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A wavelet Galerkin method applied to partial differential equations with variable coefficients *

José Roberto Linhares de Mattos & Ernesto Prado Lopes

Abstract

We consider the problem $K(x)u_{xx} = u_t$, $0 < x < 1$, $t \geq 0$, where $K(x)$ is bounded below by a positive constant. The solution on the boundary $x = 0$ is a known function g and $u_x(0,t) = 0$. This is an ill-posed problem in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists. We consider the existence of a solution $u(x,\cdot) \in L^2(R)$ and we use a wavelet Galerkin method with the Meyer multi-resolution analysis, to filter away the high-frequencies and to obtain well-posed approximating problems in the scaling spaces V_j. We also derive an estimate for the difference between the exact solution of the problem and the orthogonal projection, onto V_j, of the solution of the approximating problem defined in V_{j-1}.

1 Introduction

In this paper, we consider the following problem, for $0 < \alpha \leq K(x) < +\infty$,

$$\begin{gathered} K(x)u_{xx}(x,t) = u_t(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) = g, \quad u_x(0,\cdot) = 0 \end{gathered} \tag{1.1}$$

We assume that this problem has a solution $u(x,\cdot) \in L^2(R)$, for K continuous, and we extend $u(x,t)$ and g to R assuming that both vanish for $t < 0$.

Problem (1.1) is ill-posed in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists. This means that if the solution exists, it does not depend continuously on g (see note 1 below).

We consider the Meyer multi-resolution analysis. The advantage of using this method is that it has good localization in the frequency domain, since its Fourier transform has compact support. The orthogonal projection onto Meyer scaling spaces, can be considered as a low pass filter, cutting off the high frequencies. We get a version of the Gronwall inequality that we use to obtain an estimate for the frequency of the solution of the problem (1.1).

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¿From the variational formulation of the approximating problem on the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. An estimate obtained for the solution of this evolution problem, is used to get the stability of the wavelet Galerkin method. Using an estimate obtained for the difference between the exact solution of the problem (1.1) and its orthogonal projection onto V_j, we get an estimate for the difference between the exact solution of the problem (1.1) and the orthogonal projection, onto V_j, of the solution of the approximating problem defined on the scaling space V_{j-1}.

Our approach is similar to the one used in [4] for the sideway heat equation. The problem considered in [4] is an inverse problem for the heat equation with constant coefficient. There the variational formulation, on the scaling space V_j, of the approximating problem, produces an infinite-dimensional system of second order ordinary differential equations with constant coefficients, for which the solution is known. Stability and convergence of the method follows from form of this solution.

In section 2, we construct the Meyer multi-resolution analysis. In section 3, we get the estimates of the numerical stability and the convergence of the wavelet Galerkin method.

For a function $h \in L^1(R) \bigcap L^2(R)$ its Fourier Transform is given by $\widehat{h}(\xi) := \int_{\mathbb{R}} h(x)e^{-ix\xi}dx$.

2 Meyer multi-resolution analysis

To construct a wavelet basis from a mother wavelet, we need an structure in $L^2(\mathbb{R})$ which allows us to decompose $L^2(\mathbb{R})$ in a direct sum of mutually orthogonal spaces.

Definition A *multi-resolution analysis* is a sequence of closed subspaces V_j in $L^2(\mathbb{R})$, called *scaling spaces*, satisfying:

(M1) $V_j \subseteq V_{j-1}$ for all $j \in \mathbb{Z}$

(M2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(M4) $f \in V_j$ if and only if $f(2^j\cdot) \in V_0$

(M5) $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$

(M6) There exists $\phi \in V_0$ such that $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is an orthonormal basis in V_0, where $\phi_{j,k}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ for all $j, k \in \mathbb{Z}$. The function ϕ is called the *scaling function* of the multi-resolution analysis.

Remarks 1) M4 and M6 imply $\{\phi_{j,k} : k \in \mathbb{Z}\}$ being an orthonormal basis for the space V_j.
2) Let $\phi \in L^2(\mathbb{R})$ and $V_j = \overline{\operatorname{span}\{\phi_{jk}\}}_{k\in\mathbb{Z}}$ where $\phi_{jk}(t) := 2^{-j/2}\phi(2^{-j}t-k)$ and $j \in \mathbb{Z}$. Thus, $V_0 = \overline{\operatorname{span}\{\phi(\cdot - k)\}}_{k\in\mathbb{Z}}$. We have that V_j satisfy M1 if only if $\phi \in V_{-1}$, that is, if only if there exists a 2π-periodic square integrable function m_0, such that

$$\widehat{\phi}(\xi) = m_0(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}).$$

The Meyer multi-resolution analysis is constructed in the following way: Let φ be the scaling function defined by its Fourier transform by

$$\widehat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 2\pi/3 \\ \cos[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\xi|-1)] & \text{if } 2\pi/3 \le |\xi| \le 4\pi/3 \\ 0, & \text{otherwise,} \end{cases}$$

where ν is a differentiable function satisfying

$$\nu(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases} \tag{2.1}$$

$$\nu(x) + \nu(1-x) = 1 \tag{2.2}$$

¿From (2.2), it follows that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1$, which is equivalent to the orthonormality of $\varphi(\cdot - k)$, $k \in \mathbb{Z}$. Then M6 is satisfied. Here m_0 can be constructed on $[0, 2\pi]$, from $\widehat{\varphi}$, by

$$m_0(\xi) = \sum_{l\in\mathbb{Z}} \widehat{\varphi}(2(\xi + 2\pi l))$$

This function is 2π-periodic, square integrable, and, for $\xi \in [0, 2\pi]$,

$$m_0(\frac{\xi}{2})\widehat{\varphi}(\frac{\xi}{2}) = \sum_{l\in\mathbb{Z}} \widehat{\varphi}(\xi + 4\pi l)\widehat{\varphi}(\frac{\xi}{2}) = \widehat{\varphi}(\xi)\widehat{\varphi}(\frac{\xi}{2}) = \widehat{\varphi}(\xi)$$

The second equality above follows from

$$\widehat{\varphi}(\xi + 4\pi l)\widehat{\varphi}(\frac{\xi}{2}) = 0, \quad \forall l \ne 0$$

and the third equality follows from $\widehat{\varphi}(\xi/2) = 1$ for all $\xi \in \operatorname{supp}\widehat{\varphi}$. Then M1 is satisfied and the other conditions of the definition can also be proved. The associated mother wavelet is given by (see [2])

$$\begin{aligned} \widehat{\psi}(\xi) &= e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\widehat{\varphi}(\xi/2) \\ &= e^{i\xi/2}\sum_{l\in\mathbb{Z}} \widehat{\varphi}(\xi + 2\pi(2l+1))\widehat{\varphi}(\xi/2) \\ &= e^{i\xi/2}[\widehat{\varphi}(\xi + 2\pi) + \widehat{\varphi}(\xi - 2\pi)]\widehat{\varphi}(\xi/2) \end{aligned}$$

or equivalently,

$$\widehat{\psi}(\xi)=\begin{cases} e^{i\xi/2}\sin[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\xi|-1)], & \text{if } \frac{2\pi}{3}\le|\xi|\le\frac{4\pi}{3}\\ e^{i\xi/2}\cos[\frac{\pi}{2}\nu(\frac{3}{4\pi}|\xi|-1)], & \text{if } \frac{4\pi}{3}\le|\xi|\le\frac{8\pi}{3}\\ 0, & \text{otherwise.}\end{cases}$$

The function ψ is the Meyer wavelet.

Now, we consider the Meyer multi-resolution analysis. We have

$$\begin{aligned}\widehat{\psi_{jk}}(\xi) &= \int_{\mathbb{R}}\psi_{jk}(x)e^{-ix\xi}dx\\ &= \int_{\mathbb{R}}2^{-\frac{j}{2}}\psi(2^{-j}x-k)e^{-ix\xi}dx\\ &= \int_{\mathbb{R}}2^{j/2}\psi(y-k)e^{-i2^jy\xi}dy\\ &= 2^{j/2}\int_{\mathbb{R}}\psi(t)e^{-i2^j(t+k)\xi}dt\\ &= 2^{j/2}\int_{\mathbb{R}}\psi(t)e^{-i2^jt\xi-i2^jk\xi}dt = 2^{j/2}e^{-i2^jk\xi}\widehat{\psi}(2^j\xi)\end{aligned}$$

Since $\operatorname{supp}(\widehat{\psi})=\{\xi:\frac{2}{3}\pi\le|\xi|\le\frac{8}{3}\pi\}$ we have that

$$\operatorname{supp}(\widehat{\psi_{jk}})=\{\xi;\frac{2}{3}\pi2^{-j}\le|\xi|\le\frac{8}{3}\pi2^{-j}\}\quad\forall k\in\mathbb{Z}\tag{2.3}$$

Furthermore,

$$\operatorname{supp}(\widehat{\varphi_{jk}})=\{\xi;|\xi|\le\frac{4}{3}\pi2^{-j}\}\quad\forall k\in\mathbb{Z}\tag{2.4}$$

Now we consider the orthogonal projection onto V_j, $P_j:L^2(R)\to V_j$,

$$P_jf(t)=\sum_{k\in\mathbb{Z}}\langle f,\varphi_{jk}\rangle\varphi_{jk}(t)$$

The hypothesis M1 and M2 imply that $\lim_{j\to-\infty}P_jf=f$, for all $f\in L^2(R)$. This means that from a representation of f in a given scale, we can get f by adding details which are given at higher frequencies. ¿From (2.4), we see that P_j filters away the frequencies higher than $\frac{4}{3}\pi2^{-j}$ (low pass filter).

We have, for all $f\in L^2(R)$,

$$\begin{aligned}f &= P_jf-P_jf+f\\ &= P_jf+(I-P_j)f\\ &= \sum_{k\in\mathbb{Z}}\langle f,\varphi_{jk}\rangle\varphi_{jk}+\sum_{l\le j}\sum_{k\in\mathbb{Z}}\langle f,\psi_{lk}\rangle\psi_{lk}\end{aligned}$$

This implies

$$\widehat{P_jf}(\xi)=\widehat{f}(\xi)\quad\text{for } |\xi|\le\frac{2}{3}\pi2^{-j}\tag{2.5}$$

since, by (2.3), $\widehat{\psi}_{lk}(\xi) = 0$ for all $l \leq j$ and $|\xi| \leq \frac{2}{3}\pi 2^{-j}$.

Considering the corresponding orthogonal projections in the frequency space, $\widehat{P_j} : L^2(R) \to \widehat{V_j} = \overline{\mathrm{span}\{\widehat{\varphi_{jk}}\}}_{k\in\mathbb{Z}}$,

$$\widehat{P_j} f = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \langle f, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}}$$

we have

$$\widehat{P_j} \widehat{f} = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \langle \widehat{f}, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}} = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{jk} \rangle \widehat{\varphi_{jk}} = \widehat{P_j f}$$

Then (2.5) implies that

$$\begin{aligned} \|(I - P_j)f\| &= \frac{1}{\sqrt{2\pi}} \|[(I - P_j)f]^\wedge\| = \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P_j})\widehat{f}\| \\ &= \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P_j})\chi_j \widehat{f}\| \leq \|\chi_j \widehat{f}\| \end{aligned} \tag{2.6}$$

where χ_j is the characteristic function in $(-\infty, -\frac{2}{3}\pi 2^{-j}] \cup [\frac{2}{3}\pi 2^{-j}, +\infty)$.

3 Results of Stability and Convergence

Hereafter, the multi-resolution analysis considered corresponds to the Meyer multi-resolution analysis with scaling function φ. The next lemma is a version of the Gronwall inequality.

Lemma 3.1 *Let u and v be positive continuous functions, $x \geq a$ and $c > 0$. If $u(x) \leq c + \int_a^x \int_a^s v(\tau)u(\tau)\, d\tau ds$ then*

$$u(x) \leq c \exp\Big(\int_a^x \int_a^s v(\tau)\, d\tau ds \Big) .$$

Proof. Let $w(x) = c + \int_a^x \int_a^s v(\tau)u(\tau)\, d\tau ds$. Then $w'(x) = \int_a^x v(\tau)u(\tau)\, d\tau$. Therefore,

$$w''(x) = v(x)u(x) \leq v(x)w(x) \quad \text{and} \quad \frac{w''(x)}{w(x)} \leq v(x)$$

Now

$$\frac{w''(x)}{w(x)} = (\frac{w'}{w})'(x) + (\frac{w'(x)}{w(x)})^2$$

Thus $\big(\frac{w'}{w}\big)'(x) \leq v(x)$ which implies

$$\frac{w'(x)}{w(x)} \leq \int_a^x v(\tau)\, d\tau \quad \text{and} \quad (\ln w(x))' \leq \int_a^x v(\tau)\, d\tau;$$

Therefore,

$$\ln w(x) - \ln w(a) \leq \int_a^x \int_a^s v(\tau)\, d\tau ds$$

Since $w(a) = c$, $\ln w(x) - \ln\ c \leq \int_a^x \int_a^s v(\tau)\, d\tau ds$, which implies

$$\ln \frac{w(x)}{c} \leq \int_a^x \int_a^s v(\tau)\, d\tau ds \quad \text{and} \quad w(x) \leq c \exp\Big(\int_a^x \int_a^s v(\tau)\, d\tau ds\Big).$$

Since, by hypothesis, $u(x) \leq w(x)$, we have

$$u(x) \leq c \exp\Big(\int_a^x \int_a^s v(\tau)\, d\tau ds\Big)$$

which completes the proof. □

Applying the Fourier Transform with respect to time in Problem (1.1), we obtain the following problem in the frequency space:

$$\begin{gathered} \widehat{u}_{xx}(x,\xi) = \frac{i\xi}{K(x)}\widehat{u}(x,\xi), \quad 0 < x < 1,\ \xi \in R \\ \widehat{u}(0,\xi) = \widehat{g}(\xi), \quad \widehat{u}_x(0,\cdot) = 0 \end{gathered}$$

whose solution satisfies

$$\widehat{u}(x,\xi) = \widehat{g}(\xi) + \int_0^x \int_0^s \frac{i\xi}{K(\tau)}\widehat{u}(\tau,\xi)\, d\tau ds$$

Then, from lemma 3.1, for $\widehat{g}(\xi) \neq 0$, we have

$$|\widehat{u}(x,\xi)| \leq |\widehat{g}(\xi)| \exp\Big[|\xi| \int_0^x \int_0^s \frac{1}{K(\tau)}\, d\tau ds\Big] \tag{3.1}$$

The next lemma corresponds to proposition 3.1 in [4], when $K(x)$ is constant.

Lemma 3.2 *The operator $D_j(x)$ defined by*

$$[(D_j)_{lk}(x)]_{l\in\mathbb{Z},\ k\in\mathbb{Z}} = \Big[\frac{1}{K(x)}\langle\varphi'_{jl}, \varphi_{jk}\rangle\Big]_{l\in\mathbb{Z}, k\in\mathbb{Z}}$$

satisfies the following three conditions: 1) $(D_j)_{lk}(x) = -(D_j)_{kl}(x)$
2) $(D_j)_{lk}(x) = (D_j)_{(l-k)0}(x)$. Hence, $(D_j)_{lk}(x)$ are equal along diagonals.
3) $\|D_j(x)\| \leq \frac{\pi 2^{-j}}{K(x)}$

Proof. This proof follows quite closely the proof in [4] for $D_j(x)$ independent of x.
1) As we already know, $\widehat{\varphi_{j0}}(\xi) = 2^{j/2}\widehat{\varphi}(2^j\xi)$, $\widehat{\varphi_{jk}}(\xi) = 2^{j/2}e^{-ik\xi 2^j}\widehat{\varphi}(2^j\xi) =$

$e^{-ik\xi 2^j}\widehat{\varphi_{j0}}(\xi)$, and the Fourier transform of the scaling function φ is even. Then $\widehat{\varphi_{j0}}(\xi)=\widehat{\varphi_{j0}}(-\xi)$ and

$$\begin{aligned}
(D_j)_{lk}(x) &= \frac{1}{K(x)}\langle\varphi'_{jl},\varphi_{jk}\rangle = \frac{1}{K(x)}\frac{1}{2\pi}\langle\widehat{\varphi'_{jl}},\widehat{\varphi_{jk}}\rangle \\
&= \frac{1}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi'_{jl}}(\xi)\overline{\widehat{\varphi_{jk}}}(\xi)\,d\xi \\
&= \frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\xi\widehat{\varphi_{jl}}(\xi)\overline{\widehat{\varphi_{jk}}}(\xi)\,d\xi \\
&= \frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\xi e^{-i(l-k)\xi 2^j}|\widehat{\varphi_{j0}}(\xi)|^2 d\xi
\end{aligned}$$

Then

$$\begin{aligned}
(D_j)_{lk}(x) &= \frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}(-\omega)e^{-i(k-l)\omega 2^j}|\widehat{\varphi_{j0}}(-\omega)|^2 d\omega \\
&= -\frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\omega e^{-i(k-l)\omega 2^j}|\widehat{\varphi_{j0}}(\omega)|^2 d\omega \\
&= -\frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\omega\widehat{\varphi_{jk}}(\omega)\overline{\widehat{\varphi_{jl}}}(\omega)\,d\omega \\
&= -\frac{1}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi'_{jk}}(\omega)\overline{\widehat{\varphi_{jl}}}(\omega)\,d\omega.
\end{aligned}$$

Thus

$$(D_j)_{lk}(x) = -\frac{1}{K(x)}\frac{1}{2\pi}\langle\widehat{\varphi'_{jk}},\widehat{\varphi_{jl}}\rangle = -\frac{1}{K(x)}\langle\varphi'_{jk},\varphi_{jl}\rangle = -(D_j)_{kl}(x)$$

2) As proved above

$$(D_j)_{lk}(x) = \frac{i}{K(x)}\frac{1}{2\pi}\int_{\mathbb{R}}\xi e^{-i(l-k)\xi 2^j}|\widehat{\varphi_{j0}}(\xi)|^2 d\xi = (D_j)_{(l-k)0}(x)$$

Then $(D_j)_{lk}(x)$ is equal along diagonals.
3) We have

$$\|D_j(x)\| = \Big\|\frac{1}{K(x)}B_j\Big\| = \frac{1}{K(x)}\|B_j\|$$

where $(B_j)_{lk}=\langle\varphi'_{jl},\varphi_{jk}\rangle$. ¿From results 1) and 2), we have $(B_j)_{lk}=-(B_j)_{kl}$, $(B_j)_{lk}=\frac{1}{2\pi}\int_{\mathbb{R}}\xi e^{-i(l-k)\xi 2^j}|\widehat{\varphi_{j0}}(\xi)|^2 d\xi=(B_j)_{(l-k)0}$ and $(B_j)_{lk}$ is constant along diagonals. We will show that $\|B_j\|\le\pi 2^{-j}$. Thus, we will have

$$\|D_j(x)\| \le \frac{\pi}{K(x)}2^{-j}$$

For $|t|\le\pi 2^{-j}$,

$$\begin{aligned}
\Gamma_j(t) =& i2^{-j}\big[(t-2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t-2^{-j+1}\pi)|^2 + t|\widehat{\varphi_{j0}}(t)|^2 \\
&+ (t+2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t+2^{-j+1}\pi)|^2\big]
\end{aligned}$$

Extend Γ_j periodically to $\mathbb{R}$ and expand it in Fourier series as

$$\Gamma_j(t) = \sum_{k\in\mathbb{Z}} \gamma_k e^{ikt2^j}$$

We have $\gamma_k = b_k$ for all k, where b_k is the element in diagonal k of B_j. In fact, since $\widehat{\varphi_{j0}}(t) = 0$ for $|t| \geq \frac{4}{3}\pi 2^{-j}$, it follows that

$$\begin{aligned}
\gamma_k &= \frac{1}{2^{-j+1}\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} \Gamma_j(t)e^{-ikt2^j}\,dt \\
&= \frac{i}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t-2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t-2^{-j+1}\pi)|^2 e^{-ikt2^j}\,dt \\
&\quad +\frac{i}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&\quad +\frac{i}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t+2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t+2^{-j+1}\pi)|^2 e^{-ikt2^j}\,dt
\end{aligned}$$

Making a change of variable, we obtain:

$$\begin{aligned}
\gamma_k &= \frac{i}{2\pi}\int_{-3\pi 2^{-j}}^{-\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt + \frac{i}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&\quad +\frac{i}{2\pi}\int_{\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&= \frac{i}{2\pi}\int_{-3\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&= \frac{i}{2\pi}\int_{\mathbb{R}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt = b_k
\end{aligned}$$

Now, $\|B_j\| = \sup_{\|f\|=1}\|B_j f\|$ where $\|f\|^2 = \sum_{k\in\mathbb{Z}}|f_k|^2$. Let $F(t) = \sum_{k\in\mathbb{Z}} f_k e^{ikt2^j}$ and define $W(t) = \Gamma_j(t)F(t)$. We have

$$W(t) = \sum_{k\in\mathbb{Z}} \omega_k e^{ikt2^j} \quad \text{and} \quad \omega_k = \sum_{l\in\mathbb{Z}} b_{k-l}f_l = (B_j f)_k$$

Hence

$$\begin{aligned}
\|\omega\|^2 = \sum_{k\in\mathbb{Z}}|\omega_k|^2 &= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |W(t)|^2 dt \\
&= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |\Gamma_j(t)F(t)|^2 dt \\
&\leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |F(t)|^2 dt \\
&= \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2\|f\|^2
\end{aligned}$$

Then

$$\|B_j\| \leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2$$

On the other hand, Γ_j is an odd function. Hence

$$\sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 = \sup_{0\leq t\leq \pi 2^{-j}} |\Gamma_j(t)|^2$$

But, for $0 \leq t \leq \pi 2^{-j}$, we have $t + \pi 2^{-j+1} \geq \pi 2^{-j+1}$ and $t - \pi 2^{-j+1} \leq 0$. Hence

$$\widehat{\varphi_{j0}}(t + \pi 2^{-j+1}) = 0 \quad \text{and} \quad (t - \pi 2^{-j+1})|\widehat{\varphi_{j0}}(t - \pi 2^{-j+1})|^2 \leq 0$$

for $t \in [0, \pi 2^{-j}]$. Thus

$$\begin{aligned}
\sup_{0\leq t\leq \pi 2^{-j}} |\Gamma_j(t)|^2 &\leq \pi 2^{-j+1} \sup_{0\leq t\leq \pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 \\
&= \pi 2^{-j+1} \sup_{0\leq t\leq \pi 2^{-j}} (t2^j)|\widehat{\varphi}(2^j t)|^2 \\
&= \pi 2^{-j+1} \sup_{0\leq s\leq \pi} s|\widehat{\varphi}(s)|^2
\end{aligned}$$

By definition of $\widehat{\varphi}$ we have $|\widehat{\varphi}(s)|^2 \leq \frac{1}{2\pi}$ and therefore $s|\widehat{\varphi}(s)|^2 \leq \frac{\pi}{2\pi} = \frac{1}{2}$ for $0 \leq s \leq \pi$. Then

$$\sup_{0\leq t\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \leq \sup_{0\leq s\leq \pi} s|\widehat{\varphi}(s)|^2 \leq \frac{\pi 2^{-j+1}}{2} = \pi 2^{-j}$$

Thus

$$\|D_j(x)\| = \frac{1}{K(x)}\|B_j\| \leq \frac{1}{K(x)} \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \leq \frac{\pi 2^{-j}}{K(x)}$$

which completes the proof of lemma 3.2. □

Let us now consider the following approximating problem in V_j, where the projection in the first equation of (3.2) is due to the fact that we can have $\varphi \in V_j$ with $\varphi' \notin V_j$ (see note 2 below),

$$\begin{gathered}
K(x)u_{xx}(x,t) = P_j u_t(x,t), \quad t \geq 0,\ 0 < x < 1 \\
u(0,\cdot) = P_j g \quad u_x(0,\cdot) = 0 \\
u(x,t) \in V_j
\end{gathered} \tag{3.2}$$

Its variational formulation is

$$\begin{gathered}
\langle K(x)u_{xx} - u_t, \varphi_{jk}\rangle = 0 \\
\langle u(0,\cdot), \varphi_{jk}\rangle = \langle P_j g, \varphi_{jk}\rangle, \quad \langle u_x(0,\cdot), \varphi_{jk}\rangle = \langle 0, \varphi_{jk}\rangle, \quad k \in \mathbb{Z}
\end{gathered}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Consider u_j a solution of the approximating problem (3.2), given by $u_j(x,t) =$

$\sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$. Then, we have $(u_j)_t(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi'_{jl}(t)$ and $(u_j)_{xx}(x,t) = \sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t)$. Therefore,

$$K(x)(u_j)_{xx}(x,t) - (u_j)_t(x,t) = K(x)\sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t) - \sum_{l\in\mathbb{Z}} w_l(x)\varphi'_{jl}(t)$$

Hence

$$\begin{aligned}
&\langle K(x)(u_j)_{xx} - (u_j)_t, \varphi_{jk}\rangle = 0\\
&\iff \langle \sum_{l\in\mathbb{Z}} K(x)w''_l\varphi_{jl} - \sum_{l\in\mathbb{Z}} w_l\varphi'_{jl}, \varphi_{jk}\rangle = 0\\
&\iff \sum_{l\in\mathbb{Z}} K(x)w''_l\langle\varphi_{jl},\varphi_{jk}\rangle = \sum_{l\in\mathbb{Z}} w_l\langle\varphi'_{jl},\varphi_{jk}\rangle\\
&\iff K(x)w''_k = \sum_{l\in\mathbb{Z}} w_l\langle\varphi'_{jl},\varphi_{jk}\rangle \quad k\in\mathbb{Z}.
\end{aligned}$$

Therefore,

$$\frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l\frac{1}{K(x)}\langle\varphi'_{jl},\varphi_{jk}\rangle \quad\text{and}\quad \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l(D_j)_{lk}(x)$$

where, as defined before, $(D_j)_{lk}(x) = \frac{1}{K(x)}\langle\varphi'_{jl},\varphi_{jk}\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations

$$\begin{gathered}
\frac{d^2}{dx^2}w = -D_j(x)w\\
w(0) = \gamma, \quad w'(0) = 0
\end{gathered} \tag{3.3}$$

where γ is given by

$$P_jg = \sum_{z\in\mathbb{Z}} \gamma_z\varphi_{jz} = \sum_{z\in\mathbb{Z}}\langle g,\varphi_{jz}\rangle\varphi_{jz}$$

Lemma 3.3 *If w is a solution of the evolution problem of second order (3.3), then*

$$\|w(x)\| \le \|\gamma\|\exp\Big(2^{-j}\pi\int_0^x\int_0^s \frac{1}{K(\tau)}\,d\tau\,ds\Big)$$

Proof Since $w(x) = \gamma + \int_0^x\int_0^s(-D_j)(\tau)w(\tau)\,d\tau ds$,

$$\|w(x)\| \le \|\gamma\| + \int_0^x\int_0^s \|D_j(\tau)\|\|w(\tau)\|\,d\tau\,ds$$

By lemma 3.2 this implies

$$\|w(x)\| \le \|\gamma\| + \int_0^x\int_0^s \frac{2^{-j}\pi}{K(x)}\|w(\tau)\|\,d\tau\,ds.$$

Then by lemma 3.1 we have

$$\|w(x)\| \leq \|\gamma\| \exp\Big(2^{-j}\pi \int_0^x \int_0^s \frac{1}{K(\tau)}\, d\tau ds\Big)$$

which completes the proof. □

Theorem 3.4 (Stability of the wavelet Galerkin method) *Let u_j and v_j be solutions in V_j of the approximating problems (3.2) for the boundary specifications g and $\widetilde{g}$, respectively. If $\|g-\widetilde{g}\| \leq \epsilon$ then*

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon \exp\big(\frac{2^{-j-1}\pi}{\alpha} x^2\big)$$

where α satisfies $0 < \alpha \leq K(x) < +\infty$ as in the definition of the problem (1.1). For j such that $2^{-j} \leq \frac{2\alpha}{\pi} \log \epsilon^{-1}$ we have

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon^{1-x^2}$$

Proof. $u_j(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$, $v_j(x,t) = \sum_{l\in\mathbb{Z}} \widetilde{w}_l(x)\varphi_{jl}(t)$ where w and $\widetilde{w}$ are solutions of the Galerkin problem (3.3) with conditions $w(0) = \gamma$ and $\widetilde{w}(0) = \widetilde{\gamma}$, respectively. So, by lemma 3.3 and linearity of (3.3) we have

$$\begin{aligned}
\|u_j(x,\cdot) - v_j(x,\cdot)\| &= \|w(x) - \widetilde{w}(x)\| \\
&\leq \|\gamma - \widetilde{\gamma}\| \exp(2^{-j}\pi \int_0^x \int_0^s \frac{1}{K(\tau)}\, d\tau ds) \\
&\leq \epsilon \exp(2^{-j}\pi \int_0^x \int_0^s \frac{1}{\alpha}\, d\tau ds) \\
&= \epsilon \exp(2^{-j-1}\frac{\pi}{\alpha} x^2)
\end{aligned}$$

For $j = j(\epsilon)$ such that $2^{-j} \leq \frac{2\alpha}{\pi} \log \epsilon^{-1}$, we have

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon \exp(x^2 \log \epsilon^{-1}) = \epsilon^{1-x^2}$$

which completes the proof. □

Now, we are interested in the solutions $u(x,\cdot) \in L^2(R)$ of problem (1.1), for the functions $g \in L^2(R)$ such that $\widehat{g}(\cdot)\exp(|\cdot|/(2\alpha)) \in L^2(R)$, where $\widehat{g}$ is the Fourier Transform of g. The Inverse Fourier Transform of $\exp(-\frac{\xi^2+|\xi|}{2\alpha})$, for instance, satisfies this condition. Define

$$f := \widehat{g}(\cdot) \exp\big(\frac{|\cdot|}{2\alpha}\big) \in L^2(R) \tag{3.4}$$

Proposition 3.5 *If $u(x,t)$ is a solution of problem (1.1), then*

$$\|u(x,\cdot) - P_j u(x,\cdot)\| \leq \|f\|_{L^2(\mathbb{R})} \exp(-\frac{1}{3}\frac{\pi}{\alpha} 2^{-j}(1-x^2))$$

where f is given by (3.4).

Proof ¿From (2.6) and (3.1), we have

$$\begin{aligned}\|(I-P_j)u(x,\cdot)\| &\le \|\chi_j\widehat{u}(x,\cdot)\|\\ &= [\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{u}(x,\xi)|^2\,d\xi]^{1/2}\\ &\le [\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp[2|\xi|\int_0^x\int_0^s \frac{1}{K(\tau)}\,d\tau\,ds]\,d\xi]^{1/2}\end{aligned}$$

Then

$$\begin{aligned}\|(I-P_j)u(x,\cdot)\| &\le [\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp(|\xi|\frac{x^2}{\alpha})\,d\xi]^{1/2}\\ &\le [\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{|\xi|}{\alpha})\exp(\frac{|\xi|}{\alpha}x^2)d\xi]^{1/2}\\ &= [\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{|\xi|}{\alpha}(1-x^2))\,d\xi]^{1/2}\end{aligned}$$

For $|x|<1$,

$$\begin{aligned}\|(I-P_j)u(x,\cdot)\| &\le [\int_{\mathbb{R}} |f(\xi)|^2\,d\xi]^{1/2}\exp(-\frac{(2/3)\pi 2^{-j}}{2\alpha}(1-x^2))\\ &\le \|f\|_{L^2(\mathbb{R})}\exp(-\frac{1}{3}\frac{\pi}{\alpha}2^{-j}(1-x^2))\end{aligned}$$

which completes the proof. □

Proposition 3.6 *If u is a solution of problem (1.1) and u_{j-1} is a solution of the approximating problem in V_{j-1} then*

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \textit{for } |\xi| \le \frac{4}{3}\pi 2^{-j} \tag{3.5}$$

Consequently,

$$P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot) \tag{3.6}$$

Proof Let $\Lambda(x,\xi) = \widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)$. We will show that $\Lambda(x,\xi) = 0$ for $|\xi| \le \frac{4}{3}\pi 2^{-j}$. Consider the approximating problem in V_{j-1}:

$$\begin{gathered}K(x)(u_{j-1})_{xx} = P_{j-1}(u_{j-1})_t \quad t\in\mathbb{R},\ 0<x<1\\ u_{j-1}(0,\cdot) = P_{j-1}g, \quad (u_{j-1})_x(0,\cdot) = 0\\ u_{j-1}(x,\cdot)\in V_{j-1}\end{gathered}$$

Applying the Fourier transform with respect to time, we have

$$K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) = \widehat{P}_{j-1}[(u_{j-1})_t]\widehat{\ }(x,\xi) = \widehat{P}_{j-1}(i\xi\widehat{u}_{j-1}(x,\xi))$$

for $0 \le x < 1$, $\xi \in \mathbb{R}$, with the conditions: $\widehat{u}_{j-1}(0,\xi) = \widehat{P}_{j-1}\widehat{g}(\xi)$ and $(\widehat{u}_{j-1})_x(0,\cdot) = 0$. Now, by (2.5),

$$\widehat{P}_{j-1}(i\xi\widehat{u}_{j-1}(x,\xi)) = i\xi\widehat{u}_{j-1}(x,\xi) \quad \text{and} \quad \widehat{P}_{j-1}\widehat{u}(0,\xi) = \widehat{u}(0,\xi)$$

for $|\xi| \le \frac{4}{3}\pi 2^{-j}$. Thus, for $|\xi| \le \frac{4}{3}\pi 2^{-j}$, we have

$$\begin{aligned} &K(x)\Lambda_{xx}(x,\xi) - i\xi\Lambda(x,\xi) \\ &= K(x)\widehat{u}_{xx}(x,\xi) - K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) - i\xi[\widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)] = 0 \end{aligned}$$

$$\Lambda(0,\xi) = \widehat{u}(0,\xi) - \widehat{u}_{j-1}(0,\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{g}(\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{u}(0,\xi) = 0$$
$$\Lambda_x(0,\xi) = \widehat{u}_x(0,\xi) - (\widehat{u}_{j-1})_x(0,\xi) = 0$$

Hence, for $|\xi| \le \frac{4}{3}\pi 2^{-j}$, fixed, $\Lambda(x,\xi)$ is solution on $0 \le x < 1$ of the problem

$$\begin{gathered} K(x)\Lambda_{xx}(x,\xi) - i\xi\Lambda(x,\xi) = 0, \quad 0 < x < 1 \\ \Lambda(0,\xi) = 0, \quad \Lambda_x(0,\xi) = 0 \end{gathered}$$

This problem has an unique solution $\Lambda(x,\xi) = 0$, for all $x \in [0,1)$. Thus,

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \text{for } |\xi| \le \frac{4}{3}\pi 2^{-j}$$

Now, (3.6) is consequence of (3.5) and the definition of $\widehat{P}_j$. □

Theorem 3.7 *Let u be a solution of (1.1) with the condition $u(0,\cdot) = g$, and let f be given by (3.4). Let v_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification $\widetilde{g}$ such that $\|g - \widetilde{g}\| \le \epsilon$. If $j = j(\epsilon)$ is such that $2^{-j} = \frac{\alpha}{\pi}\log\epsilon^{-1}$, then*

$$\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \epsilon^{1-x^2} \ + \ \|f\|_{L^2(R)} \cdot \epsilon^{\frac{1}{3}(1-x^2)}$$

Proof Note that

$$\begin{aligned} \|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot) + P_j u(x,\cdot) - u(x,\cdot)\| \\ &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| + \|P_j u(x,\cdot) - u(x,\cdot)\| . \end{aligned}$$

Let u_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification g. By (3.6), $P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot)$. Thus, by theorem 3.4, we have

$$\begin{aligned} \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| &= \|P_j v_{j-1}(x,\cdot) - P_j u_{j-1}(x,\cdot)\| \\ &\le \|v_{j-1}(x,\cdot) - u_{j-1}(x,\cdot)\| \le \epsilon^{1-x^2} \end{aligned}$$

Now, by proposition 3.5,

$$\|P_j u(x,\cdot) - u(x,\cdot)\| \le \|f\|_{L^2(\mathbb{R})} \exp(-\frac{1}{3}\frac{\pi}{\alpha}2^{-j}(1-x^2)) \le \|f\|_{L^2(\mathbb{R})} \cdot \epsilon^{\frac{1}{3}(1-x^2)}$$

Then $\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \epsilon^{1-x^2} + \|f\|_{L^2(R)}\epsilon^{\frac{1}{3}(1-x^2)}$ □

Conclusion

We had considered solutions $u(x,\cdot) \in L^2(R)$ of the problem $K(x)u_{xx} = u_t$, $0 < x < 1$, $t \geq 0$, with boundary specification g and $u_x(0,\cdot) = 0$, where $K(x)$ is bounded below by a positive constant. The inequality (3.1) implies that a solution of the problem above will be in $L^2(\mathbb{R})$ if $\widehat{g}$ has a rapid decay at high frequencies. Since the Meyer wavelet has compact support in the frequency domain, it cuts the high frequencies. Utilizing a wavelet Galerkin method with the Meyer multi-resolution analysis, we regularize the ill-posedness of the problem, approaching it by well-posed problems in the scaling spaces, as shown by theorem 3.4. We had shown the convergence of the wavelet Galerkin method applied to our problem, with an estimate error, in theorem 3.7. A more direct result would be to have a similar estimate for the difference between the exact solution of the problem and the solution of the approximating problem defined on the scaling space V_j. We are working towards this goal at the moment.

Notes: 1) Consider the problem

$$\begin{gathered} u_{xx}(x,t) = u_t(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0\,, \end{gathered}$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos 2n^2 t, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(2n^2 t + j\frac{\pi}{2})\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

Note that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero. This example was inspired by [1].

2) Note that $(\varphi_{jl})' \notin V_j$. In fact, if $(\varphi_{jl})' \in V_j$ then $(\varphi_{jl})' = \sum_{k\in Z}\alpha_k\varphi_{jk}$. Hence

$$\widehat{(\varphi_{jl})'} = \sum_{k\in Z}\alpha_k\widehat{\varphi_{jk}}$$

So, we would have

$$i2^{j/2}e^{-i2^j l\xi}\xi\widehat{\varphi}(2^j\xi) = \sum_{k\in Z}\alpha_k 2^{j/2}e^{-i2^{j/2}\xi}\widehat{\varphi}(2^j\xi)$$

This equality implies $\xi = \sum_{k\in Z}\alpha_k e^{-i[2^j(k-l)\xi+\frac{\pi}{2}]}$.

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José Roberto Linhares de Mattos
Rural Federal University of Rio de Janeiro,
Exact Sciences Institute, Department of Mathematics,
BR465 Km 7, Seropédica RJ, CEP 23890-000, Brazil
email: linhares@cos.ufrj.br

Ernesto Prado Lopes
Federal University of Rio de Janeiro,
COPPE, Systems and Computing Engineering Program,
Tecnology Center, Bloco H
and
Institute of Mathematics, Tecnology Center, Bloco C, Ilha do Fundão,
Rio de Janeiro RJ, CEP 21945-970, Brazil
email: lopes@cos.ufrj.br