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AN INFINITE-HARMONIC ANALOGUE OF A LELONG THEOREM AND INFINITE-HARMONICITY CELLS

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Abstract. We consider the problem of finding a function f in the set of ∞-harmonic functions, satisfying

$$\lim_{w\to\zeta} |\widetilde{f}(w)| = \infty, \quad w \in \mathcal{H}(D), \quad \zeta \in \partial\mathcal{H}(D)$$

and being a solution to the quasi-linear parabolic equation

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \quad \text{in } D \subset \mathbb{R}^2\,,$$

where D is a simply connected plane domain, $\mathcal{H}(D) \subset \mathbb{C}^2$ is the harmonicity cell of D, and $\widetilde{f}$ is the holomorphic extension of f. As an application, we show a p-harmonic behaviour of the modulus of the velocity of an arbitrary stationary plane flow near an extreme point of the profile.

1. Introduction

The complexification problems for partial differential equations in a domain $\Omega \subset \mathbb{R}^n$ include the introduction of a common domain $\widetilde{\Omega} \supset \Omega$ in $\mathbb{C}^n$ to which all the solutions of a specified p.d.e. extend holomorphically. The complex domains in question are the so-called harmonicity cells $\mathcal{H}(\Omega)$, in [4], for the following set of $2m$-order elliptic operators:

$$\Delta^m u = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \frac{\partial^{2|\alpha|} u}{\partial x_1^{2\alpha_1} \dots \partial x_n^{2\alpha_n}} = 0, \quad m = 1, 2, 3 \dots \tag{1.1}$$

They often describe properties of physical processes which are governed by such a p.d.e [19]. The operator Δ^2 has been widely studied in the literature, frequently in the contexts of biharmonic functions [3].

Motivation. Our objective is to introduce the complex domain $\widetilde{D}$, and the adequate solution $f = f_\zeta$ in the space of ∞-harmonic functions $\mathbf{H}_\infty(D)$, for equation (1.5), below. In view of Theorem 2.5, part 2, we assign a domain $\widetilde{D} \subset \mathbb{C}^2$, denoted by $\mathcal{H}_\infty(D)$, to the class $\mathbf{H}_\infty(D)$. The definition of $\mathcal{H}_\infty(D)$, is similar to the definition of $\mathcal{H}(D)$, although less explicit. Equation (1.5) is actually the formal limit, as

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$p \to +\infty$, of the p-harmonic equation in $D \subset \mathbb{R}^2$

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \Delta u) = 0, \quad 1 < p < +\infty, \tag{1.2}$$

For every finite real $p > 1$, the hodograph method transforms $\Delta_p u = 0$ into a linear elliptic p.d.e. in the hodograph plane. Due to [5], the pull-back operation is possible from $\mathbb{R}^2(u_x, u_y)$ to the physical plane. Although linear, the obtained equation is not easily computed since its limit conditions become more complicated.

Preliminaries. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega \neq \emptyset$, $\partial\Omega \neq \emptyset$. In 1935, Aronszajn [3] introduced the notion of harmonicity cells in order to study the singularities of m-polyharmonic functions. These functions, used in elasticity calculus of plates, are C^∞-solutions in Ω of (1.1). Recall that $\mathcal{H}(\Omega)$ is the domain of $\mathbb{C}^n$, whose trace $\operatorname{Tr}\mathcal{H}(\Omega)$ on $\mathbb{R}^n$ is Ω, and represented by the connected component containing Ω of the open set $\mathbb{C}^n - \cup_{t\in\partial\Omega}\Gamma(t)$, where $\Gamma(t) = \{z \in \mathbb{C}^n : (z_1 - t_1)^2 + \cdots + (z_n - t_n)^2 = 0\}$ is the isotropic cone of $\mathbb{C}^n$, with vertex $t \in \mathbb{R}^n$. Lelong [16] proved that $\mathcal{H}(\Omega)$ coincides with the set of points $z \in \mathbb{C}^n$ such that there exists a path γ satisfying: $\gamma(0) = z$, $\gamma(1) \in \Omega$ and $T[\gamma(\tau)] \subset \Omega$ for every τ in $[0, 1]$, where T is the Lelong transformation, mapping points $z = x + iy \in \mathbb{C}^n$ to Euclidean $(n-2)$-spheres $S^{n-2}(x, \|y\|)$ of the hyperplane of $\mathbb{R}^n$ defined by: $\langle t - x, y\rangle = 0$. If Ω is starshaped at $a_0 \in \Omega, \mathcal{H}(\Omega) = \{z \in \mathbb{C}^n; T(z) \subset \Omega\}$ is also starshaped at a_0. Furthermore, for bounded convex domains Ω of $\mathbb{R}^n$, we get

$$\mathcal{H}(\Omega) = \big\{z = x + iy \in \mathbb{C}^n : \max_{t\in T(iy)} \max\big[\max_{\xi\in S^{n-1}} \big(\langle x + t, \xi\rangle - \max_{s\in\Omega}\langle \xi, s\rangle\big)\big] < 0\big\} \tag{1.3}$$

where S^{n-1} is the Euclidean unit sphere of $\mathbb{R}^n$ [4, 6]. The harmonicity cell of the Euclidean unit ball B_n of $\mathbb{R}^n$ gives a central example, since $\mathcal{H}(B_n)$ coincides with the Lie ball $LB = \{z \in \mathbb{C}^n; L(z) = [\|z\|^2 + \sqrt{\|z\|^4 - |z_1^2 + \cdots + z_n^2|^2}]^{1/2} < 1\}$, where $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. Besides, representing also the fourth type of symmetric bounded homogenous irreducible domains of $\mathbb{C}^n$, $\mathcal{H}(B_n)$ has been studied (specially in dimension $n = 4$) by theoretical physicits interested in a variety of different topics: particle physics, quantum field theory, quantum mechanics, statistical mechanics, geometric quantization, accelerated observers, general relativity and even harmony and sound analysis (For more details, see [11, 18, 19, 20].

From the point of view of complex analysis, Jarnicki [14] proved that if D_1 and D_2 are two analytically homeomorphic plane domains of $\mathbb{C} \simeq \mathbb{R}^2$ then their harmonicity cells $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ are also analytically homeomorphic in $\mathbb{C}^2$. A generalization in $\mathbb{C}^n, n \geq 2$, of this Jarnicki Theorem is established by the author [8], as well as a characterization of polyhedric harmonicity cells in $\mathbb{C}^2$ [10]. Furthermore, recall that if $\mathbf{A}(\Omega)$ and $\mathbf{Ha}(\Omega)$ denote the spaces of all real analytic and harmonic functions (respectively) in Ω, then $\mathcal{H}(\Omega)$ is characterized by the following feature

$$[\cap_{f\in\mathbf{Ha}(\Omega)}\Omega^f]^0 = \mathcal{H}(\Omega), \tag{1.4}$$

while $[\cap\Omega^f]^0 = \emptyset$, when f runs through $\mathbf{A}(\Omega)$, where Ω^f is the greatest domain of $\mathbb{C}^n$ to which f extends holomorphically. We emphasize that in (1.4), Ω is actually required to be star-shaped at some point a_0, or a C-domain (that is, Ω contains the convex hull $\operatorname{Ch}(S^{n-2})$ of any $(n-2)$-Euclidean sphere S^{n-2} included in Ω) or $\Omega \subset \mathbb{R}^{2p}$ with $2p \geq 4$, or Ω is a simply connected domain in $\mathbb{R}^2$ (cf. [4]). The technique of holomorphic extension, used for harmonic functions in [22], has been generalized for solutions of partial differential equations with constant coefficients by Kiselman [15]. In a recent paper, Ebenfelt [12] considers the holomorphic extension to the so-called kernel $\mathcal{NH}(\Omega)$ of Ω's harmonicity cell, for solutions in simply connected domains Ω in $\mathbb{R}^n$, of linear elliptic partial differential equations of type: $\Delta^k u + \sum_{|\alpha|<2k} a_\alpha(x) D^\alpha u = g$, where $\mathcal{NH}(\Omega) = \{z \in \mathcal{H}(\Omega); \mathrm{Ch}[T(z)] \subset \Omega\}$. It can be observed that one of the central results in the theory of harmonicity cells is the following Lelong theorem (stated here in the harmonic case)

Theorem 1.1. *Let Ω be a non empty domain in $\mathbb{R}^n$, $n \geq 2$, with non empty boundary and $\mathcal{H}(\Omega)$ its harmonicity cell in $\mathbb{C}^n$. For every $\zeta \in \partial\mathcal{H}(\Omega)$ there exists $f = f_\zeta$, a harmonic function in Ω, which is the restriction to $\Omega = \mathcal{H}(\Omega) \cap \mathbb{R}^n$ of a (unique) holomorphic function $\widetilde{f_\zeta}$ defined in $\mathcal{H}(\Omega)$ such that $\widetilde{f_\zeta}$ can not be extended holomorphically in any open neighborhood of ζ.*

Statement of the problem. In this paper we consider the simpler case of a non-empty plane domain D (with $\partial D \neq \emptyset$) which we set to be simply connected and look for a suitable ∞-harmonic function f_ζ in D. We state the problem as follows:

Let ζ be a boundary point of $\mathcal{H}(D)$ and put $T(\zeta) = \{\zeta_1 + i\zeta_2, \bar{\zeta}_1 + i\bar{\zeta}_2\}$. We will assume first that ζ belongs to $\Gamma(\zeta_1 + i\zeta_2)$. The problem is to find a solution f_ζ in the classical sense, i.e. $f_\zeta \in C^2(D)$ and f_ζ a.e. continuous on ∂D of the quasi-elliptic system:

$$u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0 \quad \text{in} D \tag{1.5}$$

$$\frac{\partial}{\partial \bar{w}_j}\widetilde{u} = 0 \quad j = 1,2 \qquad \text{in } \mathcal{H}(D) \tag{1.6}$$

$$\lim_{w\to\zeta,\, w\in\mathcal{H}(D)} |\widetilde{u}(w)| = \infty\,. \tag{1.7}$$

This problem has already been considered in [16] in the harmonic case, and in [7] in the p-polyharmonic case. It has also been solved in the (non linear) p-harmonic case with $1 < p < +\infty$ and $n = 2$ [9]. We used in [9] radial p-harmonic functions and their stream functions, centered at points of ∂D; but this approach limited our results to *finite* real p (with $p > 1$) and to *real valued* p-harmonic functions. Our main result in the present paper consists of introducing infinite-harmonicity cells and proving an existence theorem for the ∞-Laplace equation. In Theorem 2.5, we prove that to $\zeta \in \partial\mathcal{H}(D)$ corresponds a $f_\zeta \in \mathbf{H}_\infty(D)$ such that $\widetilde{f_\zeta}$ is holomorphic in $\mathcal{H}(D)$ and satisfies $|\widetilde{f_\zeta}(w)| \to \infty$, when $w \to \zeta$ with w inside $\mathcal{H}(D)$.

2. Infinite-harmonicity cells

The next four propositions are used in this work and their proofs are found in the references as cited.

Proposition 2.1 ([16])**.** *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega \neq \emptyset$, $\partial\Omega \neq \emptyset$, and $\mathcal{H}(\Omega) \subset \mathbb{C}^n$ be its harmonicity cell. For every point $\zeta \in \partial\mathcal{H}(\Omega)$, the topological boundary of $\mathcal{H}(\Omega)$, one can associate a point $t \in \partial\Omega$, the topological boundary of Ω, such that $\zeta \in \Gamma(t)$, the isotropic cone of $\mathbb{C}^n$ with vertex t.*

Proposition 2.2 ([2, 17])**.** *A classical solution $u = u(x_1, x_2) \in \mathbf{C}^2$ of the partial differential equation*

$$\Delta_\infty u = u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0,$$

in every non-empty domain $D \subset \mathbb{R}^2$, is real analytic in D, and cannot have a stationary point without being constant

Proposition 2.3 ([4]). *To every couple (Ω, f), where Ω is an open set of $\mathbb{R}^n = \{x + iy \in \mathbb{C}^n; y = 0\}$ (equipped with the induced topology from $\mathbb{C}^n$), f is a real analytic function on D, one can associate a couple $(\widetilde{\Omega}, \widetilde{f})$ such that $\widetilde{\Omega}$ is an open set of $\mathbb{C}^n$ whose trace $\widetilde{\Omega} \cap \mathbb{R}^n$ with $\mathbb{R}^n$ is the starting domain Ω, and $\widetilde{f}$ is a holomorphic function in $\widetilde{\Omega}$ whose restriction $\widetilde{f}|\Omega$ to Ω coincides with f. Furthermore, (i) if Ω is connected, so is $\widetilde{\Omega}$; (ii) Among all the $\widetilde{\Omega}$'s above, there exists a unique domain, denoted Ω^f, which is maximal in the inclusion meaning.*

Proposition 2.4 ([13]). *Let $A \subset \mathbb{C}^n$ be a connected open set, f and g be two holomorphic functions in A with values in a complex Banach space E. If there exists an open subset U of A such that $f(z) = g(z)$ for every z in $U \cap \mathbb{R}^n$, then $f(z) = g(z)$ for every z in A.*

Theorem 2.5. *Let D be a simply connected domain of $\mathbb{R}^2 \simeq \mathbb{C}$, with $D \neq \emptyset$, and $\partial D \neq \emptyset$. Let $\mathcal{H}(D) = \{z \in \mathbb{C}^2; z_1 + iz_2 \in D \text{ and } \bar{z}_1 + i\bar{z}_2 \in D\}$ be the harmonicity cell of D. Then*

(1) For every $\zeta \in \partial\mathcal{H}(D)$, and every open neighbourhood V_ζ of ζ in $\mathbb{C}^2$, there exists a classical ($\in C^2$) ∞-harmonic function f_ζ on D, whose complex extension is holomorphic in $\mathcal{H}(D)$, but cannot be analytically continued through V_ζ.

(2) For the given domain D, let us denote by $\mathcal{H}_\infty(D)$ the interior in $\mathbb{C}^2$ of $\cap\{D^u; u \in \mathbf{H}_\infty(D)\}$. The set $\mathcal{H}_\infty(D)$ which may be called the infinite-harmonicity cell of D, satisfies:

- (a) *The trace of $\mathcal{H}_\infty(D)$ with $\mathbb{R}^2$ is D, under the hypothesis that $\mathcal{H}_\infty(D) \neq \emptyset$*
- (b) *$\mathcal{H}_\infty(D)$ is a connected open of $\mathbb{C}^2$*
- (c) *The inclusion $\mathcal{H}_\infty(D) \subset \mathcal{H}(D)$ always holds*
- (d) *If D is such that every $u \in \mathbf{H}_\infty(D)$ extends holomorphically to $\mathcal{H}(D)$ then $\mathcal{H}_\infty(D) \neq \emptyset$, and both the cells $\mathcal{H}(D)$ and $\mathcal{H}_\infty(D)$ coincide.*
- (e) *Suppose D is bounded and covered by a finite union of open rectangles $P_2^r(a_j; \rho_{j1}, \rho_{j2})$, centered at $a_j \in D$, $j = 1, \dots, m$, such that for every $u \in \mathbf{H}_\infty(D)$*
$$\limsup_{n_k \to +\infty} \Big[\frac{1}{(n_k)!}\Big|\frac{\partial^{n_k} u}{\partial x_k^{n_k}}(a_j)\Big|\Big]^{1/n_k} \le \frac{1}{\rho_{jk}}, \quad k = 1, 2,\ 1 \le j \le m\,.$$
Then $\mathcal{H}_\infty(D) \supset \cup_{j=1}^m P_2^c(a_j, \rho_j)$, and therefore $\mathcal{H}_\infty(D) \neq \emptyset$.

In the proof of Theorem 2.5, we will use the following two lemmas.

Lemma 2.6. *In every sector $-\pi < \theta < \pi$, the ∞-Laplace equation $\Delta_\infty u = 0$ has a solution in the form $u = \frac{v(\theta)}{\rho}$, where $\theta = \operatorname{Arg} z$, $\rho = |z|$, and v satisfies the ordinary differential equation (not containing θ)*
$$(v')^2 v'' + 3v(v')^2 + 2v^3 = 0 \tag{2.1}$$

Proof. It is clear that we have to use polar coordinates. With $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$ in (1.5), we get by a simple calculation: $u_{x_1} = u_\rho\cos\theta - \frac{1}{\rho}u_\theta\sin\theta$, $u_{x_2} = u_\rho\sin\theta + \frac{1}{\rho}u_\theta\cos\theta$, $u_{x_1x_1} = u_{\rho\rho}\cos^2\theta + \frac{1}{\rho^2}u_{\theta\theta}\sin^2\theta - \frac{1}{\rho}u_{\theta\rho}\sin 2\theta + \frac{1}{\rho}u_\rho\sin\theta + \frac{1}{\rho^2}u_\theta\sin 2\theta$, $u_{x_2x_2} = u_{\rho\rho}\sin^2\theta + \frac{1}{\rho^2}u_{\theta\theta}\cos^2\theta + \frac{1}{\rho}u_{\theta\rho}\sin 2\theta + \frac{1}{\rho}u_\rho\cos^2\theta - \frac{1}{\rho^2}u_\theta\sin 2\theta$, $u_{x_1x_2} = \frac{1}{2}u_{\rho\rho}\sin 2\theta - \frac{1}{2\rho^2}u_{\theta\theta}\sin 2\theta + \frac{1}{\rho}u_{\theta\rho}\cos 2\theta - \frac{1}{2\rho}u_\rho\sin 2\theta - \frac{1}{\rho^2}u_\theta\cos 2\theta$. Finally, after expanding the terms and rearranging, the ∞-Laplace equation (1.5) takes the form (in polar coordinates)

$$\Delta_\infty u = u_\rho^2 u_{\rho\rho} + \frac{2u_\rho u_\theta u_{\rho\theta}}{\rho^2} + \frac{u_\theta^2 u_{\theta\theta}}{\rho^4} - \frac{u_\rho u_\theta^2}{\rho^3} = 0 \tag{2.2}$$

Putting $u = \frac{v(\theta)}{\rho}$ in (2.2) we find that v satisfies the non-linear o.d.e. (2.1). □

Lemma 2.7. *Let D be a simply connected domain in $\mathbb{C}$, $D \neq \emptyset$, $\partial D \neq \emptyset$. For every $t \in \partial D$, there exists a complex valued ∞-harmonic function in D which cannot be extended continuously in any given open neighborhood of t.*

Proof. Let us look for a solution of (1.5) in D in the form $u(z) = \frac{v(\theta)}{|z-t|}$, where the argument θ is the unique angle in $]-\pi,\pi[$ satisfying $z - t = e^{i\theta}|z-t|, v$ is assumed to be C^2 in $]-\pi,\pi[$. Note here that the simple connexity of D guarantees that u is uniform in D. As it can be shown that the $\infty-$Laplacien operator: $\Delta_\infty u = u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_xx_2} + u_{x_2}^2 u_{x_2x_2}$ is invariant under translations τ_a of $\mathbb{C} \simeq \mathbb{R}^2$, $z = x_1 + ix_2$, $a = a_1 + ia_2$ - that is $\Delta_\infty(u \circ \tau_a) = (\Delta_\infty u) \circ \tau_a$ - we may assume without loss of generality that $t = 0$. Insertion of $v = e^{\gamma\theta}$, where $\gamma \in \mathbb{C}$ is a constant, in (2.1) gives: $\gamma^4 + 3\gamma^2 + 2 = 0$ or $(\gamma^2+1)(\gamma^2+2) = 0$. Take $\gamma = i$ and consider the ∞-harmonic function in D defined by: $u(z) = \frac{e^{i\theta}}{|z-t|}$, or more explicitly:

$$u(z) = \begin{cases} \frac{1}{|z-t|}\exp(i \arcsin\frac{x_2-t_2}{|z-t|}) & \text{if } x_1 \ge t_1 \\ \frac{\pi}{|z-t|} - \frac{1}{|z-t|}\exp(i\arcsin\frac{x_2-t_2}{|z-t|}) & \text{if } x_1 < t_1 \text{ and } x_2 > t_2 \\ \frac{-\pi}{|z-t|} - \frac{1}{|z-t|}\exp(i\arcsin\frac{x_2-t_2}{|z-t|}) & \text{if } x_1 < t_1 \text{ and } x_2 < t_2 \end{cases}$$

The result follows immediately by taking the principal argument $z \mapsto \operatorname{Arg} z \in]-\pi,\pi[$; here for $\delta \in [-1,1]$, $\arcsin\delta$ signifies the unique number β in $[-\frac{\pi}{2},\frac{\pi}{2}]$ satisfying $\sin\beta = \delta$. □

Proof of Theorem 2.5. For $\zeta \in \partial\mathcal{H}(D)$ there exists, due to Proposition 2.1, a boundary point t of D, such that $\zeta \in \Gamma(t)$ (which is equivalent to $t \in T(\zeta)$). As $T(w)$ reduces in the two-dimensional case to the pair $\{w_1 + iw_2, \quad \bar{w}_1 + i\bar{w}_2\}$, we have $t = \zeta_1 + i\zeta_2$ or $t = \bar{\zeta}_1 + i\bar{\zeta}_2$.
1.a. Suppose at first that $t = \zeta_1 + i\zeta_2$. By Lemma 2.6, we deduce that (1.5) has a solution $u(z)$ in D in the form $|z-t|^{-1}e^{i\theta}$. We conclude then as the solutions of (1.5) are in particular real analytic in D (Proposition 2.2), that the so-defined function $u(z)$ (given by Lemma 2.7) has a holomorphic extension $\widetilde{u}$ to a maximal domain $A_1 = D^u$ in $\mathbb{C}^2$ (Proposition 2.3). Since $\mathcal{H}(D)$ is the connected component containing D of the open $\mathbb{C}^2 - \cup_{t' \in \partial D}\{w \in \mathbb{C}^2; (w_1 - t_1')^2 + (w_2 - t_2')^2\}$, we have $A_1 \supset \mathcal{H}(D)$. Substituting in $u(z)$ complex variables w_1, w_2 to real ones and putting $h(w) = \sqrt{(w_1-t_1)^2 + (w_2-t_2)^2}$, we obtain

$$\widetilde{u}(w) = \begin{cases} \frac{1}{h(w)}\exp(i\arcsin\frac{w_2-t_2}{h(w)}) & \text{if } \operatorname{Re} w_1 \ge t_1 \\ \frac{\pi}{h(w)} - \frac{1}{h(w)}\exp(i\arcsin\frac{w_2-t_2}{h(w)}(& \text{if } \operatorname{Re} w_1 < t_1 \text{ and } \operatorname{Re} w_2 > t_2 \\ \frac{-\pi}{h(w)} - \frac{1}{h(w)}\exp(i\arcsin\frac{w_2-t_2}{h(w)}) & \text{if } \operatorname{Re} w_1 < t_1 \text{ and } \operatorname{Re} w_2 < t_2, \end{cases}$$

where the branches are taken such that the square root is positive when it is restricted to D, and for arcsin the branch is chosen such that its values are real (in

$]-\pi,\pi[)$ whenever z belongs to D. To see that $\widetilde{u}(w)$ is holomorphic in $\mathcal{H}(D)$, we consider

$$F(w)=\begin{cases}\frac{1}{g(w)}\exp(i\arcsin\frac{w_2-\operatorname{Im}(\zeta_1+i\zeta_2)}{g(w)}) & \text{if } \operatorname{Re}w_1\geq\operatorname{Re}(\zeta_1+i\zeta_2)\\ \frac{\pi}{g(w)}-\frac{1}{g(w)}\exp(i\arcsin\frac{w_2-\operatorname{Im}(\zeta_1+i\zeta_2)}{g(w)}) & \text{if } \operatorname{Re}w_1<\operatorname{Re}(\zeta_1+i\zeta_2),\\ & \operatorname{Re}w_2>\operatorname{Im}(\zeta_1+i\zeta_2)\\ \frac{-\pi}{g(w)}-\frac{1}{g(w)}\exp(i\arcsin\frac{w_2-\operatorname{Im}(\zeta_1+i\zeta_2)}{g(w)}) & \text{if } \operatorname{Re}w_1<\operatorname{Re}(\zeta_1+i\zeta_2),\\ & \operatorname{Re}w_2<\operatorname{Im}(\zeta_1+i\zeta_2),\end{cases}$$

where $g(w)=\sqrt{[(w_1+iw_2)-(\zeta_1+i\zeta_2)][(\bar{w}_1+i\bar{w}_2)-(\zeta_1+i\zeta_2)]}$, and the branches are chosen as in $\widetilde{u}(w)$. Seeing that by [16], $\mathcal{H}(D)=\{w\in\mathbb{C}^2;T(w)\subset D\}$, and noting that $g(w)=0$ if and only if $w\in\Gamma(t)$ with $t\in\partial D$, the function $F(w)$ is well defined in some open $A_2\supset\mathcal{H}(D)$. Observe that $\widetilde{u}$ and F are both holomorphic in $A=A_1\cap A_2$ - since $\frac{\partial\widetilde{u}}{\partial\bar{w}_j}=\frac{\partial F}{\partial\bar{w}_j}=0$ in $A\supset\mathcal{H}(D)$, $w_j=x_j+iy_j$, $j=1,2$- having the same restriction on $D=U\cap\mathbb{R}^2$, with $U=\mathcal{H}(D)$: $\widetilde{u}|_D(z)=F|_D(z)=u(z)$, with $z=x_1+ix_2$. By Proposition 2.4, we deduce that $\widetilde{u}=F$ in $\mathcal{H}(D)$. Furthermore, since ζ and t satisfies $(\zeta_1-t_1)^2+(\zeta_2-t_2)^2=0$, one has by letting $w\in\mathcal{H}(D)$ tend to ζ: $|\widetilde{u}(w)|=|h(w)^{-1}|\to\infty$; consequently the function $\widetilde{u}(w)$ cannot be extended holomorphically across $\zeta\in\partial\mathcal{H}(D)$.
1.b If $t=\bar{\zeta}_1+i\bar{\zeta}_2$, the function $G(w)$ defined in the same way by substituting $\bar{\zeta}_1+i\bar{\zeta}_2$ to $\zeta_1+i\zeta_2$ in $F(w)$ (with similar branches) satisfies: (i) $G(w)$ exists for every $w\in\mathcal{H}(D)$, (ii) $G(w)$ is holomorphic in $\mathcal{H}(D)$, (iii) $G(w)$ cannot be extended holomorphically to any open neighborhood of ζ in $\mathbb{C}^2$ (since $|G(w)|\to\infty$ when $w\in\mathcal{H}(D)\to\zeta$), (iv) The restriction of $G(w)$ on D is an infinite-harmonic function on D.
2) It might happen that the set $\cap\{D^u;u\in\mathbf{H}_\infty(D)\}$ reduces to only the starting domain D, we would obtain thus an empty ∞-harmonicity cell, and consequently (b), (c) are held to be true if this eventual case occur.
(a) Suppose the above intersection is of non empty interior in $\mathbb{C}^2$. Since D is considered as a relative domain in $\mathbb{R}^2$, with respect to the induced topology from $\mathbb{C}^2$, and since $D^u\cap\mathbb{R}^2=D$ for every $u\in\mathbf{H}_\infty(D)$, we have: $\cap\{D^u;u\in\mathbf{H}_\infty(D)\}\cap\mathbb{R}^2=\cap\{D^u\cap\mathbb{R}^2;u\in\mathbf{H}_\infty(D)\}=D$; so $\operatorname{Tr}\mathcal{H}_\infty(D)=\mathcal{H}_\infty(D)\cap\mathbb{R}^2\subset D$. On the other hand, since $D\subset D^u$ for every $u\in\mathbf{H}_\infty(D)$, we have $D\subset(\cap_{u\in\mathbf{H}_\infty(D)}D^u)\cap\mathbb{R}^2$. Moreover, from the real analyticity of a function $u\in\mathbf{H}_\infty(D)$ in D, we deduce that for every point $a\in D$, there exist radius $\rho_j^u=\rho_j^u(a)>0, j=1,2$, small enough such that $u(z)=\sum_{\alpha\in\mathbb{N}^2}a_\alpha(z-a)^\alpha$, for all z in the rectangle $P_2^r(a,\rho_j^u(a))=\{x\in\mathbb{R}^2;|x_j-a_j|<\rho_j^u(a),\ j=1,2\}\subset D$, where $(z-a)^\alpha=(x_1-a_1)^{\alpha_1}(x_2-a_2)^{\alpha_2}$. Substituting $w\in\mathbb{C}^2$ to z, we obtain $\widetilde{u}(w)=\sum_{\alpha\in\mathbb{N}^2}a_\alpha(w-a)^\alpha$ which is of course holomorphic in the complex bidisk $P_2^c(a,\rho_j^u(a))=\{w\in\mathbb{C}^2;|w_j-a_j|<\rho_j^u(a),j=1,2\}\subset\mathbb{C}^2$, where $(w-a)^\alpha=(w_1-a_1)^{\alpha_1}(w_2-a_2)^{\alpha_2}$, the chosen branch being such that the restriction of $(w-a)^\alpha$ to $\mathbb{R}^2$ is >0. Thus the domain of holomorphic extension of u is nothing else but the union of all the $P_2^c(a,\rho_j^u(a))$'s with a running through D. The above construction involves $D\subset[\cap\{D^u;u\in\mathbf{H}_\infty(D)\}]^0\cap\mathbb{R}^2$; so one has $\operatorname{Tr}\mathcal{H}_\infty(D)=D$.
(b) Let w,w' be two arbitrary points in $B=\cap\{D^u;u\in\mathbf{H}_\infty(D)\}$. By (a), $B=\cap_{u\in\mathbf{H}_\infty(D)}\cup_{a\in D}P_2^c(a,\rho_j^u(a))$, where $\rho_j^u(a)$, $j=1,2$, are the greatest radius corresponding to the power series expansion of u at a. Note that the set B

is connected in case the above intersection reduces to D. Suppose then $B \neq D$ and take w, w' in B. Since w, w' are in D^u for every $u \in \mathbf{H}_\infty(D)$, there exist, by construction of D^u, $a, a' \in D$, such that $w \in P_2^c(a, \rho_j^u(a))$, and $w' \in P_2^c(a', \rho_j^u(a'))$. Putting $\rho_j(a) = \inf\{\rho_j^u(a); u \in \mathbf{H}_\infty(D)\}, \rho_j(a') = \inf\{\rho_j^u(a'); u \in \mathbf{H}_\infty(D)\}$, we obtain $w \in P_2^c(a, \rho_j(a))$, $w' \in P_2^c(a', \rho_j(a'))$, with $\rho_j(a) \geq 0$ and $\rho_j(a') \geq 0$. Let then β denote a path in D joining $\operatorname{Re} w \in P_2^r(a, \rho_j(a)) \subset D$ to $\operatorname{Re} w' \in P_2^r(a', \rho_j(a')) \subset D$. The path γ, constituted successively with the paths $[w, \operatorname{Re} w]$, β, and $[\operatorname{Re} w', w']$ joins w to w' and is included into the union $P_2^c(a, \rho_j(a)) \cup D \cup P_2^c(a', \rho_j(a')) \subset D^u$. We conclude that $\gamma \subset B$, B is connected and therefore so is $\mathcal{H}_\infty(D) = B^0$.

(c) By contradiction, suppose that $\mathcal{H}(D)$ does not contain $\mathcal{H}_\infty(D)$. Take $w_0 \in \mathcal{H}_\infty(D)$ with $w_0 \notin \mathcal{H}(D)$. Since $\mathcal{H}_\infty(D)$ is connected and $D \subset \mathcal{H}_\infty(D)$, there would exist a continuous path $\gamma_{w_0,a}$ joining w_0 to some point $a \in D$, with $\gamma_{w_0,a} \subset \mathcal{H}_\infty(D)$. Next, due to the inclusion $D \subset \mathcal{H}(D)$, we ensure the existence of a point ζ_0 belonging to $\gamma_{w_0,a} \cap \partial\mathcal{H}(D)$. Due to Part 1 above, to the boundary point ζ_0 of $\mathcal{H}(D)$ corresponds some function f_{ζ_0} which is ∞-harmonic in D and whose extension $\widetilde{f_{\zeta_0}}$ in $\mathbb{C}^2$ is a holomorphic function in $\mathcal{H}(D)$ which can not be holomorphically continued beyond ζ_0. Now, the ∞-harmonicity cell $\mathcal{H}_\infty(D)$ is characterized by: (i) Every $u \in \mathbf{H}_\infty(D)$ is the restriction on D of a holomorphic function $\widetilde{u} : \mathcal{H}_\infty(D) \to \mathbb{C}$; (ii) $\mathcal{H}_\infty(D)$ is the maximal domain of $\mathbb{C}^2$, in the inclusion sense, whose trace on $\mathbb{R}^2$ is D, and satisfying (i). Then $\widetilde{f_{\zeta_0}}$ is not holomorphic at ζ_0 with ζ_0 inside $\mathcal{H}_\infty(D)$, which contradicts the property (i). Consequently, the inclusion $\mathcal{H}_\infty(D) \subset \mathcal{H}(D)$ always holds.

(d) By Proposition 2.3, given $u \in \mathbf{H}_\infty(D)$, there exists a maximal domain $D^u \subset \mathbb{C}^2$ to which u extends holomorphically. The domain D^u is then a domain of holomorphy of $\widetilde{u}$ (also called domain of holomorphy of u). Suppose that every $u \in \mathbf{H}_\infty(D)$ extends holomorphically to $\mathcal{H}(D)$. One has then $\mathcal{H}(D) \subset D^u$, for every $u \in \mathbf{H}_\infty(D)$; therefore, $\mathcal{H}(D) = \mathcal{H}(D)^0 \subset [\cap_{u \in \mathbf{H}_\infty(D)} D^u]^0 = \mathcal{H}_\infty(D)$. The result follows by (c).

(e) Due to Proposition 2.2, every ∞-harmonic function u in D is in particular real analytic in D, and thereby partially real analytic in D. Since $D \subset \cup_{j=1}^m P_2^r(a_j, \rho_j)$, there exist open rectangles $P_2^r(a_j, \rho_j^u) \subset D$ in which u writes as the sum of a power series in $(x_1 - a_{j1})(x_2 - a_{j2})$. More, the convergence radius ρ_{j1}^u, ρ_{j2}^u corresponding to the development of $x_1 \mapsto u(x_1, a_{j2})$ and $x_2 \mapsto u(a_{j1}, x_2)$ at a_{j1} and a_{j2} (respectively) are given by $\rho_{jk}^u = \{\limsup_{n_k \to +\infty} [\frac{1}{(n_k)!} |\frac{\partial^{n_k} u}{\partial x_k^{n_k}}(a_j)|]^{1/n_k}\}^{-1}$ $k = 1, 2$, $1 \leq j \leq m$. By assumption, the given covering of D satisfies $\inf_{u \in \mathbf{H}_\infty(D)} \rho_{jk}^u \geq \rho_{jk}$, that is for every $x \in P_2^r(a_j, \rho_j)$:

$$u(x) = \sum_{n_1 \in \mathbb{N}} \sum_{n_2 \in \mathbb{N}} \frac{1}{n_1!\, n_2!} \frac{\partial^{n_1+n_2} u}{\partial x_1^{n_1}\, \partial x_2^{n_2}}(a_j)(x_1 - a_{j1})^{n_1}(x_2 - a_{j2})^{n_2},$$

where $x = (x_1, x_2)$, $a_j = (a_{j1}, a_{j2})$ and $\rho_j = (\rho_{j1,} \rho_{j2})$. It is clear that the complex series obtained by substituting $w_1, w_2 \in \mathbb{C}$ to $x_1, x_2 \in \mathbb{R}$ is convergent on every complex bidisk $P_2^c(a_j, \rho_j) = \{w \in \mathbb{C}^2; |w_1 - a_{j1}| < \rho_{j1}$ and $|w_2 - a_{j2}| < \rho_{j2}\}$. Due to the maximality of D^u, we have $\cup_{j=1}^m P_2^c(a_j, \rho_j) \subset D^u$ for every $u \in \mathbf{H}_\infty(D)$, and thereby $\cup_{j=1}^m P_2^c(a_j, \rho_j) \subset \cap\{D^u; u \in \mathbf{H}_\infty(D)\}$. The last union being an open set, one deduces that $\mathcal{H}_\infty(D) \supset \cup_{j=1}^m P_2^c(a_j, \rho_j)$; this mean in particular that $\mathcal{H}_\infty(D) \neq \emptyset$. $\square$

Remark 2.8. The significant fact of the inclusion $\mathcal{H}_\infty(D) \subset \mathcal{H}(D)$ is that the common complex domain $\widetilde{D}$, denoted $\mathcal{H}_\infty(D)$, for the whole class $\mathbf{H}_\infty(D)$, cannot pass beyond $\mathcal{H}(D)$. Nevertheless, given a specified $\infty-$harmonic function u in D, we may have: $D^u \supset \mathcal{H}(D)$ with $D^u \neq \mathcal{H}(D)$.

Example 2.9. Consider $D = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_2 > 0\}$, and look for a C^2 solution u in D of $\Delta_\infty u = 0$ in the form $u = Ax_1^\alpha + Bx_2^\beta$ (where A, B, α, β are constant). Since $\Delta_\infty u = A^3\alpha^3(\alpha-1)x_1^{3\alpha-4} + B^3\beta^3(\beta-1)x_2^{3\beta-4}$, we deduce that $u = x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}$ is a classical ∞-harmonic function u in D. Putting $w_j = x_j + iy_j$, $j = 1, 2$ and $\widetilde{u}(w_1, w_2) = w_1^{\frac{4}{3}} - w_2^{\frac{4}{3}}$, where the branch is chosen such that the restriction of $\widetilde{u}$ to $D \subset \mathbb{R}^2$ is a real valued function, we observe that $\widetilde{u}$ is holomorphic in $\mathbb{C}^2 - (L_1 \cup L_2)$, where $L_1 = \mathbb{C} \times \{0\}$, $L_2 = \{0\} \times \mathbb{C}$, and $\widetilde{u}|D = u$. Since $\mathbb{C}^2 - (L_1 \cup L_2) = \mathbb{C}^* \times \mathbb{C}^*$ is a domain (connected open) in $\mathbb{C}^2$, we deduce that $D^u = \mathbb{C}^* \times \mathbb{C}^*$. The harmonicity cell of D is given explicitly by the set of all $w \in \mathbb{C}^2$ satisfying: $w_1 + iw_2 = x_1 - y_2 + i(x_2 + y_1) \in D$ and $\bar{w}_1 + i\bar{w}_2 = x_1 + y_2 + i(x_2 - y_1) \in D$ (here $\mathbb{R}^2 \simeq \mathbb{C}$). Thus $\mathcal{H}(D) = \{w \in \mathbb{C}^2;\ x_1 > |y_2| \text{ and } x_2 > |y_1|\} \subset D^u$, and $\mathcal{H}(D) \neq D^u$.

Remark 2.10. The inclusion $\mathcal{H}_\infty(D) \subset \mathcal{H}(D)$ can be strengthened. Indeed, let $D \subset \mathbb{C}$ be a simply connected domain, with smooth boundary, and let $\mathbf{H}_{qr}(D)$ denote the sub-class of all ∞-harmonic functions which are quasi-radial with respect to some boundary point of D. A function $u \in \mathbf{H}_{qr}(D)$ if there exists $t \in \partial D$ such that $u(z) = \rho^m f(\theta)$, where $z = t + \rho e^{i\theta} \in D$, f is a real or complex-valued C^2 function in $]-\pi, \pi[$, and m is a constant (no restriction on m also). Note that by Aronsson [2], $\mathbf{H}_{qr}(D)$ is not empty. For instance, for $m > 1$, one can find functions $Z = f(\theta)$ in parametric representation: $Z = \frac{C}{m}(1 - \frac{1}{m}\cos^2\tau)^{\frac{m-1}{2}}\cos\tau$, $\theta = \theta_0 + \int_{\tau_0}^{\tau} \frac{\sin^2\tau'}{m - \cos^2\tau'} d\tau'$, $\tau_1 < \tau < \tau_2$ (C, θ_0, τ_0, τ_1, τ_2 are constants). Similarly, let $\mathcal{H}_{qr}(D)$ denote the complex domain $\widetilde{D}$ corresponding to $\mathbf{H}_{qr}(D)$. Since $\mathcal{H}_{qr}(D) = [\bigcap_{u \in \mathbf{H}qr(D)} D^u]^0$, $\mathbf{H}_{qr}(D) \subset \mathbf{H}_\infty(D)$, and the constructed function f_ζ in the proof of Theorem 2.5 is quasi-radial, we have: $\mathcal{H}_\infty(D) \subset \mathcal{H}_{qr}(D) \subset \mathcal{H}(D)$.

Remark 2.11. To see that the property: $\lim_{w\to\zeta} |\widetilde{f}(w)| = \infty$, ($w \in \mathcal{H}(D)$, $\zeta \in \partial\mathcal{H}(D)$) may fail, we give the following example.

Example 2.12. Let D be an arbitrary simply connected plane domain, $D \neq \emptyset$, $\partial D \neq \emptyset$. For a fixed $\zeta \in \partial\mathcal{H}(D)$, take $t = \zeta_1 + i\zeta_2 \in T(\zeta)$ and consider

$$F(w) = \sqrt{(w_1 - t_1)^2 + (w_2 - t_2)^2}\exp(\frac{1}{2}\arctan\frac{w_2 - t_2}{w_1 - t_1}),$$

where the branches are taken such that their restriction to $D \subset \mathbb{R}^2$ is positive for the square root and in $]-\frac{\pi}{2}, \frac{\pi}{2}[$ for $arctg$). This function verifies: $F(w)$ is well defined and holomorphic on $\mathcal{H}(D)$, its restriction f to D is ∞-harmonic in D since $f(z) = \sqrt{\rho}e^{\theta/2}$ where $z - t = \rho e^{i\theta}$; nevertheless $\lim_{w\to\zeta} |F(w)| = 0$. Indeed, if ζ is assumed in $\partial\mathcal{H}(D) - \partial D$, one has $w_1 + iw_2 \to \zeta_1 + i\zeta_2$, so that $(w_1 - t_1)^2 + (w_2 - t_2)^2 = [(w_1 + iw_2) - (\zeta_1 + i\zeta_2)][(\bar{w}_1 + i\bar{w}_2) - (\zeta_1 + i\zeta_2)] \to 0$; on the other hand, by definition of $T(\zeta)$, $(\zeta_1 - t_1)^2 + (\zeta_2 - t_2)^2 = 0$, thus $\arctan\frac{w_2 - t_2}{w_1 - t_1} \to \arctan\frac{\zeta_2 - t_2}{\zeta_1 - t_1} = \arctan \pm i = \pm i\infty$, and $|\exp(\frac{1}{2}arctg\frac{w_2 - t_2}{w_1 - t_1})| \to 1$. Otherwise, the result is immediate if $\zeta \in \partial D \subset \partial\mathcal{H}(D)$.

sectionHolomorphic extension in Fluids dynamic

In this section, we consider two general examples where the above techniques, of complexification and analytic continuation to $\mathbb{C}^n$, are used for the study of some physical problems. The main application we are interested in is the problem of the behaviour of a flow near an extreme point. In the following, $\mathbf{H}_p(D)$ denotes the class of all p-harmonic functions on D.

Proposition 2.13. *Let $D \subset \mathbb{C}$ be an arbitrary profile limited by a connected closed curve C, and consider a stationary plane flow round D defined by the data of a vanishing point and its velocity V_∞ at the infinite. Suppose that C contains two straight segments $[a, z_1]$, $[a, z_2]$ originated at $a = a_1 + ia_2$ and forming an angle $\nu\pi, 0 < \nu < 1$. Then there exist a suitable real $p > 1$ and an open simply connected neighborhood U of a, such that the quasi-linear p.d.e: $\Delta_p u = |\nabla u|^2 \Delta u + (p-2)\Delta_\infty u = 0$, has a radial (with respect to a) positive solution φ in U, which approximates the modulus of the velocity $V(z)$ of the fluid. More precisely:*

(i) *$|V(z)| \sim \varphi(z)$ as $z \to a$, $(z \in U)$.*
(ii) *$\varphi \in \mathbf{H}_{(3\nu-4)/(2\nu-2)}(U)$.*
(iii) *Let $C > 0$ be a constant, and put $\delta = (\frac{2-\nu}{\nu} C)^{(\nu-2)/(2\nu-2)}$; then a stream function φ_c associated with a function φ of the form $C|z-a|^{\nu/(2-\nu)}$ is given by*

$$\varphi_c(x_1 + ix_2) = \begin{cases} \delta \arcsin \frac{x_2 - a_2}{|z-a|} & \text{if } x_1 \geq a_1 \\ \delta\pi - \delta \arcsin \frac{x_2 - a_2}{|z-a|} & \text{if } x_1 < a_1 \text{ and } x_2 > a_2 \\ -\delta\pi - \delta \arcsin \frac{x_2 - a_2}{|z-a|} & \text{if } x_1 < a_1 \text{ and } x_2 < a_2 \end{cases}$$

which is q-harmonic in U, with $\frac{1}{p} + \frac{1}{q} = 1$,; that is, $\varphi_c \in \mathbf{H}_{(3\nu-4)/(\nu-2)}(U)$.

Proof. Since D is connected and simply connected, there exists an injective holomorphic transformation f_1 sending the closed lower half-plane

$$P^- - \{-i\} = \{\beta \in \mathbb{C} : \operatorname{Im} \beta \leq 0, \beta \neq -i\},$$

sharpened at $-i$, onto the the exterior of D, with $a = f_1(\beta_0)$ and $\operatorname{Im} \beta_0 = 0$. The composed map $\gamma = g_1(\beta) = [f_1(\beta) - a]^{1/(2-\nu)}$ is holomorphic, injective, and sends an open simply connected neighborhood $\mathcal{V}(\beta_0) \subset P^- - \{-i\}$ of β_0 onto a neighborhood $\mathcal{V}(0) \subset P_1$, where $P_1 \subset \mathbb{C}[\gamma]$ is one of the half-planes limited by the straight line passing through $\gamma_1 = (z_1 - a)^{1/(2-\nu)}$ and $\gamma_2 = (z_2 - a)^{1/(2-\nu)}$. Due to the symmetry principle of Schwartz [13], the function g_1 extends as a holomorphic function $\widetilde{g_1}$ in some open simply connected neighborhood $\widetilde{\mathcal{V}}(\beta_0) \subset \mathbb{C} - \{-i\}$, $\widetilde{\mathcal{V}}(\beta_0) \supset \mathcal{V}(\beta_0)$. Thus, for every $\beta \in \widetilde{\mathcal{V}}(\beta_0)$, one has the absolutely convergent expansion for $\widetilde{g_1}$: $\widetilde{g_1}(\beta) = \sum_{j=1}^{+\infty} A_j(\beta - \beta_0)^j$. Moreover, seeing that $\widetilde{g_1}$ is holomorphic and injective in a neighborhood of β_0, the first coefficient $A_1 = [\widetilde{g_1}]'(\beta_0) = g_1'(\beta_0)$ is $\neq 0$. This implies in particular: $f_1(\beta) = a + (\beta - \beta_0)^{2-\nu} f_2(\beta)$, for $\beta \in \mathcal{V}(\beta_0)$, where the function $f_2(\beta) = [\sum_{j=1}^{+\infty} A_j(\beta - \beta_0)^{j-1}]^{2-\nu}$ is a holomorphic function in $\mathcal{V}(\beta_0)$ and may be taken uniform (seeing that A_1 is $\neq 0$) and holomorphic in a certain open neighborhood $\widetilde{\mathcal{V}}_1(\beta_0)$. Thus, for $\beta \in \mathcal{V}_1(\beta_0)$, one has $f_1(\beta) = a + (\beta - \beta_0)^{2-\nu} \sum_{j=0}^{+\infty} B_j(\beta - \beta_0)^j$, and

$$f_1(\beta) - a \sim B_0(\beta - \beta_0)^{2-\nu} \quad \text{as } \beta \to \beta_0$$

where $B_0 = A_1^{2-\nu}$. Recall that the flow is supposed to be held round a profile D with an angular point a. Due to the well known Chaplygine condition, the vanishing point of the current moves under the effect of viscosity and the formation of whirlpools, to the extreme point a of $\bar{D}$. As a simple calculus reveals, the complex potential of the flow round D is given by:

$$w = f(z) = Re^{-i\theta} g(z) + \frac{Re^{i\theta} r^2}{g(z)} - [2irR\sin(\psi - \theta)]\ln(z),$$

where $\mu = g(z)$ is the bijective holomorphic function from D^c, the exterior of D, onto the domain $|\mu| > r$. The values of r and ψ are such that $\lim_{z\to\infty} g(z) = \infty$, $\lim_{z\to\infty} g'(z) = 1$, $\mu_0 = g(a) = re^{i\psi}$ and $V_\infty = Re^{i\theta}$ is the velocity at the infinite. The holomorphic bijection $f_3 = f_1^{-1} \circ g^{-1}$ maps $\{|\mu| \geq r\}$ onto $P^- - \{-i\}$. Thus (2) gives

$$f_1 \circ f_3(\mu) - f_1 \circ f_3(\mu_0) \sim B_0[f_3(\mu) - f_3(\mu_0)]^{2-\nu} \quad \text{as } \mu \to \mu_0\,. \tag{2.3}$$

Since $f_3'(\mu_0) \neq 0$ one has $f_3(\mu) - f_3(\mu_0) \sim f_3'(\mu_0)(\mu - \mu_0)$ as $\mu \to \mu_0$, so that (2.3) implies $g^{-1}(\mu) - g^{-1}(\mu_0) \sim C_0(\mu - \mu_0)^{2-\nu}$, where

$$C_0 = B_0 f_3'(\mu_0)^{2-\nu} = [\frac{g_1'(\beta_0).g'(a)}{f_1'(\beta_0)}]^{2-\nu};$$

that is, $g(z) - g(a) \sim C_0^{1/(\nu-2)}(z-a)^{1/(2-\nu)}$ as $z \to a$. Consequently, near the vanishing point a of the flow, the derivative of g satisfies

$$g'(z) \sim \frac{g(z) - g(a)}{z - a} \sim C_0^{1/(\nu-2)}(z-a)^{1/(2-\nu)-1} = C_0^{1/(\nu-2)}(z-a)^{(\nu-1)/(2-\nu)} \tag{2.4}$$

as $z \to a$. On the other hand, putting $\mu = g(z)$, we obtain

$$\frac{df}{d\mu} = R\, e^{-i\theta} - R\, e^{i\theta}\frac{r^2}{\mu^2} - \frac{2irR}{\mu}\sin(\psi - \theta) \tag{2.5}$$

Since the velocity satisfies $V(z) = \bar{f'}(z)$, for $z \in D^c$, Equality (2.5) at μ_0 gives

$$Re^{-i\theta} - Re^{i\theta}\frac{r^2}{\mu_0^2} - \frac{2irR}{\mu_0}\sin(\psi - \theta) = 0$$

From the above equation and (2.5), we get for $|\mu| \geq r$: $\frac{df}{d\mu}(\mu) - \frac{df}{d\mu}(\mu_0) = (\mu - \mu_0)h(\mu)$, with $h(\mu) = r^2 R\, e^{i\theta}\frac{\mu+\mu_0}{\mu^2\mu_0^2} + \frac{2irR}{\mu\mu_0}\sin(\psi - \theta)$. By a simple calculus, $\lim_{\mu\to\mu_0} h(\mu) = \frac{2R}{r}e^{-2i\psi}\cos(\theta - \psi) \neq 0$, here we will have to suppose that V_∞ is such that $\theta \neq \psi \pm \frac{\pi}{2}$ (otherwise, if $\theta = \psi \pm \frac{\pi}{2}$, a direct calculus will do). Hence,

$$\frac{df}{d\mu} \sim D_0(z-a)^{1/(2-\nu)} \quad \text{as } \mu \to \mu_0\,, \tag{2.6}$$

with $D_0 = 2C_0^{1/(\nu-2)}R\cos(\theta-\psi)/(re^{2i\psi})$. Writing $\frac{df}{dz} = \frac{df}{d\mu}.\frac{d\mu}{dz}$ and combining (2.4) and (2.6), we obtain the equivalence $\frac{df}{dz} \sim C_0^{1/(\nu-2)}D_0(z-a)^{1/(2-\nu)}(z-a)^{\frac{\nu-1}{2-\nu}}$ as $z \to a$. Consequently $|V(z)| \sim C|z-a|^{\nu/(2-\nu)}$, where

$$C = \frac{2R|\cos(\theta - \psi)|}{r}|\frac{f_1'(\beta_0)}{g_1'(\beta_0).g'a)}|.$$

Therefore, (i) and (ii) may be obtained by taking $p = \frac{3\nu-4}{2\nu-2}$, $\eta = 0$, $\varepsilon = \frac{\nu C}{2-\nu}$ in the following lemma. □

Lemma 2.14. *For every real $p > 1$ and fixed complex point $z_0 \in \mathbb{C}$, the p-Laplace equation* (1.2) *has radial solutions (with respect to the origin point z_0) defined in any sharpened disk X^* at z_0: $X^* = \{z \in \mathbb{C};\, 0 < |z - z_0| < R_0\}$. All these functions may be given by: $\varepsilon\frac{p-1}{p-2}|z - z_0|^{\frac{p-2}{p-1}} + \eta$, if $p \neq 2$, and $\varepsilon \ln|z - z_0| + \eta$, if $p = 2$, where a, b are arbitrary in $\mathbb{R}$.*

Proof of Lemma 2.14. Since $\Delta_p(u \circ \tau_{z_0}) = (\Delta_p u) \circ \tau_{z_0}$, we may assume that $z_0 = 0$. Firstly, the case $p = 2$ is well known since a 2-harmonic function u in a domain Ω of $\mathbb{R}^n$ is also harmonic outside the zeros of *grad* u. If $p \neq 2$ and if $x = \rho\cos\theta$, $y = \rho\sin\theta$ is used in the p-Laplace equation

$$\Delta_p u = (p-2)[u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}] + (u_x^2 + u_y^2)(u_{xx} + u_{yy}) = 0 \qquad (2.7)$$

we observe, via a simple substitution of u_x, u_y, u_{xx}, u_{yy}, u_{xy}, expressed by means of the polar coordinates $(\rho,\ \theta)$, and taking into account that the usual Laplace operator Δ, and the gradient of u give in polar form: $\Delta u = u_{\rho\rho} + \rho^{-1}u_\rho + \rho^{-2}u_{\theta\theta}$, $|\nabla u|^2 = u_\rho^2 + \rho^{-2}u_\theta^2$, that (2.7) takes the form

$$\Delta_p u = (p-2)[u_\rho^2 u_{\rho\rho} + \frac{2u_\rho u_\theta u_{\rho\theta}}{\rho^2} + \frac{u_\theta^2 u_{\theta\theta}}{\rho^4} - \frac{u_\rho u_\theta^2}{\rho^3}] + (u_\rho^2 + \frac{u_\theta^2}{\rho^2})[u_{\rho\rho} + \frac{u_\rho}{\rho} + \frac{u_{\theta\theta}}{\rho^2}] = 0 \qquad (2.8)$$

To look for a radial solution u of (2.7), it suffices to put $u(x + iy) = h(\rho)$ in (2.8). We obtain $\Delta_p u = (h')^2[(p-1)h'' + \frac{1}{\rho}h'] = 0$ which is computed without difficulty and gives the result stated in Lemma 2.14.
(iii) Writing (2.7) in the divergence form, and seing that U is simply connected, one can associate to $\varphi = C|z - a|^{\nu/(2-\nu)}$ a conjugate q-harmonic function φ_c in U, defined by $(\varphi_c)_{x_1} = -|\nabla\varphi|^{\nu/(2-\nu)}\varphi_{x_2}$ and $(\varphi_c)_{x_2} = |\nabla\varphi|^{\nu/(2-\nu)}\varphi_{x_1}$. ☐

Remark 2.15. There is a physical interpretation of the p-Laplace equation (1.2) in terms of the laminar pipe flow of so-called power-law fluids [1]. Using the terminology of non-linear fluid mechanic, one is motivated to call the stream function v, corresponding to the potential u, the solution of $\Delta_q v = \operatorname{div}(|\nabla v|^{q-2}\Delta v) = 0$, $1 < q < +\infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. In the language of Potential theory we say that u and v are conjugate functions.

Proposition 2.16. *Under the same hypothesis than proposition above, suppose that a is a non angular point, $V(a + i\gamma) \neq 0$ for γ real $\neq 0$ sufficiently small, and $\frac{\partial^r V}{\partial x_2^r}(a) \neq 0$ for some integer $r \geq 1$. Then in some neighborhood U' of a, the velocity of the fluid writes as*

$$V(z) = [(x_2 - a_2)^r + (x_2 - a_2)^{r-1}h_1(x_1) + \dots + h_r(x_1)]h(z) = W(x_2 - a_2)h(z) \quad (2.9)$$

where $z = x_1 + ix_2$, $a = a_1 + ia_2$, h is a real analytic function in some neighborhood U' of a with $h(z) \neq 0$ for every $z \in U'$, and $h_1, \dots, h_r$, appearing in the Weierstrass' unitary polynomial in $(x_2 - a_2)$, are real analytic functions in some interval $]a_1 - \varepsilon, a_1 + \varepsilon[, \varepsilon > 0$.

Proof. Due to Propositions 2.3 and 2.4 above, we can extend holomorphically in $\mathbb{C}^2$ the velocity function $V : \Omega = (\bar{D})^c \to \mathbb{C}$, $(x_1, x_2) \mapsto V(x_1 + ix_2)$, which is real analytic (in fact even antiholomorphic) in Ω. Using the same technique above, and putting: $w = (w_1, w_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$, we find a maximal domain Ω^V in $\mathbb{C}^2$ whose trace with $\mathbb{R}^2$ is Ω, and to which V extends holomorphically. Let then $\widetilde{V}$ denote the unique complexified function of V with $\widetilde{V}|_\Omega = V$ and $\widetilde{V}$ is holomorphic in

Ω^V. Since $\widetilde{V}:\Omega^V\to\mathbb{C}$ satisfies also $\widetilde{V}(a)=V(a)=0$, $\widetilde{V}(a_1,a_2+\gamma)=V(a_1,a_2+\gamma)$ is $\neq 0$ for some $(a_1,a_2+\gamma)\in\Omega\subset\Omega^V$ with $\gamma\neq 0$, and $\frac{\partial^r\widetilde{V}}{\partial w_2^r}(a)\neq 0$ -seing that $\frac{\partial^r\widetilde{V}}{\partial w_2^r}(a)=\frac{\partial^r\widetilde{V}}{\partial w_2^r}|_\Omega(a)=\frac{\partial^r V}{\partial x_2^r}(a)$ - there exist, owing to Weierstass' preparation Theorem in $\mathbb{C}^n$ [21, p.290] with $n=2$, r functions $H_1(w_1),\dots,H_r(w_1)$ which are holomorphic in some open neighborhood $\widetilde{\Omega}_1$ of a_1 in $\mathbb{C}$, and a function $H(w)$ which is holomorphic in some open neighborhood $\widetilde{\Omega}\subset\Omega^V$ of a in $\mathbb{C}^2$ with $H(w)\neq 0$ in $\widetilde{\Omega}$, such that

$$\widetilde{V}(w)=[(w_2-a_2)^r+(w_2-a_2)^{r-1}H_1(w_1)+\cdots+H_r(w_1)]H(w) \tag{2.10}$$

for every w in some open neighborhood $(\widetilde{\Omega})'$ of a in $\mathbb{C}^2$ with $(\widetilde{\Omega})'\subset\widetilde{\Omega}\subset\Omega^V$. Taking now the restriction of Equality (2.10) to $\mathbb{R}^2$, and seeing that the restriction $h_1,\dots,h_r$ of each holomorphic function $H_1(w_1),\dots,H_r(w_1)$ is (real) analytic in $\widetilde{\Omega}_1\cap\mathbb{R}$, we find the announced result (2.9) by putting $H_j|_{\mathbb{R}^2}=h_j$, $H|_{\mathbb{R}^2}=h$, and $(\widetilde{\Omega})'\cap\mathbb{R}^2=U\subset\Omega$. Note also that the restriction h is analytic in U. □

Some concrete examples and physical interpretations of the above results will be discussed in a further paper; nevertheless, the determination of the h_j 's rests heavily upon an identification process and a residue formula. These functions stand for the analytic coefficients of what we will call the Weierstrass polynomial associated to the velocity of the flow in a neighborhood of a vanishing point.

Following Lelong's method who introduced the transformation T in 1954 (which was useful for constructing the harmonicity cells defined by Aronszajn in 1936), it seems advisable now that an analogue T_∞ of T must be precise in order to give explicitly some infinite-harmonicity cells.

References

[1] G. Aronsson, P. Lindqvist; *On p-harmonic functions in the plane and their stream functions*, J. Differential Equations 74.1, 157-178 (1988).

[2] G. Aronsson; *On certain singular solutions of the partial differential equation* $u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0$, manuscripta math. 47, p. 133-151 (1984).

[3] N. Aronszajn; *Sur les décompositions des fonctions analytiques uniformes et sur leurs applications*, Acta. math. 65 1-156 (1935).

[4] V. Avanissian; *Cellule d'harmonicité et prolongement analytique complexe*, Travaux en cours, Hermann, Paris (1985).

[5] L. Bers; *Mathematical Aspects of Subsonic and Transonic Gaz Dynamics.* Surveys in Applied Mathematics III. Wiley New-York (1958).

[6] M. Boutaleb; *Sur la cellule d'harmonicité de la boule unité de* $\mathbb{R}^n$, Publications de L'I.R.M.A. Doctorat de 3^0cycle, U.L.P. Strasbourg France (1983).

[7] M. Boutaleb; *A polyharmonic analogue of a Lelong theorem and polyhedric harmonicity cells*, Electron. J. Diff. Eqns, Conf. 09 (2002), pp. 77-92.

[8] M. Boutaleb; *Généralisation à* $\mathbb{C}^n$ *d'un théorème de M. Jarnicki sur les cellules d'harmonicité*, à paraître dans un volume of the Bulletin of Belgian Mathematical Society Simon Stevin (2003).

[9] M. Boutaleb; *On the holomorphic extension for p-harmonic functions in plane domains*, submitted for publication (2003).

[10] M. Boutaleb; *On polyhedric, and topologically homeomorphic, harmonicity cells in* $\mathbb{C}^n$, submitted for publication (2003).

[11] R. Coquereaux , A. Jadczyk; *Conformal Theories, Curved phase spaces, Relativistic wavelets and the Geometry of complex domains*, Centre de physique théorique, Section 2, Case 907. Luminy, 13288. Marseille. France.(1990).

[12] P. Ebenfelt; *Holomorphic extension of solutions of elliptic partial differential equations and a complex Huygens' principle*, Trita Math 0023, Royal Institute of Technology, S-100 44 Stockholm, Sweden (1994).

[13] M. Hervé; *Les fonctions analytiques Presses Universitaires de France*, 1982.

[14] M. Jarnicki; *Analytic Continuation of harmonic functions*, Zesz. Nauk.U J, Pr. Mat 17, 93-104 (1975).

[15] C. O. Kiselman; *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France 97 (4) 328-356 (1969).

[16] P. Lelong; *Prolongement analytique et singularités complexes des fonctions harmoniques*, Bull. Soc. Math. Belg.710-23 (1954-55).

[17] J. L. Lewis; *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. 32, 849-858 (1983).

[18] Z. Moudam; *Existence et Régularité des solutions du p-Laplacien avac poids dans* $\mathbb{R}^n$, D. E. S. Univ. S. M. Ben Abdellah, Faculté des Sciences Dhar Mehraz (1996).

[19] E. Onofri; *SO(n,2)- Singular orbits and their quantization.* Colloques Internationaux C.N.R.S. No. 237. Géométrie symplectique et Physique mathématique. Instituto di Fisica de l'Universita. Parma. Italia (1976).

[20] M. Pauri; *Invariant Localization and Mass-spin relations in the Hamiltonian formulation of Classical Relativistic Dynamics*, University of Parma, IFPR-T 019, (1971).

[21] W. Rudin; *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag New -York Heidelberg Berlin (1980).

[22] J. Siciak; *Holomorphic continuation of harmonic functions*, Ann. Pol. Math. XXIX 67-73 (1974).

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