j\rangle,\ 1 \le j \le m, \\ u_m(0) = u_{om}. \end{gathered} \tag{2.2}$$

It is easy to see that (2.2) has a unique solution $u_m$ according to hypotheses (H1)–(H2) and Cartan's existence theorem concerning ordinary differential equation (see [3]). This solution is shown to exist on a maximal interval $[0; t_m[$. The following estimates enable us to assert that it can be continued on the hole interval $[0; T]$. We shall denote by $C_i$ different positive constants, depending on data, but not on $m$. □

**Lemma 2.2.** *For any $\tau > 0$, there exists a constant $c_3(\tau), c_4(\tau)$ such that*

$$\|u_m(t)\|_{k_0+2} \le c_3(\tau), \forall t \ge \tau, \tag{2.3}$$

$$\|u_m(t)\|_\infty \le c_4(\tau), \forall t \ge \tau. \tag{2.4}$$

*Proof.* (i) Multiplying the first equation of (3.2) by $|u_m|^k g_{jm}$, integrating on $\Omega$, adding from $j = 1$ to $m$ and using (H1)-(H2), yields

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{4}{(k+2)^2}\|\nabla |u_m|^{\frac{k}{2}} u_m\|_2^2 \le c_5 \|u_m\|_{k+\alpha+2}^{k+\alpha+2} + c_6. \tag{2.5}$$

By using well-known Sobolev's and Gagliardo-Nirenberg's inequalities, we have

$$\|u_m\|_{k_0+\alpha+2}^{k_0+\alpha+2} \le c_7 \|u_m\|_{k_0+2}^\alpha \|\nabla |u_m|^\gamma u_m\|_2^2, \tag{2.6}$$

Thus, from (2.5) and (2.6), we obtain

$$\frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} \le (c_8\|u_m\|_{k_0+2}^\alpha - \frac{4}{(k_0+2)^2})\|\nabla |u_m|^\gamma u_m\|_2^2 + c_6. \tag{2.7}$$

We shall make the following compatibility condition on $u_0$

$$\|u_0\|_{k_0+2} < \Big(\frac{4}{c_8(k_0+2)^2}\Big)^{1/\alpha} = d_0. \tag{2.8}$$

Then, there exists a small $\tau > 0$ such that

$$\|u_m(t)\|_{k_0+2} < d_0 \text{ for } t \in ]0, \tau[. \tag{2.9}$$

Hence

$$\frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} + c_9 \|\nabla |u_m|^\gamma u_m\|_2^2 \le c_6 \quad \forall \quad 0 < t < \tau. \tag{2.10}$$

By Poincaré's inequality and after integrating, it follows that

$$\|u_m(t)\|_{k_0+2} \le c_{10}, \quad \forall \quad 0 < t < \tau,$$

Therefore, relation (3.3) is achieved by iterating successively the same process with initial condition calculated at the last one.
(ii) By using Hôlder's inequality, we get

$$\|u_m\|_{k+\alpha+2}^{k+\alpha+2} \le c_{11}\|u_m\|_{k+2}^{\theta_1}\|u_m\|_{k_0+2}^{\theta_2}\|u_m\|_q^{\theta_3}, \tag{2.11}$$

with $\theta_1, \theta_2$ and $\theta_3$ satisfying

$$\frac{\theta_1}{k+2} + \frac{\theta_2}{k_0+2} + \frac{\theta_3}{q} = 1 \quad \text{and} \quad \theta_1 + \theta_2 + \theta_3 = k + \alpha + 2.$$

We require moreover

$$\frac{\theta_1}{k+2} + \frac{\theta_3}{2(\gamma+1)} = 1.$$

Using the boundedness of $\|u_m\|_{k_0+2}$, the choice of $q$, Sobolev's inequality and young's inequality, we have from (2.11) that

$$\begin{aligned} c_5\|u_m\|_{k+\alpha+2}^{k+\alpha+2} &\le c_{12}\|u_m\|_{k+2}^{\theta_1}\|\nabla|u_m|^\gamma u_m\|_2^{\frac{\theta_3}{\gamma+1}} \\ &\le c_{13}(k+2)^{\theta_4}\|u_m\|_{k+2}^{k+2} + \frac{2}{(k+2)^2}\|\nabla|u_m|^\gamma u_m\|_2^2, \end{aligned}$$

where $\theta_4$ is some positive constant. Hence (2.5) becomes

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{c_{14}}{(k+2)^2}\|\nabla|u_m|^\gamma u_m\|_2^2 \le c_{15}(k+2)^{\theta_4}\|u_m\|_{k+2}^{k+2} + c_5.$$

Therefore, by applying [8, lemma 4] we conclude to (3.4).

**Passage to the limit in (3.2) as $m \to \infty$.** Multiplying the jth equation of system (3.2) by $g_{jm}(t)$, adding these equations for $j = 1, \dots, m$ and integrating with respect to the time variable, we deduce the existence of a subsequence of $u_m$ such that

$$\begin{gathered} u_m \to u \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)), \\ u_m \to u \quad \text{weak in } L^2(0,T;H_0^1(\Omega)), \\ u_{mt} \to u_t \quad \text{weak in } L^2(0,T;H^{-1}(\Omega)), \\ u_m \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \\ mboxanda.einQ_T. \end{gathered}$$

Straightforward standard compactness arguments allow us to assert that $u$ is a solution of problem (1.1)

**Uniqueness.** Consider $u_1$ and $u_2$ two weak solutions of the problem (1.1) and define $w = u_1 - u_2$. Substracting the equations verified by $u_1$ and $u_2$, we obtain

$$\begin{aligned} \frac{dw}{dt} - \triangle w &= \frac{\lambda}{\big(\int_\Omega f(u_1)\,dx\big)^2}\Big(f(u_1) - f(u_2)\Big) \\ &\quad + \lambda\frac{\Big(\int_\Omega f(u_2) - f(u_1)\,dx\Big)\Big(\int_\Omega f(u_2) + f(u_1)\,dx\Big)}{\big(\int_\Omega f(u_1)\,dx\big)^2\big(\int_\Omega f(u_2)\,dx\big)^2} f(u_2). \end{aligned} \tag{2.12}$$

Taking the inner product of (2.12) by $w$ and using (H1) and (2.4), we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 \le c_{16}\|w(t)\|_2^2,$$

which implies that $w = 0$. Hence the solution is unique. □

## 3. The semi-discrete problem

**Existence and uniqueness.** We consider the Euler scheme (1.3), with $N\tau = T$, $T > 0$ fixed and $1 \le n \le N$. In the sequel, $(\cdot,\cdot)$ will denote the associated inner product in $L^2(\Omega)$ or the duality product between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$.

**Theorem 3.1.** *Assume (H1)-(H2). Then, for each n, there exists a unique solution $U^n$ of (1.3) in $H_0^1(\Omega) \cap L^\infty(\Omega)$ provided that $\tau$ is small enough.*

*Proof.* For simplicity, we write $U = U^n$, $h(x) = U^{n-1}$. Then (1.3) becomes

$$\begin{gathered} U - \tau\triangle U = h(x) + \lambda\frac{f(U)}{(\int_\Omega f(U)\,dx)^2}, \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

**Existence.** Define the map $S(\mu,.)$ by $U = S(\mu,v), \mu \in [0,1]$ if

$$\begin{gathered} U - \tau\triangle U = \mu g(x,v) \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega, \\ U^0 = \mu u_0, \end{gathered} \tag{3.2}$$

where $g(x,v) = h(x) + \lambda f(v)/\big(\int_\Omega f(v)\,dx\big)^2$.

For a fixed $v \in H_0^1(\Omega)$, (3.2) has a unique solution $U \in H_0^1(\Omega)$. Then, for each $\mu \in [0,1]$, the operator $S(\mu,.)$ is well defined. Moreover, $S(\mu,.)$ is compact from $H_0^1(\Omega)$ into it self. Indeed, using (H2), we have the estimate

$$|U|_2^2 + \tau|\nabla U|_2^2 \le c_{17}.$$

We can easily see that $\mu \to S(\mu,v)$ is continuous and that $S(0,v) = U$, for any $v$, if and only if $U = 0$. From the Leray-Schauder fixed point theorem, there exists therefore a fixed point $U$ of $S(\mu,.)$. □

Now, we derive an a priori estimate.

**Lemma 3.2.** *If $u_0 \in L^\infty(\Omega)$, then for all $n \in \{1,\dots,N\}$, $U^n \in L^\infty(\Omega)$.*

The proof of the above lemma is similar to the one used by de Thelin in [4] in a different problem; we shall give here only a sketch. Suppose $d \ge 2$ and define

$$\delta = \begin{cases} \frac{2d}{d-2} & \text{if } 2 < d, \\ 2(\alpha+2) & \text{if } d = 2. \end{cases}$$

Let $q_1 = \delta$ and let

$$q_k = \{(\frac{\delta}{2})^{k-1}(\delta-\gamma) - (2-\gamma)\}\frac{\delta}{\delta-2}, \quad k \ge 2\,.$$

Then we have

$$q_{k+1} = (q_k + 2 - \gamma)\frac{\delta}{2} \text{ with } \gamma = \alpha+2, \quad \text{for all } k \in \mathbb{N}^*.$$

**Lemma 3.3.** *For $k$ in $\mathbb{N}^*$, $U^n \in L^{q_k}(\Omega)$ and*
$$|U^n|_\infty = \limsup |U^n|_{q_k} < +\infty. \tag{3.3}$$

*Proof.* We prove by recurrence that $U \in L^{q_k}$. This property is true for $k = 1$, since $H_0^1(\Omega) \subset L^\delta(\Omega)$. Now we show that $U \in L^{q_{k+1}}$. Let $m \in \mathbb{N}$, $1 \le m \le k$. Multiplying (2.1) by $|U|^{q_m-\gamma}U$, using (H2) and Young's inequality, we get
$$(q_m - \gamma + 1)\int_\Omega |\nabla U|^2 |U|^{q_m-\gamma}\,dx \le c_{18}|U|_{q_m}^{q_m} + c_{19}.$$
On the other hand,
$$|U|_{q_{m+1}}^{q_m+2-\gamma} \le c_{20}(1 + \frac{q_m-\gamma}{2})^2 \int_\Omega |\nabla U|^2 |U|^{q_m-\gamma}\,dx.$$
Therefore,
$$|U|_{q_{m+1}}^{q_m+2-\gamma} \le (c_{21} + c_{22}|U|_{q_m}^{q_m})(q_m + 2 - \gamma).$$
Thus,
$$(|U|_{q_{k+1}}^{q_{k+1}})^{2/\delta} \le (c_{21} + c_{22}|U|_{q_k}^{q_k})(q_k + 2 - \gamma).$$
The rest of the proof follows the same lines as in [4, p. 383-384]. □

**Uniqueness.** Consider $U$ and $V$ two different solutions of (2.1) and define $w = U - V$. Then, we have
$$\begin{aligned} w - \tau\triangle w &= \frac{\lambda\tau}{(\int_\Omega f(U)\,dx)^2}\big(f(U) - f(V)\big) \\ &\quad + \lambda\tau\frac{\big(\int_\Omega f(U) - f(V)\,dx\big)\big(\int_\Omega f(V) + f(U)\,dx\big)}{(\int_\Omega f(U)\,dx)^2(\int_\Omega f(V)\,dx)^2} f(V). \end{aligned} \tag{3.4}$$
Multiplying (3.4) by $w$, integrating on $\Omega$ and using the $L^\infty-$estimate obtained in lemma 3.2, we obtain
$$|w|_2^2 + \tau|\nabla w|_2^2 \le c_{30}\tau|w|_2^2.$$
Therefore, $w = 0$ when $\tau \le 1/c_{30}$.

## 4. Stability

**Theorem 4.1.** *Assume (H1)-(H2). Then, there exists $c(T, u_0) > 0$ depending on the data but not on $N$ such that for any $n \in \{1, \dots, N\}$*
$$|U^n|_{L^\infty(\Omega)} \le c(T, u_0),$$
$$|U^n|_2^2 + \tau\sum_{k=1}^n |\nabla U^k|_2^2 \le c(T, u_0),$$
$$\sum_{k=1}^n |U^k - U^{k-1}|_2^2 \le c(T, u_0).$$

*Proof.* (i) Multiplying (1.3) by $|U^k|^m U^k$ for some integer $m \ge 1$, using lemma 3.2 and Hôlder's inequality, we obtain after simplification
$$|U^k|_{m+2} \le |U^{k-1}|_{m+2} + c_{31}\tau. \tag{4.1}$$
By induction and taking the limit in the resulting inequality as $m \to +\infty$, we get
$$|U^k|_{L^\infty(\Omega)} \le c(T, u_0).$$

(ii) Multiplying the first equation of (1.3) by $U^k$ and using the hypotheses on $f$, one easily has

$$(U^k - U^{k-1}, U^k) + \tau|\nabla U^k|_2^2 \le c_{32}\,\tau|U^k|_1.$$

Using the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and summing from $k = 1$ to $n$, we obtain

$$|U^n|_2^2 + \sum_{k=1}^{n}|U^k - U^{k-1}|_2^2 + \tau\sum_{k=1}^{n}|\nabla U^k|_2^2 \le |u_0|_2^2 + \tau\, c_{33}\sum_{k=1}^{n}|U^k|_1.$$

Then, the inequalities(b) and (c) of the lemma hold by using relation (3.3) and (a). $\square$

## 5. Error estimates for solutions

We shall adopt the following notation concerning the time discretization for problem (1.1). Let us denote the time step by $\tau = \frac{T}{N}$, $t^n = n\tau$ and $I_n = (t^n, t^{n-1})$ for $n = 1, \dots, N$. If $z$ is a continuous function (respectively summable), defined in $(0,T)$ with values in $H^{-1}(\Omega)$ or $L^2(\Omega)$ or $H_0^1(\Omega)$, we define $z^n = z(t^n, .)$, $\overline{z}^n = \frac{1}{\tau}\int_{I_n} z(t,.)dt$, $\overline{z}^0 = z^0 = z(0,.)$; the error $e_n = u(t) - U^n$ for all $t \in I_n$ and the local errors $e_u^n$ and $e^n$ defined by $e_u^n = \overline{u}^n(t) - U^n$, $e^n = u^n - U^n$.

**Theorem 5.1.** *Let (H1)-(H2) hold. Then, the following error bounds are satisfied*

$$\|e_n\|^2_{L^\infty(0,T,H^{-1}(\Omega))} + \int_0^T |e_n|^2 dt \le c_{34}\,\tau,$$

$$\|e^m\|_{H^{-1}(\Omega)} \le c_{35}\,\tau^{1/2},$$

$$|\nabla\int_0^T e_n(t)\,dt|_2 \le c_{36}\,\tau^{1/4}.$$

*Proof.* We consider the following variational formulation of discrete problem (1.3):

$$(U^n - U^{n-1}, \varphi) + \tau(\nabla U^n, \nabla\varphi) = \frac{\lambda\tau}{\big(\int_\Omega f(U^n)\,dx\big)^2}(f(U^n), \varphi), \tag{5.1}$$

for all $\varphi \in H_0^1(\Omega)$. Integrating the continuous problem (1.1) over $I_n$, we get

$$(u^n - u^{n-1}, \varphi) + \tau(\nabla\overline{u}^n, \nabla\varphi) = \lambda\tau\int_{I_n}\frac{(f(u^n), \varphi)}{\big(\int_\Omega f(u^n)\,dx\big)^2}, \quad \forall\,\varphi \in H_0^1(\Omega) \tag{5.2}$$

Subtracting (5.2) from (5.1) and adding from $n = 1$ to $m$ with $m \le N$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{m}(e^n - e^{n-1}, \varphi) + \tau\sum_{n=1}^{m}(\nabla e_u^n, \nabla\varphi)\\
&\le c_{37}\,\tau|\sum_{n=1}^{m}(\overline{f(u)}^n - f(U^n), \varphi)| + c_{38}\,\tau|\sum_{n=1}^{m}(f(U^n), \varphi)|.
\end{aligned}\tag{5.3}$$

Let $(-\triangle)^{-1}$ the green operator satisfying

$$(\nabla(-\triangle)^{-1}\psi, \nabla\varphi) = (\psi, \varphi)_{H^{-1}(\Omega), H_0^1(\Omega)}$$

for all $\psi \in H^{-1}(\Omega), \varphi \in H_0^1(\Omega)$. Choosing $\varphi = (-\triangle)^{-1}(e^n)$ as test function, we then obtain

$$I_1 + I_2 \le I_3 + I_4, \tag{5.4}$$

where

$$I_1=\sum_{n=1}^{m}(e^n-e^{n-1},(-\triangle)^{-1}(e^n)),\quad I_2=\tau\sum_{n=1}^{m}(e_u^n,e^n),$$
$$I_3\le c_{37}\tau|\sum_{n=1}^{m}(\overline{f(u)}^n-f(U^n),(-\triangle)^{-1}(e^n))|,$$
$$I_4=c_{38}\tau|\sum_{n=1}^{m}(f(U^n),(-\triangle)^{-1}(e^n))|.$$

With the aid of the elementary identity $2a(a-b)=a^2-b^2+(a-b)^2$ and the property of $(-\triangle)^{-1}$, $I_1$ reduces after straightforward calculations to

$$I_1=\frac{1}{2}\|e^m\|^2_{H^{-1}(\Omega)}+\frac{1}{2}\sum_{n=1}^{m}\|e^n-e^{n-1}\|^2_{H^{-1}(\Omega)}.$$

On the other hand

$$\begin{aligned}
I_2&=\tau\sum_{n=1}^{m}(e_u^n,e^n)\\
&=\sum_{n=1}^{m}\int_{I_n}(u(t)-U^n,u(t)-U^n)\,dt+\sum_{n=1}^{m}\int_{I_n}(u(t)-U^n,u^n-u(t))\,dt\\
&=I_{21}+I_{22}.\\
I_{22}&=\sum_{n=1}^{m}\int_{I_n}(u(t),u^n-u(t))\,dt-\sum_{n=1}^{m}\int_{I_n}(U^n,u^n-u(t))\,dt\\
&=I_{22}^1+I_{22}^2.
\end{aligned}$$

We now estimate $I_{22}^1$.

$$\begin{aligned}
|I_{22}^1|&=|\sum_{n=1}^{m}\int_{I_n}(u(t),\int_t^{t^n}\frac{\partial u}{\partial s}\,ds)\,dt|\\
&\le\sum_{n=1}^{m}\int_{I_n}(\int_t^{t^n}\|\frac{\partial u}{\partial s}\|_{H^{-1}(\Omega)}\,ds)\|u(t)\|_{H_0^1(\Omega)}\,dt\\
&\le\tau\|\frac{\partial u}{\partial s}\|_{L^2(0,t^m,H^{-1}(\Omega))}\,\|u\|_{L^2(0,t^m,H_0^1(\Omega))}\\
&\le c_{39}\,\tau.
\end{aligned}$$

In the same manner,

$$|I_{22}^2|\le\tau\|\frac{\partial u}{\partial s}\|_{L^2(0,t^m,H^{-1}(\Omega))}(\tau\sum_{n=1}^{m}\|U^n\|^2_{H_0^1(\Omega))})^{1/2}\le c_{40}\,\tau.$$

Next, we estimate the first term on the right-hand side of (5.4) by using Hôlder's and Young's inequalities and (H1)

$$
\begin{aligned}
|I_3| &\le |\sum_{n=1}^{m}(\int_{I_n}[f(u)-f(U^n)]\,dt,(-\triangle)^{-1}(e^n))| \\
&\le c_{41}\,\tau^{1/2}\sum_{n=1}^{m}(\int_{I_n}|f(u)-f(U^n)|_2^2\,dt)^{1/2}\|e^n\|_{H^{-1}(\Omega)} \\
&\le \eta\sum_{n=1}^{m}(\int_{I_n}|f(u)-f(U^n)|_2^2\,dt)+\frac{c_{42}}{\eta}\,\tau\sum_{n=1}^{m}\|e^n\|^2_{H^{-1}(\Omega)} \\
&\le c_{43}\,\eta\sum_{n=1}^{m}(\int_{I_n}|e_n|_2^2\,dt)+\frac{c_{42}}{\eta}\,\tau\sum_{n=1}^{m}\|e^n\|^2_{H^{-1}(\Omega)}.
\end{aligned}
$$

Moreover, we have

$$
|I_4|\le c_{44}\,\tau+c_{45}\,\tau\sum_{n=1}^{m}\|e^n\|^2_{H^{-1}(\Omega)}.
$$

Choosing suitably $\eta$, we conclude that

$$
\begin{aligned}
&\|e^m\|^2_{H^{-1}(\Omega)}+\sum_{n=1}^{m}\|e^n-e^{n-1}\|^2_{H^{-1}(\Omega)}+\sum_{n=1}^{m}\int_{I_n}|e_n|_2^2\,dt \\
&\le c_{46}\,\tau+c_{47}\,\tau\sum_{n=1}^{m}\|e^n\|^2_{H^{-1}(\Omega)}.
\end{aligned}
\tag{5.5}
$$

On the other hand, setting $y^m=\sum_{n=1}^{m}\|e^n\|^2_{H^{-1}(\Omega)}$, from (5.5), we get

$$
y^m-y^{m-1}\le c_{46}\,\tau+c_{47}\,\tau y^m.
$$

By applying the discrete Gronwall inequality, we deduce that $y^m\le c(T)$.Therefore,

$$
\|e^m\|_{H^{-1}(\Omega)}\le c_{48}\,\tau^{1/2}.
$$

On the other hand, we have

$$
\sup_{t\in(0,t_m)}\|e_n(t)\|_{H^{-1}(\Omega)}-c_{48}\tau^{1/2}\le\max_{1\le n\le m}\|e_n(t^n)\|_{H^{-1}(\Omega)}=\max_{1\le n\le m}\|e^n\|_{H^{-1}(\Omega)}.
$$

Thus,

$$
\|e_n\|_{L^\infty(0,T,H^{-1}(\Omega))}-c_{48}\,\tau^{1/2}\le\max_{1\le n\le m}\|e^n\|_{H^{-1}(\Omega)}.
$$

From the last inequality, we obtain

$$
\begin{gathered}
\|e_n\|^2_{L^\infty(0,T,H^{-1}(\Omega))}+\int_0^T|e_n|_2^2\,dt\le c_{49}\,\tau, \\
\sum_{n=1}^{m}\|e^n-e^{n-1}\|^2_{H^{-1}(\Omega)}\le c_{49}\,\tau.
\end{gathered}
$$

Choosing $\varphi = \tau \sum_{n=1}^{m} (\overline{u}^n - U^n)$ in (5.3) , we obtain

$$\begin{aligned} &\tau \int_\Omega (u^m - U^m)(\sum_{n=1}^{m} (\overline{u}^n - U^n)\,dx) + \tau^2 |\sum_{n=1}^{m} \nabla(\overline{u}^n - U^n)|_2^2 \\ &\le c_{50}\tau^2 |\int_\Omega \sum_{n=1}^{m} (\overline{f(u)}^n - f(U^n))(\sum_{n=1}^{m} (\overline{u}^n - U^n))dx| \\ &\quad + c_{51}\,\tau^2 |\sum_{n=1}^{m} (f(U^n), \sum_{n=1}^{m} (\overline{u}^n - U^n))|. \end{aligned}$$

This implies

$$\begin{aligned} &\tau^2 |\sum_{n=1}^{m} \nabla(\overline{u}^n - U^n)|_2^2 = |\nabla \int_0^{t^m} e_n\,dt|_2^2 \le \tau |\int_\Omega (u^m - U^m)(\sum_{n=1}^{m} (\overline{u}^n - U^n)\,dx)| \\ &+ c_{50}\tau^2 |\int_\Omega \sum_{n=1}^{m} (\overline{f(u)}^n - f(U^n))(\sum_{n=1}^{m} (\overline{u}^n - U^n)dx| \\ &+ c_{51}\,\tau^2 |\sum_{n=1}^{m} (f(U^n), \sum_{n=1}^{m} (\overline{u}^n - U^n))|. \\ &\le I + II + III. \end{aligned}$$

Clearly

$$\begin{aligned} I &\le \|e^m\|_{H^{-1}(\Omega)} (\sum_{n=1}^{m} \int_{I_n} \|u(t)\|_{H_0^1(\Omega)}\,dt + \tau \sum_{n=1}^{m} \|U^n\|_{H_0^1(\Omega)}) \\ &\le c_{52} \|e^m\|_{H^{-1}(\Omega)} \le c_{53}\tau^{1/2}. \end{aligned}$$

We get also

$$\begin{aligned} II &\le (\int_\Omega (\sum_{n=1}^{m} \int_{I_n} (f(u) - f(U^n))\,dt)^2\,dx)^{1/2} \times (\int_\Omega (\sum_{n=1}^{m} \int_{I_n} (u(t) - U^n)\,dt)^2\,dx)^{1/2} \\ &\le T^2 (\sum_{n=1}^{m} \int_{I_n} |f(u) - f(U^n|_2^2\,dt)^{1/2} \times (\sum_{n=1}^{m} \int_{I_n} |u(t) - U^n|_2^2\,dt)^{1/2} \\ &\le T^2 (\sum_{n=1}^{m} \int_{I_n} |f(u) - f(U^n|_2^2\,dt)^{1/2} \times (2\|u\|^2_{L^2(0,T,H_0^1(\Omega))} + 2\tau \sum_{n=1}^{m} |U^n|_2^2)^{1/2} \\ &\le c_{54}\,\tau^{1/2}. \end{aligned}$$

The last inequality follows by using simultaneously the $L^\infty$−estimate of $u(t)$ , $U^n$ and the error bound given in (4.1). Arguing as in the previous estimate, we get

$$III \le T^2 (\sum_{n=1}^{m} \int_{I_n} |f(U^n|_2^2\,dt)^{1/2} \times (2\|u\|^2_{L^2(0,T,H_0^1(\Omega))} + 2\tau \sum_{n=1}^{m} |U^n|_2^2)^{1/2}.$$

Using again the hypothesis (H1) and the estimates above, we obtain

$$III \le c_{55}\,\tau^{1/2}.$$

Finally collecting these results, it follows that

$$|\nabla \int_0^T e_n\,dt|_2^2 \le c_{56}\,\tau^{1/2}.$$

This completes the proof. ☐

**Corollary 5.2.** *Under hypotheses (H1)-(H2), problem* (1.3) *generates a continuous semi-group $S_\tau$ defined by $S_\tau U^{n-1} = U^n$.*

## 6. The semi-discrete dynamical system

The aim here is to study the discrete dynamical system (1.3) via the concepts of absorbing sets and global attractors (see Temam [14]).

**Theorem 6.1.** *The semi-group associated with* (1.3) *possesses a compact attractor $\mathbb{A}_\tau$ which is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $\tau$ small enough.*

*Proof.* We begin by showing the existence of an absorbing set in $H_0^1(\Omega) \cap L^\infty(\Omega)$.
(i) Denoting $y_m^n = |U^n|_{m+2}$ and $y^n = |U^n|_{L^\infty(\Omega)}$, then from (4.1), we have

$$y_m^n \leq c_{57}\, y_m^{n-1} + c_{58}\tau.$$

Letting $m$ approach infinity, we deduce that

$$y^n \leq c_{57}\, y^{n-1} + c_{58}\tau.$$

On the other hand, we have

$$\tau \sum_{n=n_0}^{n_0+N} y^n \leq a_1, \quad \forall n_0 \geq n_\tau,$$

for some positive real number $a_1$ which do not depend on $n_0$.
Applying the discrete uniform Gronwall's lemma ([14]), we get

$$|U^n|_{L^\infty(\Omega)} \leq c_{59}, \quad \forall\, n \geq n_\tau,$$

which implies the existence of absorbing sets in $L^\infty(\Omega)$.
(ii) To obtain existence of absorbing sets in $H_0^1(\Omega)$, multiply (1.3) by $U^n - U^{n-1}$. By using Hôlder's and Poincaré's inequalities, we have

$$|\nabla U^n|_2^2 \leq |\nabla U^{n-1}|_2^2 + c_{60}\tau, \quad \forall\, n \geq n_\tau.$$

Using again the relation $(b)$ and the discrete uniform Gronwall's lemma, we get

$$\|U^n\|_{H_0^1(\Omega)} \leq c_{61}, \quad \forall n \geq n_\tau.$$

Therefore, the existence of absorbing sets in $H_0^1(\Omega)$ is proved. Applying Temam [14, Theorem 1.1], we therefore get the result. ☐

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