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On the average value for nonconstant eigenfunctions of the p-Laplacian assuming Neumann boundary data *

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Abstract

We show that nonconstant eigenfunctions of the p-Laplacian do not necessarily have an average value of 0, as they must when $p = 2$. This fact has implications for deriving a sharp variational characterization of the second eigenvalue for a general class of nonlinear eigenvalue problems.

1 Introduction

In this paper we show that the nonconstant solutions of

$$\begin{gathered} -\Delta_p u - \lambda |u|^{p-2} u = 0 \quad \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

do not necessarily satisfy $\int_\Omega u = 0$. This fact has implications for deriving a sharp variational characterization of the second eigenvalue for a broad class of nonlinear eigenvalue problems including (1.1). We assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, λ is a real number, and Δ_p is the p-Laplacian, i.e. $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, for some $p \in (1, \infty)$.

In some respects (1.1) is already well understood. Since Neumann boundary conditions are assumed, it is straightforward to see that the principle eigenvalue is $\lambda_1 = 0$ with simple eigenspace $W := span\{1\}$. Recent work in [2], [3], [4], and [6] has provided a detailed description of the second eigenvalue, λ_2, which is defined as the smallest real number greater than λ_1 such that (1.1) has a nontrivial solution. In particular, it is known that $\lambda_2 > 0$, and that eigenfunctions associated with λ_2 are sign-changing with exactly two nodal domains and are in the set $V_p := \{v \in W^{1,p}(\Omega) : \int_\Omega |v|^{p-2} v = 0\}$. Also, λ_2 satisfies variational characterizations that generalize from the linear case in a natural way. We should point out that the references above impose Dirichlet boundary

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conditions, but provide a framework that works just as well for (1.1). In section 2 we will provide a sketch of how some of these facts can be proved.

There are several situations where it is straightforward to see that second eigenfunctions have an average value of 0. Of course, if $p = 2$, then (1.1) reduces to the standard eigenvalue problem for the Laplace operator with Neumann boundary data, and it is clear that every nonconstant eigenfunction lies in $V_2 = W^{\perp} = \{u \in W^{1,2}(\Omega) : \int_\Omega u = 0\}$. For arbitrary p, if we examine the ODE case, then it is possible to exploit the symmetry of $\Omega = (a, b)$ to prove that nonconstant eigenfunctions once again satisfy $\int_a^b u = 0$. This ODE argument can be extended to eigenfunctions on "boxes" in $\mathbb{R}^N$ with $N > 1$, i.e. $\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N)$. But what of the average value for second eigenfunctions over more general domains?

This question arose while studying eigenvalue problems for a class of quasilinear operators that generalize the p-Laplacian, i.e.

$$Q(u) := \sum_{1 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_m'(u)),$$

where Q is a $2m$-th order quasilinear operator satisfying general growth, ellipticity and monotonicity conditions. For boundary value problems associated with such operators some interesting existence theorems have been proved by Shapiro, et.al., where a *second eigenvalue* is defined and used as an upper bound in certain key growth estimates. This second eigenvalue is obtained via the minimization of an appropriate functional, essentially a Raleigh quotient, over the space $V_2 = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$. (More details are provided in section 2 and in the references [7] and [9].) This allows something like an orthogonal splitting of the Banach Space $W^{1,p}(\Omega)$ so that saddle point theorems can be applied in a standard way. An open question that arose as a result of these papers was whether or not this orthogonal splitting leads to a sharp characterization of the second eigenvalue. Our main result in this paper shows that it does not. It follows that an improved characterization should lead to an improvement of the existence results in the papers listed above. These improved existence theorems are described in subsequent work.

2 Preliminaries

We begin with a standard variational formulation of the problem, and briefly present some straightforward properties and definitions. Details can be checked in the references. Let $W^{1,p}(\Omega)$ be defined in the usual way, as in [1]. Let

$$E(u) := \int_\Omega |\nabla u|^p, \text{ for } u \in W^{1,p}(\Omega).$$

It is well known that E is a C^1 functional with

$$E'(u) \cdot v = p \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v.$$

Moreover, if we consider E constrained to the surface $\mathcal{S} := \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$, then any critical point, ϕ, satisfies

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla v = \lambda\int_\Omega |\phi|^{p-2}\phi v \tag{2.1}$$

for some $\lambda \in \mathbb{R}$ and all $v \in W^{1,p}(\Omega)$. Hence, the critical points of the constrained functional correspond to eigenfunctions, and the associated Lagrange multipliers correspond to eigenvalues. (Substitute $v = \phi$ into (2.1) to see that $\lambda = E(\phi)$.) Notice that by constraining the functional to the L^p unit sphere we are simply recognizing that all nontrivial eigenfunctions can be rescaled so that they are elements of $\mathcal{S}$.

E clearly attains a global minimum of 0 at $\pm\phi_1 = \pm(\frac{1}{|\Omega|})^{\frac{1}{p}}$. Also, it is clear that $E(u) > 0$ for any nonconstant u. Thus $\lambda_1 = 0$ is a simple eigenvalue with eigenspace $W := span\{1\}$.

If $\lambda > 0$ is an eigenvalue with associated eigenfunction ϕ, then we can substitute $v = 1$ into (2.1) to see that $\phi \in V_p$. Hence our search for critical points can be restricted to the set $V_p \bigcap \mathcal{S}$. Members of this set are clearly sign-changing. Using the fact that $V_p \bigcap \mathcal{S}$ is weakly closed, and that E is bounded below and weakly lower semicontinuous, we see that E attains its positive infimum on $\mathcal{S}$. Hence, one variational characterization of λ_2 is

$$\lambda_2 := \inf_{\mathcal{S}\cap V_p} E. \tag{2.2}$$

Let ϕ_2 represent an associated eigenfunction and consider the curve

$$h : \mathbb{R} \to \mathcal{S} : h(t) = \frac{\phi_2 + t}{||\phi_2 + t||_{L^p}}.$$

Then

$$E(h(t)) = \frac{\int_\Omega |\nabla\phi_2|^p}{\int_\Omega |\phi_2 + t|^p}, \text{ and } \frac{d}{dt}E(h(t)) = \frac{-p\int_\Omega |\nabla\phi_2|^p \int_\Omega |\phi_2 + t|^{p-2}(\phi_2 + t)}{\left(\int_\Omega |\phi_2 + t|^p\right)^2}.$$

Thus $E(h(t))$ reaches a global maximum of λ_2 only at $t = 0$. Moreover, $\lim_{t\to\pm\infty} h(t) = \pm\phi_1$ and $\lim_{t\to\pm\infty} E(h(t)) = 0$. Thus $h(t)$ can be modified to create a continuous curve $\gamma : [-1, 1] \to \mathcal{S}$ such that $\gamma(\pm 1) = \pm\phi_1$, $\gamma(0) = \phi_2$, and such that $E(\gamma(t))$ achieves a maximum value of λ_2 precisely when $t = 0$. Conversely, any continuous curve on $\mathcal{S}$ connecting $\pm\phi_1$ must cross V_p and hence must contain a point, $\gamma(t)$, where $E(\gamma(t)) \geq \lambda_2$. Thus we deduce a second, equivalent, variational characterization of λ_2 which is

$$\lambda_2 := \inf_{\gamma\in\Gamma} \sup_{-1\leq t\leq 1} E(\gamma(t)), \tag{2.3}$$

where $\Gamma := \{\gamma : [-1, 1] \to \mathcal{S} : \gamma \text{ is continuous}, \gamma(\pm 1) = \pm\phi_1\}$. The proof that ϕ_2 has exactly 2 nodal domains relies on the fact that if ϕ_2 has more than 2 nodal domains then a curve can be constructed that contradicts (2.3) . Details can be found in [4] or [6].

Let μ_2 represent the parameter characterized in [9] and [7]. For homogeneous problems, such as (1.1), this reduces to

$$\mu_2 := \inf_{\mathcal{S} \cap V_2} E. \tag{2.4}$$

If we compare the characterization (2.3) with (2.4), we observe that every curve in Γ must cross at least one point in V_2, and thus the maximum value of E over any such curve is at least as large as μ_2. It follows that $\mu_2 \leq \lambda_2$. Now suppose that we can show that $\phi_2 \notin V_2$. If we examine the special curve γ, constructed above, we see that γ crosses V_2 at a point $\gamma(t) \neq \phi_2$, so $E(\gamma(t)) < \lambda_2$, and thus $\mu_2 < \lambda_2$. This would show that (2.4) is not a sharp characterization of λ_2. In section 3 we will prove that $\phi_2 \notin V_2$ for certain asymmetric domains. An interesting open question might be to classify the domains where $\mu_2 = \lambda_2$, and it is reasonable to conjecture that this depends upon a symmetry condition.

It is important to note that the quasilinear operators in [7] and [9] are not assumed to be homogeneous, so the associated eigenvalue problems could not be restricted to $\mathcal{S}$. Hence, the more general characterization had to consider the infimum of $\frac{E(u)}{\int_\Omega |u|^p}$ over $V_2 \bigcap r\mathcal{S}$ and then compute a lim inf as $r \to \infty$.

3 Comparing λ_2 and μ_2

Theorem 3.1 *There is at least one domain $\Omega \subset \mathbb{R}^N$ such that the associated second eigenvalue, λ_2, has an associated eigenfunction, ϕ_2, that does not lie in V_2.*

Proof Consider the problem

$$\begin{gathered} -\Delta_p u - \lambda_2^\epsilon |u|^{p-2} u = 0 \quad \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\epsilon, \end{gathered} \tag{3.1}$$

where $\Omega_\epsilon := ((0,2) \times (0,2)) \bigcup ((2,3) \times (0,\epsilon)) \bigcup ((3,4) \times (0,1))$ for $0 \leq \epsilon \leq 1$, and where λ_2^ϵ is characterized by (2.2) and (2.3). Ω_0 will refer to the limiting case which is simply the union of the two disjoint rectangles. Let $\phi_{2,\epsilon} \in V_p \bigcap \mathcal{S}$ represent an associated second eigenfunction. When $\epsilon = 0$ this will simply indicate a function that is a positive constant over one rectangle and a negative constant over the other, where the constants are balanced to fit the constraints.

First, we find an upper bound for λ_2^ϵ. Let

$$u_2 := \begin{cases} 1 & \text{for } (x,y) \in [0,2] \times [0,2], \\ -2x + 5 & \text{for } (x,y) \in [2,3] \times [0,\epsilon], \\ -1 & \text{for } (x,y) \in [3,4] \times [0,1] \end{cases}$$

Also, let $\gamma(\alpha,\beta) = \alpha u_2^+ - \beta u_2^-$, where $u_2^+ := \max\{u_2, 0\}$, $u_2^- := \max\{-u_2, 0\}$, and α and β are nonnegative scalars such that $\alpha^p ||u_2^+||^p_{L^p} + \beta^p ||u_2^-||^p_{L^p} = 1$. Notice that γ is a curve on $\mathcal{S}$ connecting the points $\frac{u_2^+}{||u_2^+||_{L^p}}$ and $-\frac{u_2^-}{||u_2^-||_{L^p}}$. By the Intermediate Value Theorem γ crosses the surface V_p. Hence the maximum of $E(\gamma(\alpha,\beta))$ must be greater than λ_2^ϵ. However,

$$\nabla\gamma(\alpha,\beta) = \begin{cases} (0,0) & \text{for } (x,y) \in [0,2] \times [0,2], \\ (-2\alpha, 0) & \text{for } (x,y) \in [2, \frac{5}{2}] \times [0,\epsilon], \\ (-2\beta, 0) & \text{for } (x,y) \in (\frac{5}{2}, 3] \times [0,\epsilon], \\ (0,0) & \text{for } (x,y) \in (3,4] \times [0,1] \end{cases}$$

Thus $\int_{\Omega_\epsilon} |\nabla\gamma(\alpha,\beta)|^p \leq 2^p \max\{\alpha^p, \beta^p\}\epsilon$. But $||u_2^+||^p_{L^p} \geq 4$ and $||u_2^-||^p_{L^p} \geq 1$, so $\alpha^p \leq \frac{1}{4}$ and $\beta^p \leq 1$. Therefore $\int_{\Omega_\epsilon} |\nabla\gamma(\alpha,\beta)|^p \leq 2^p\epsilon$. It follows that $\lambda_2^\epsilon \leq \max_{(\alpha,\beta)} E(\gamma(\alpha,\beta)) \leq 2^p\epsilon$, so $\lim_{\epsilon\to 0} \lambda_2^\epsilon = 0$.

We will now show that $\int_{\Omega_\epsilon} \phi_{2,\epsilon} \neq 0$ for some ϵ. Since $\lambda_2^\epsilon \to 0$, straightforward estimates now show that $\phi_{2,\epsilon} \to \phi_{2,0}$ in $W^{1,p}(\Omega_0)$, where $\nabla\phi_{2,0} \equiv 0$ and $\int_{\Omega_0} |\phi_{2,0}|^p = 1$. Moreover, $\int_{\Omega_0} |\phi_{2,0}|^{p-2}\phi_{2,0} = \lim_{\epsilon\to 0} \int_{\Omega_\epsilon} |\phi_{2,\epsilon}|^{p-2}\phi_{2,\epsilon} = 0$. It must be that there are constants $a, b \in \mathbb{R}$ such that $\phi_{2,0} \equiv a$ in $[0,2] \times [0,2]$ and $\phi_{2,0} \equiv b$ in $[3,4] \times [0,1]$. Moreover, it follows that $4|a|^p + |b|^p = 1$, a and b have opposite signs, and $4|a|^{p-1} - |b|^{p-1} = 0$. Thus $|b| = 4^{\frac{1}{p-1}}|a|$. It can now be checked that $\int_{\Omega_0} \phi_{2,0} = \pm(4 - 4^{\frac{1}{p-1}})|a| \neq 0$ for $p \neq 2$. Hence $\int_{\Omega_\epsilon} \phi_{2,\epsilon} \neq 0$ for some $\epsilon > 0$. ☐

As an immediate consequence we have the following statement.

Corollary 3.2 *If Ω is the domain given in Theorem 3.1, then $\mu_2 < \lambda_2$.*

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