Fifth Mississippi State Conference on Differential Equations and Computational Simulations,
Electronic Journal of Differential Equations, Conference 10, 2003, pp 193–210.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Cross diffusion systems on n spatial dimensional domains *

Dung Le

Abstract

We show that there exists a global attractor for a triangular cross diffusion system with Lotka-Volterra reaction given on a two dimensional domain.

1 Introduction

In population dynamics, Shigesada, Kawasaki, Teramoto [24] proposed to study the cross diffusion system

$$
\begin{gathered}
\frac{\partial u}{\partial t} = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\
\frac{\partial v}{\partial t} = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2 u - c_2 v), \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u^0(x), \quad v(x,0) = v^0(x), \quad x \in \Omega.
\end{gathered}
\tag{1.1}
$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ and the initial data u^0, v^0 are nonnegative functions.

When $\alpha_{ij} = 0$, the above system is the well known Lotka-Volterra competition-diffusion system which has been studied intensively. For nonzero α_{ij}, (1.1) is a strongly coupled parabolic system which has attracted attention in recent years and reopened many fundamental questions. In a series of papers [2, 3, 4], Amann considered a general class of strongly coupled parabolic systems and established local existence and uniqueness results. Roughly speaking, he showed that, for u^0, v^0 in $W^{1,p}$ with $p > n$, there exist $\varepsilon > 0$ and a unique solution u, v defined in $(0, \varepsilon)$.

Yagi [26, 27] investigated global existence problem for (1.1) which is given on a two dimensional domain. Under certain conditions on α_{ij}'s, he proved that solutions to (1.1) cease to exist in finite time if and only if their L^p norms blow

---

* *Mathematics Subject Classifications:* 35K50, 35K45, 35B40, 35B41.
Key words: Cross diffusion systems, global attractors.

Published February 28, 2003.

up. Recently, Lou, Ni and Wu in [21] studied the case when $\alpha_{21} = 0$ and $n = 2$ and established global existence results for the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ \frac{\partial v}{\partial t} &= \Delta[(d_2 + \alpha_{22}v)v] + v(a_2 - b_2 u - c_2 v), \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, t > 0. \end{aligned} \tag{1.2}$$

To the best of our knowledge, there has been no work on the dynamics or long time behavior of solutions to the above systems. In [22], Redlinger proved the existence of global attractors for certain triangular systems but his result does not apply to ours. This is the purpose of this paper to discuss not only global existence but also long time dynamics of solutions to a class of cross diffusion systems which includes (1.2).

On a bounded domain $\Omega \subset \mathbb{R}^n$ where $n \geq 1$, let us consider the parabolic system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla(P(u,v)\nabla u + R(u,v)\nabla v) + g(u,v), \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} &= \nabla(Q(v)\nabla v) + f(u,v), \quad x \in \Omega,\ t > 0, \end{aligned} \tag{1.3}$$

with Neumann or Robin type boundary conditions

$$\begin{aligned} Q(v)\frac{\partial v}{\partial n} + r_0(x)v(x) &= 0, \\ P(u,v)\frac{\partial u}{\partial n} + R(u,v)\frac{\partial v}{\partial n} + r(x)u(x) &= 0, \end{aligned} \tag{1.4}$$

and initial conditions

$$v(x,0) = v^0(x), \quad u(x,0) = u^0(x), \quad x \in \Omega.$$

The functions v^0, u^0 are nonnegative functions in $W^{1,p}(\Omega)$ for some $p > n$ (see [2]). In (1.3), P and Q represent the *self-diffusion* pressures, and R is the *cross-diffusion* pressure acting on the population u by v. It is easy to see that (1.2) is a special case of (1.3).

System of the form (1.3) is strongly coupled and of triangular form because the cross diffusion terms occur only in one equation and therefore the diffusion matrix is triangular. Such system was investigated by Amann in [4] where he established necessary conditions for the global existence of solutions. In particular, he proved that if one can control the L^∞ norms of *every components* of the solution then the solution exists globally in time.

In Section 2, under certain structural conditions and for any dimension $n \geq 2$, we will show that global existence as well as the existence of the global attractor for (1.3) can be proven if one can control the L^∞ norm of one component and the L^p norm (for some finite $p \geq n$) of the other component of the solutions. Moreover, our main point here is to show that if the L^p norms of the solutions can be estimated appropriately then their Hölder norms are *ultimately uniformly bounded* (see Definition 2.1 and Theorem 2.2). This fact is important in establishing the existence of global attractors. The result of this type is well known for reaction diffusion systems (see [8, 14]). However, in our case, the presence of the cross diffusion term causes enormous difficulties and the proof of this assertion becomes much more complicated. To this end, we first estimate the term ∇v and then u and reduce the problem to an integro-differential inequality. This inequality is a special case of a functional inequality whose solution dynamics gives our desired estimate. We believe that this functional inequality is interesting in itself and can be useful to other problems.

In Section 3, we will consider (1.2)(with general P, Q, R) on 2 dimensional domains and show that Theorem 2.2 can apply here. We thus sharpen Yagi's and Lou, Ni and Wu's results by showing that the system defines a dynamical system which possesses an absorbing set. Therefore, the global attractor with finite Hausdörf dimension for (1.3) exists and attracts all solutions (see Theorem 3.1).

We mention here that steady state solutions of (1.3) were studied in [6, 13, 20, 23]. When $n = 1$, the dynamics of the solutions of (1.3) was investigated in [7, 28]. If (1.3) satisfies more restrictive conditions on the structure of the system as well as on the initial data, global existence can be obtained via certain invariant principles as in [18]. Recently, duality methods were used in [5] to obtain global existence results for certain coupled systems whose diffusion terms are linear. This method is not applicable to (1.3)and does not seem to provide uniformly boundedness estimates of Theorem 2.2. In a forthcoming paper, we will establish that the L^p assumption of Theorem 2.2 can be relaxed to certain L^1 estimates if additional assumptions on the structure of (1.3) are satisfied.

2 Uniformly Boundedness

Throughout this work, in order to simplify the statements of our theorems and proof, we will make use of the following terminology.

Definition 2.1 Consider the initial-boundary problem (1.3),(1.4). Assume a priori that there exists a solution (u, v) defined on a subinterval I of $\mathbb{R}_+$. Let $\mathcal{P}$ be the set of functions on I such that there exists a positive constant C_0, which may generally depend on the parameters of the system and the $W^{1,p}$ norm of the initial value (u^0, v^0), such that

$$\omega(t) \le C_0, \quad \forall t \in I. \tag{2.1}$$

However, if $I = (0, \infty)$ then there exists a positive constant C_∞ that depends only on the parameters of the system but does not depend on the initial value of (u^0, v^0) such that

$$\limsup_{t\to\infty} \omega(t) \le C_\infty. \tag{2.2}$$

If $\omega \in \mathcal{P}$ and $I = (0, \infty)$, we will say that ω is *ultimately uniformly bounded.*

For example, if $\|u(\cdot,t)\|_\infty, \|v(\cdot,t)\|_\infty$, as functions in t, belong to the class $\mathcal{P}$ then (2.1) says that the supremum norms of the solutions to (1.3) do not blow up in any finite time interval and are bounded by some constant that may depend on the initial conditions. This implies that the solutions exist globally (see [2]). Moreover, for t sufficiently large, (2.2) says that the norms of the solutions can be majorized by a universal constant independent of the initial data. This property implies that there is an absorbing ball for the solutions and therefore shows the existence of the global attractor if certain compactness is proven.

We will consider the following conditions on the parameters of the system.

(H.1) The functions P, Q, R are differentiable in their variables. Moreover, there exist positive constants C, d and a continuous function Φ such that

$$Q(v) \geq d > 0, \tag{2.3}$$

$$P(u,v) \geq d > 0, \tag{2.4}$$

$$|R(u,v)| \leq \Phi(v)u. \tag{2.5}$$

Moreover, their partial derivatives with respect to u, v can be majorized by some powers of u, v.

We will be interested only in nonnegative solutions, which are relevant in many applications. Therefore, we will assume that the solution u, v stay nonnegative if the initial data u^0, v^0 are nonnegative functions. Conditions on f, g guarantee such positive invariance can be found in [18]. Moreover, we will impose the following assumption on the reaction terms.

(H.2) There exists a nonnegative continuous function $C(v)$ such that

$$|f(u,v)| \leq C(v)(1+u), \quad g(u,v)u^p \leq C(v)(1+u^{p+1}), \tag{2.6}$$

for all $u, v \geq 0$ and $p > 0$. In addition, the functions r_0, r are nonnegative Hölder continuous functions on $\partial\Omega$.

Our main result is the following.

Theorem 2.2 *Assume (H.1) and (H.2). Let (u,v) be a nonnegative solution to* (1.3) *with its maximal existence interval I. If $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_n$ are in $\mathcal{P}$ then there exists $\nu > 1$ such that*

$$\|v(\cdot,t)\|_{C^\nu(\Omega)}, \quad \|u(\cdot,t)\|_{C^\nu(\Omega)} \in \mathcal{P}. \tag{2.7}$$

Remark 2.3 The assumption $\|v(\cdot,t)\|_\infty \in \mathcal{P}$ can be weakened by assuming only that $\|v(\cdot,t)\|_r \in \mathcal{P}$ for some r sufficiently large such that $\|f(u,v)(\cdot,t)\|_q \in \mathcal{P}$ for some $q > n/2$. This is due to the results of [19, 12] which assert that the weaker assumption implies the stronger one. We also remark that the assumption on g in (2.6) could be relaxed to $g(u,v)u^p \leq C(v)(1+u^{p+1+\lambda})$ for some appropriate $\lambda > 0$. A simple use of Sobolev imbedding inequality in the proof of Lemma 2.6 will cover this case.

To simplify the presentation, instead of (1.4), we will consider the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.8}$$

Our results continue to hold for (1.3) with the boundary condition (1.4) as we will briefly indicate in Remark 2.11.

In the proof we will use $\omega(t), \omega_1(t), \dots$ to denote various continuous functions in the class $\mathcal{P}$. The proof of Theorem 2.2 will be based on several lemmas. We first state some standard facts from the theory of parabolic equations.

For any $t > \tau \geq 0$, we denote $Q_t = \Omega \times [0,t]$ and $Q_{\tau,t} = \Omega \times [\tau, t]$. For $r \in (1,\infty)$ and Q as one of the cylinders $Q_t, Q_{\tau,t}$, let $W_r^{2,1}(Q)$ be the Banach space of functions $u \in L^r(Q)$ having generalized derivatives $u_t, \partial_x u, \partial_{xx} u$ with finite $L^r(Q)$ norms (see [19, page 5]). For $s \geq 0$ and $r \in (1,\infty)$, we also make use of the fractional order Sobolev spaces $W_r^s(\Omega)$ (see, e.g., [1, 19] for the definition).

Let us consider the parabolic equation

$$\begin{gathered}
\frac{\partial v}{\partial t} = A(t)v + f_0(x,t), \quad x \in \Omega,\ t > 0, \\
\frac{\partial v}{\partial n}(x,t) = 0 \quad x \in \partial\Omega,\ t > 0, \\
v(x,0) = v_0(x) \quad x \in \Omega
\end{gathered} \tag{2.9}$$

where $A(t)$ is a uniformly regular elliptic operator of divergence form, with domain of definition $W_r^2(\Omega)$. If the coefficients of the operator $A(t)$ are uniformly Hölder continuous in a cylinder $Q_{\tau,t}$ and $(\lambda I + A(s))^{-1}$ exists for all $\lambda \geq 0$ and $s \in [\tau, t]$ then it is well known that (see, e.g., [15, Sections II.16-17]) there exists an evolution operator $U(t,s)$ for (2.9) such that the abstract integral version of (2.9) in L^r is

$$v(t) = U(t,\tau)v(\tau) + \int_\tau^t U(t,s)F(s)\,ds, \tag{2.10}$$

where $F(s)(x) = f_0(x,t)$. Moreover, for each $t > 0$, $r > 1$ and any $\beta \geq 0$, the fractional power $A^\beta(t)$, with its domain of definition $D(A_r^\beta(t))$ in $L^r(\Omega)$, of $A(t)$ is well defined ([15]). We recall the following imbeddings (see [17]).

$$D(A_r^\beta(t)) \subset C^\mu(\Omega), \quad \text{for } 2\beta > \mu + n/r \tag{2.11}$$

and

$$D(A_r^\beta(t)) \subset W^{1,p}(\Omega), \quad \text{if } 2\beta \geq 1 - n/p + n/r. \tag{2.12}$$

Next, we collect some well known facts about (2.9).

Lemma 2.4 *Let $r \in (1,\infty)$. For any solution v of (2.9) we have*

i) For $t > \tau \geq 0$, assume that the coefficients of $A(t)$ are bounded and continuous and $f \in L^r(Q_{\tau,t})$ for some $r > 3$. We have

$$\|v\|_{W_r^{2,1}(Q_{\tau,t})} \leq C(t-\tau)\left(\|f_0\|_{L^r(Q_{\tau,t})} + \|v(\cdot,\tau)\|_{W_r^{2-2/r}(\Omega)}\right) \tag{2.13}$$

where the constant $C(t-\tau)$ remain bounded if the length $t-\tau$ of the cylinder $Q_{\tau,t}$ is bounded and the coefficients of $A(t)$ are uniformly bounded in $Q_{\tau,t}$.

ii) Let $r>1$ and $f(\cdot,t)\in L^r(\Omega)$. Assume that the coefficients of the operator $A(t)$ are Hölder continuous. Moreover, there exists $\delta_0>0$ such that $(\lambda I+A(t))^{-1}$ exists for all $\lambda\geq-\delta_0$ and all $t>0$. For some fixed $t_0>0$ and any $\beta\in[0,1]$, we have

$$\|A^\beta(t_0)v(t)\|_r\leq C_\beta t^{-\beta}e^{-\delta t}\|v_0\|_r+C_\beta\int_0^t(t-s)^{-\beta}e^{-\delta(t-s)}\|f_0(\cdot,s)\|_r ds \tag{2.14}$$

for some constants $\delta, C_\beta>0$.

Proof The proof of i) can be found in [19, Theorem 9.1, chapter IV] where Dirichlet boundary condition was considered but the result holds as well for Neumann boundary condition (see [19, page 351]). For ii), we apply $A^\gamma(t)$ to both sides of (2.10), take the L^r norm and then make use the inequality [15, (16.38)]. □

Going back to the solutions of Theorem 2.2, we first have the following estimates for the component v and its spatial derivative.

Lemma 2.5 *The followings hold for v*

i) For some $\alpha>0$, $v\in C^{\alpha,\alpha/2}(\Omega\times(0,\infty))$ with uniformly bounded norm.

ii) For some $\omega_0,\omega\in\mathcal{P}$ and $\delta>0$, $r>1$, $\beta\in(0,1)$ such that $2\beta>\mu+n/r$, we have

$$\|v(\cdot,t)\|_{C^\mu(\Omega)}\leq\omega_0(t)+\int_0^t(t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r ds. \tag{2.15}$$

Proof Since we assume that $\|v(\cdot,t)\|_\infty\in\mathcal{P}$ and (2.6) holds, we see that $f(u,v)\in L_p(\Omega)$ for $p=n>n/2$. Moreover, $\|f(u(\cdot,t),v(\cdot,t)\|_p\in\mathcal{P}$. The regularity theory for quasilinear parabolic equations (see [19, 9]) asserts i).

Setting $A(t)=\nabla\cdot(Q(v)\nabla v)-kv$ and $\hat{f}_0(x,t)=f(u,v)+kv$ for $k>0$ sufficiently large, we see that v satisfies (2.9). Since v satisfies a parabolic equation with Hölder continuous coefficients (by i) above), we find that the conditions in ii) of Lemma 2.4 are verified. Since $\|v(\cdot,t)\|_\infty\in\mathcal{P}$, we have $\|\hat{f}_0\|_r\leq\omega(t)(1+\|u(\cdot,s)\|_r)$, for some function $\omega(t)\in\mathcal{P}$. Hence, (2.14) of Lemma 2.4 gives

$$\|A_0^\beta v(t)\|_r\leq C_\beta t^{-\beta}e^{-\delta t}\|v_0\|_r+C_\beta\int_0^t(t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)(1+\|u(\cdot,s)\|_r)ds$$

for any fixed $t_0>0$. From the imbedding (2.11), (2.15) now follows. □

Next, we will show that the L^p norm of u is in the class $\mathcal{P}$ for any $p\geq1$. In fact, this is the crucial step in proving Theorem 2.2.

Lemma 2.6 *For any finite $p \geq 1$, there exists a function $\omega_p \in \mathcal{P}$ such that*

$$\|u(\cdot,t)\|_p \leq \omega_p(t). \tag{2.16}$$

The idea of the proof is to derive certain differential inequalities for the L^p norm of u. To this end, we have to control the norm of ∇v that occurs in the equation of u by using the equation for v . This then leads us to certain functional differential inequalities which we will study next.

For a function $y : \mathbb{R}^+ \to \mathbb{R}$, let us consider the inequality

$$y'(t) \leq \mathcal{F}(t,y), \quad y(0) = y_0, \quad t \in (0,\infty), \tag{2.17}$$

where $\mathcal{F}$ is a functional from $\mathbb{R}^+ \times C(\mathbb{R}^+,\mathbb{R})$ into $\mathbb{R}$. The following lemma is standard and gives a global estimate for y but the estimate is still dependent on the initial data. Consider the assumptions:

F.1 Suppose that there is a function $F(y,Y) : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathcal{F}(t,y) \leq F(y(t),Y)$ if $y(s) \leq Y$ for all $s \in [0,t]$.

F.2 There exists a real M such that $F(Y,Y) < 0$ if $Y \geq M$.

Lemma 2.7 *Assume* (2.17), F.1, *and* F.2 *Then there exists finite M_0 such that $y(t) \leq M_0$ for all $t \geq 0$.*

The proof of this lemma is elementary, and therefore will be omitted.

Remark 2.8 *In (F.1), the inequality $\mathcal{F}(t,y) \leq F(y(t),Y)$ is not pointwise. It requires that $y(s) \leq Y$ on the interval $s \in [0,t]$ not just that $y(t) \leq Y$. Such situation usually happens when $f(t,y)$ contains integrals of $y(t)$ over $[0,t]$.*

The above constant M_0 still depends on the initial data y_0. Moreover, the function F may depend on y_0 too. Next, we consider conditions which guarantee uniform estimates for $y(t)$.

Consider the following assumptions:

(G.1) There exists a continuous function $G(y,Y) : \mathbb{R}^2 \to \mathbb{R}$ such that for τ sufficiently large, if $t > \tau$ and $y(s) \leq Y$ for every $s \in [\tau,t]$ then there exists $\tau' \geq \tau$ such that

$$\mathcal{F}(t,y) \leq G(y(t),Y) \quad \text{if } t \geq \tau' \geq \tau. \tag{2.18}$$

(G.2) The set $\{z : G(z,z) = 0\}$ is not empty and $z_* = \sup\{z : G(z,z) = 0\} < \infty$. Moreover, $G(M,M) < 0$ for all $M > z_*$.

(G.3) For $y, Y \geq z_*$, $G(y,Y)$ is increasing in Y and decreasing in y.

Proposition 2.9 *Assume* (2.17), (G.1), (G.2), *and* (G.3). *If* $\limsup_{t\to\infty} y(t) < \infty$ *then*

$$\limsup_{t\to\infty} y(t) \leq z_*. \tag{2.19}$$

Proof If $M_0 = \limsup_{t\to\infty} y(t) \le z_*$ then there is nothing to prove. So, let us assume that $M_0 > z_*$.

First, let $M > z_*$. Since $G(z_*, z_*) = 0$ we have $G(z_*, M) > 0$ (because $G(z_*, \cdot)$ is increasing, by (G.3)). This and the fact that $G(M, M) < 0$ implies the existence of a number $z \in (z_*, M)$ such that $G(z, M) = 0$. Let $z(M)$ be the largest of such z in (z_*, M). By (G.3), we have

$$G(z(M), M) = 0 \quad \text{and} \quad G(y, M) < 0, \quad \forall y \in (z(M), M). \tag{2.20}$$

Now, for t large, says $t \ge T$, we have that $y(t) \le M$ for some $M > z_*$. By (G.1), we can find $T_0 \ge T$ such that

$$y'(t) \le G(y(t), M), \quad t \ge T_0, \quad y(T_0) \le M.$$

Comparing $y(t)$ with the solution of $Y'(t) = G(Y(t), M)$, $t > T_0$ and $Y(T_0) = M$, we conclude that $y(t) \le Y(t)$ for all $t \ge T_0$. ¿From (2.20), $Y' < 0$. We see that $Y(t) \to z(M)$, the steady state, as $t \to \infty$. Thus, for any given $\varepsilon > 0$, there exist $T_1 > T_0$ and $\varepsilon_1 \in (0, \varepsilon)$ such that $z(M) + \varepsilon_1 < M$ and $y(t) \le Y(t) \le z(M) + \varepsilon_1$ for all $t > T_1$.

Since $z(M) > z_*$, the above argument can be repeated with $z(M) + \varepsilon_1$ in place of M to show that there exist sequences of positive numbers $\{T_j\}, \{\varepsilon_j\}$ and $\{k_j\}$ such that $k_0 = M$, $\lim_{j\to\infty} \varepsilon_j = 0$, $\lim_{j\to\infty} T_j = \infty$ and

$$k_{j+1} = z(k_j) + \varepsilon_j < k_j, \quad y(t) \le k_j, \quad \forall t \ge T_j.$$

Since k_j is decreasing and bounded from below by z_*, k_j converges to some $z \ge z_*$ satisfying $G(z, z) = 0$ (because $G(k_{j+1} - \varepsilon_j, k_j) = 0$ for all j and $\varepsilon_j \to 0$). Since z_* is the largest of such solutions, we must have $z = z_*$. Thus, $\limsup_{t\to\infty} y(t) \le z_*$. □

Remark 2.10 *Condition (G.3) is only used to guarantee the existence of* $z(M)$ *that has the property* (2.20)*. One can see that the proof works as well for functions satisfying* (2.20) *for any given* $M > z_*$.

We are now ready to give the proof of Lemma 2.6.

Proof The proof is by induction on p. We suppose that (2.16) holds for some $p \ge 1$. Let us denote $U = u^p$. We multiply the equation for u by u^{2p-1} and integrate over Ω. Using integration by parts and the boundary condition of u, we see that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega U^2 dx + \int_\Omega P(u, v)|\nabla U|^2\, dx \\ &\qquad \le C_p \int_\Omega (-R(u, v)\nabla(u^{2p-1})\nabla v + g(u, v)u^{2p-1})\, dx. \end{aligned}$$

Using the conditions (2.4), (2.5) and (2.6), we derive

$$\frac{d}{dt}\int_\Omega U^2\, dx + d\int_\Omega |\nabla U|^2\, dx \le C_p \int_\Omega (|U\nabla U|\Phi(v)|\nabla v| + U^2)\, dx. \tag{2.21}$$

Set $y(t)=\int_\Omega U^2(x,t)\,dx$. By [8, Lemma 2.4], for any $\varepsilon>0$, we have that

$$\int_\Omega U^2 dx \le \varepsilon\{\int_\Omega |\nabla U|^2 dx + \|U\|_1^2\} + C\varepsilon^{-n/2}\|U\|_1^2 \tag{2.22}$$

for some positive constants C. We use the above inequality with $\varepsilon = d/(2C_p)$ in the integrals of U^2 on the right hand side of (2.21). Recalling the induction assumption $\|U(\cdot,t)\|_1\in\mathcal{P}$, we obtain

$$y'(t)+\frac{d}{2}\int_\Omega |\nabla U|^2 dx \le C_p\int_\Omega |U\nabla U|\Phi(v)|\nabla v| dx + \omega_0(t), \tag{2.23}$$

for some $\omega_0\in\mathcal{P}$. We next estimate the integral of $|U\nabla U|\Phi(v)|\nabla v|$. By our assumption on L^∞ norm of v, $\Phi(v)\le\omega_1(t)$ for some $\omega_1\in\mathcal{P}$. Using the Young inequality, we have

$$\begin{aligned} C_p\int_\Omega |U\nabla U|\Phi(v)|\nabla v|\,dx &\le \frac{d}{8}\int_\Omega |\nabla U|^2 dx + C(d)\omega_1(t)\int_\Omega U^2|\nabla v|^2 dx \\ &\le \frac{d}{8}\int_\Omega |\nabla U|^2 dx + C(d)\omega_1(t)\|\nabla v\|_\infty^2\int_\Omega U^2 dx. \end{aligned} \tag{2.24}$$

We now use (2.22) with $\varepsilon = d/(8C(d)\omega_1(t)\|\nabla v\|_\infty^2)$ to get

$$\begin{aligned} &C(d)\omega_1(t)\|\nabla v\|_\infty^2\int_\Omega U^2 dx \\ &\le \frac{d}{8}\int_\Omega (|\nabla U|^2+U^2)\,dx + C(d)\omega_2(t)\|\nabla v\|_\infty^n\|U\|_1^2 \\ &\le \frac{d}{8}\int_\Omega |\nabla U|^2 dx + C(d)\omega_3(t)\|\nabla v\|_\infty^n. \end{aligned}$$

Since $p\ge n$, we can choose $\beta\in(0,1)$ and $r\in(p,2p)$ such that $2\beta>1+n/r$. Using (2.15), we get

$$\|\nabla v(\cdot,t)\|_\infty \le \omega_4(t) + C\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega_4(s)\|u(\cdot,s)\|_r ds \tag{2.25}$$

for some $\omega_4\in\mathcal{P}$. By Hölder inequality,

$$\|u\|_r = \|U\|_{r/p}^{1/p} \le \|U\|_1^{1/p-\theta}\|U\|_2^\theta, \quad \theta = \frac{1/p-1/r}{1-1/2}.$$

Observe that θ can be arbitrarily small if r is close to p. ¿From now on, we will choose $r>p$ such that $n\theta<1$. Using the above in (2.25) we obtain

$$\|\nabla v(\cdot,t)\|_\infty \le \omega_4(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega_4(s)y^\theta(s)ds.$$

Applying this and (2.24) in (2.23), we see that

$$y'(t)+\frac{d}{4}\int_\Omega |\nabla U|^2 dx \le \omega_6(t)+\omega_6(t)K^n(t), \tag{2.26}$$

where $K(t) = \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega_4(s) y^\theta(s) ds$ and $\omega_6 \in \mathcal{P}$. By (2.22) and the induction assumption, $\int_\Omega U^2 dx \le \frac{d}{4} \int_\Omega |\nabla U|^2 dx + \omega_7(t)$ for some function $\omega_7 \in \mathcal{P}$. We thus deduce the following integro-differential inequality

$$y'(t) \le -y(t) + \omega_8(t) + \omega_8(t) K^n(t). \tag{2.27}$$

We will show that Lemma 2.7 and Proposition 2.9 can be used here to assert that y is globally bounded and, more importantly, ultimately uniformly bounded. This implies that $\|u\|_{2p} \in \mathcal{P}$ and completes the proof by induction. We define the functional

$$\mathcal{F}(t,y) = -y(t) + \omega_8(t) + \omega_8(t) K^n(t). \tag{2.28}$$

Since $\omega_8 \in \mathcal{P}$, we can find positive constants C_ω, which may still depend on the initial data, such that $\omega_8(t) \le C_\omega$ for all $t > 0$. Let

$$C_1 := \sup_{t>0} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} ds \le \int_0^\infty s^{-\beta} e^{-\delta s} ds < \infty,$$

because $\beta \in (0,1)$ and $\delta > 0$. We then set

$$F(y,Y) = -y + C_\omega + C_\omega (C_1 Y^\theta)^n.$$

It is easy to check that $\mathcal{F}, F$ satisfy the conditions (F.1), (F.2) if $n\theta < 1$. Hence, Lemma 2.7 applies and gives

$$y(t) \le C_0(v^0, u^0), \quad \forall t > 0. \tag{2.29}$$

For some constant $C_0(v^0,u^0)$ which may still depend on the initial data since F does. We have shown that $y(t)$ is globally bounded.

We now seek for uniform estimates. By Definition 2.1, we can find $\tau_1 > 0$ such that $\omega(s) \le \bar{C}_\infty = C_\infty + 1$ if $s > \tau_1$. We emphasize the fact that $\bar{C}_\infty$ is independent of the initial data. Let $t > \tau \ge \tau_1$ and assume that $y(s) \le Y$ for all $s \in [\tau, t]$. Let us write

$$K(t) = \int_0^\tau (t-s)^{-\beta} e^{-\delta(t-s)} \omega_4(s) y^\theta(s) ds + \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega_4(s) y^\theta(s) ds = J_1 + J_2.$$

By (2.29), there exists some constant $C(v^0,u^0)$ such that $\omega_4(s) y^\theta(s) \le C(v^0,u^0)$ for every s. Hence, we can find $\tau' > \tau$ such that $J_1 \le 1$ if $t > \tau'$. Hence,

$$K(t) \le 1 + \bar{C}_\infty C_* Y^\theta, \quad \text{where} \quad C_* = \sup_{t>\tau, \tau>0} \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} ds < \infty.$$

Therefore, for $t > \tau'$ we have $f(t,y) \le G(y(t),Y)$ with

$$G(y(t),Y) = -y(t) + \bar{C}_\infty + \bar{C}_\infty (1 + \bar{C}_\infty C_* Y^\theta)^n. \tag{2.30}$$

We see that G is independent of the initial data and satisfies (G.1)-(G.3) if $n\theta < 1$. Finally, Proposition 2.9 applies here to give (2.16). $\square$

Having shown that (2.16) holds for any p large we now go further in proving that the C^ν norm of u, for some $\nu > 1$, is ultimately uniformly bounded.

Proof of Theorem 2.2: We first apply i) of Lemma 2.4 to the equation for v in (1.3). Since $\|u(\cdot,t)\|_p \in \mathcal{P}$ for any p large, we see that $f(u,v) \in L^q(Q_{\tau,t})$ for any $q>1$. In fact, with $\tau = t-1$, $\|f(u,v)\|_{L^q(Q_{\tau,t})}$, as a function in t, is in the class $\mathcal{P}$. Hence,

$$\|v\|_{W_q^{2,1}(Q_{\tau,t})} \le C\left(\|f(u,v)\|_{L^q(Q_{\tau,t})} + \|v(\cdot,\tau)\|_{W_q^{2-2/q}(\Omega)}\right). \tag{2.31}$$

Choosing $\beta \in (0,1)$ (close to 1) and r sufficiently large such that $2\beta > 2 - 1/q + n/r$, Lemma 2.5 states that the norm of $v(\cdot,t)$ in $C^{2-1/q}(\Omega)$, and therefore $W_q^{2-2/q}(\Omega)$, is in the class $\mathcal{P}$ for any $q>1$. We then conclude that $\|v\|_{W_q^{2,1}(Q_{\tau,t})} \in \mathcal{P}$ for any $q>1$. So,

$$\int_{t-1}^{t}\int_\Omega \left(|\frac{\partial v}{\partial t}(x,s)|^q + |\Delta v(x,s)|^q\right) dx\, ds \le \omega(t), \quad \forall t \in I \tag{2.32}$$

for some $\omega \in \mathcal{P}$. We now write the equation for u as follows

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(x,t)\nabla u) + B(x,t)\nabla u + \hat{F}(x,t),$$

where $A(x,t) = P(u,v)$, $B = R_u \nabla v$ and $\hat{F}(x,t) = g(u,v)R(u,v)\Delta v + R_v|\nabla v|^2$. Using (2.32), we easily see that $b(x,t)$ and $\hat{F}(x,t)$ belong to $L^{q,q}$ for any q large. Standard regularity theories for quasilinear parabolic equations (see [9]) can be applied here to conclude that $u(x,t)$ is in class $C^{\alpha,\alpha/2}$ for some $\alpha>0$.

Set $U = \hat{P}(u,v)$ where $\hat{P}(u,v) = \int_0^u P(s,v)ds$. Because $\nabla u = (\nabla U - \hat{P}_v \nabla v)/P(u,v)$, the ellipticity condition (2.4) and the Hölder regularity of $u, v, \nabla v$ show that Hölder continuity of ∇U implies that of ∇u. Therefore, we will study the regularity of U.

It is easy to see that U satisfies the equation

$$U_t = a(x,t)\Delta U + b(x,t)\nabla U - kU + \hat{f}(x,t),$$

where $a(x,t) = P(u,v)$, $b(x,t) = (R_u - \hat{P}_{u,v})\nabla v$, k is a positive constant and

$$\begin{aligned}\hat{f}(x,t) =& P(R-\hat{P}_v)\Delta v + |\nabla v|^2(-\hat{P}_v R_u + \hat{P}_v\hat{P}_{u,v} + PR_v - P\hat{P}_{v,v})\\ &+ Pg(u,v) + \hat{P}_v v_t + kU.\end{aligned}$$

From the regularity of u, v and ∇v we see that $a(x,t)$ and $b(x,t)$ are Hölder continuous with ultimately uniformly bounded norms. Hence, the above equation is regular with Hölder coefficients whose Hölder norms, as functions of t, are in the class $\mathcal{P}$. Let $\hat{A}(t)$ be the operator corresponding to the above equation. By choosing k sufficiently large, we see that $\hat{A}(t)$ is a regular elliptic operator with Hölder continuous coefficient and satisfies the conditions of ii) of Lemma 2.4. Moreover, U satisfies the Neumann boundary condition as u, v do. Therefore, for any fixed $t_0 > 0$ and $\tau = t-1 > 0$

$$\|\hat{A}^\beta(t_0)u(t)\|_r \le C\|u(\tau)\|_r + C_\beta\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}\|\hat{f}(\cdot,s)\|_r ds \tag{2.33}$$

for some fixed constants $C, \delta, C_\beta > 0$. By Hölder inequality we can estimate the second term as follows:

$$\int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \|\hat{f}(\cdot,s)\|_r ds \le \Big(\int_\tau^t (t-s)^{-q\beta} e^{-q\delta(t-s)} ds\Big)^{1/q} \|\hat{f}\|_{L^r(Q_{\tau,t})}, \tag{2.34}$$

where $1/q + 1/r = 1$. From the definition of $\hat{f}$ and the facts that $\|v(\cdot,t)\|_\infty$, $\|\nabla v(\cdot,t)\|_\infty$ are functions in the class $\mathcal{P}$, $\|u(\cdot,t)\|_p \in \mathcal{P}$ for any p, and (2.32) holds for any q, we see that $\hat{f} \in L^r(Q_{\tau,t})$ and $\|\hat{f}\|_{L^r(Q_{\tau,t})} \in \mathcal{P}$ for any r large. Therefore, given any $\beta \in (0,1)$, if we choose r large enough such that $q = r/(r-1)$ sufficiently close to 1 then it is easy to see that the integral on the right hand side of (2.34) is finite. Moreover, the quantity on the right hand side is in the class $\mathcal{P}$. Using this in (2.33), we have shown that, for $Y = D(\hat{A}_r^\beta(t_0))$, $\|U(t)\|_Y \in \mathcal{P}$ for any $\beta \in (0,1)$ and $r > 1$. Using the imbedding (2.11) with $\nu = 2\beta - n/r > 0$ and β, r chosen such that $\nu > 1$, we obtain estimate for the Hölder norm of ∇U and prove (2.7). $\square$

Remark 2.11 We briefly indicate here that Theorem 2.2 continues to hold if the boundary conditions are now of the form (1.4). Indeed, Lemma 2.4 and Lemma 2.5 are still in force if one makes a change of variables to reduce the homogeneous Robin condition for v into a homogeneous Neumann one. The proof of our main technical lemma, Lemma 2.6, continues to hold if one drops the nonpositive boundary integrals result in the integrations by parts. Finally, the proof of Theorem 2.2 remains as ii) of Lemma 2.4 continues to hold for equations with Robin boundary condition and sufficiently regular parameters. Such regularity of parameters is granted as we have shown that $v(\cdot,t) \in C^{1,\gamma}(\Omega)$ for any $\gamma \in (0,1)$.

3 The 2-dimensional case

In this section we will show that the assumption on L^n boundedness of Theorem 2.2 is verified for (1.3) if the dimension $n = 2$ and the reaction terms are of Lotka-Volterra type

$$f(u,v) = v(c_1 - c_{11}v - c_{12}u), \quad g(u,v) = u(c_2 - c_{21}v - c_{22}u), \tag{3.1}$$

where c_{ij} are given constants. Furthermore, since it will not complicate much the presentation, we shall consider here the nonlinear boundary condition (1.4).

For any given $p > 2$, let

$$X = \big\{(u,v) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) : u(x),\ v(x) \ge 0,\ \forall x \in \Omega\big\}.$$

For given nonnegative initial data $u^0, v^0 \in X$, it is standard to show that the solution stays nonnegative (see [18]). We consider the dynamical system associated with (1.3),(1.4) on X (see [4]).

The main result of this section is the following.

Theorem 3.1 *Assume (H.1) and that* $c_{11}, c_{12}, c_{22} > 0$. *The system* (1.3), (1.4) *with* (3.1) *possesses a global attractor with finite Hausdorff dimension in* X.

Clearly, the functions f, g satisfy the condition (H.2). Thus, the above theorem is a consequence of Theorem 2.2 and the well known theory of dissipative dynamical systems (see [16]) if we can show that the norms $\|v\|_\infty, \|u\|_2$ are in the class $\mathcal{P}$.

First of all, since $c_{11}, c_{12} > 0$, using invariant principle for scalar parabolic equation or test the equation of v by $(v-k)_+$ for some k large we easily derive

Lemma 3.2 $\|v(\cdot,t)\|_\infty \in \mathcal{P}$.

The fact that $\|u(\cdot,t)\|_2$ is in $\mathcal{P}$ is more difficult to prove and this will be done in several steps. We start with the following simple lemma.

Lemma 3.3 *For the component* u *we have*

$$\|u(\cdot,t)\|_1 \in \mathcal{P}, \tag{3.2}$$

$$\int_t^{t+1}\int_\Omega u^2 dx \in \mathcal{P}. \tag{3.3}$$

Proof Integrating the equation for u over Ω. Using the boundary condition (1.4) and the fact that $u, v \geq 0$ we can drop the boundary integrals result in the integration by parts to obtain

$$\frac{d}{dt}\int_\Omega u dx = \int_\Omega g(u,v)\,dx \leq c_2\int_\Omega u\,dx - c_{22}\int_\Omega u^2\,dx \tag{3.4}$$

this implies

$$\frac{d}{dt}\int_\Omega u\,dx \leq c_2\int_\Omega u\,dx - c_{22}(\int_\Omega u\,dx)^2 \tag{3.5}$$

It is easy to see that (3.5) gives (3.2) (see also Proposition 2.9). Integrating (3.4) from t to $t+1$ and using (3.2), we get (3.3). $\square$

Next, by multiplying the equation of u by u, we have

$$\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega P(u,v)|\nabla u|^2 dx = -\int_\Omega R(u,v)\nabla v\nabla u\,dx + \int_\Omega g(u,v)u\,dx$$

Using (2.3), (2.5) and (3.1) we get

$$\frac{d}{dt}\int_\Omega u^2 dx + d\int_\Omega |\nabla u|^2 dx \leq \omega(t)\int_\Omega |u\nabla v\nabla u|\,dx + \omega(t)\int_\Omega u^2 dx, \tag{3.6}$$

for some $\omega \in \mathcal{P}$. Hereafter, $\omega(t)$ or C will denote a function in $\mathcal{P}$ or a generic positive constant which can be different from line to line but they depend on the previously obtained estimates. By (3.2) and the Gagliardo-Nirenberg inequality we can absorb the last term in the above inequality into the left hand side. Hence,

$$\frac{d}{dt}\int_\Omega u^2 dx + \frac{d}{2}\int_\Omega |\nabla u|^2 dx \le \omega(t)\int_\Omega |u\nabla v \nabla u| dx + \omega(t). \tag{3.7}$$

We need to investigate the first integral on the right. For any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\int_\Omega |u\nabla v\nabla u| dx \le \varepsilon \int_\Omega |\nabla u|^2 dx + C_\varepsilon \int_\Omega u^2 |\nabla v|^2 dx. \tag{3.8}$$

Next, since $n = 2$, we have the interpolation inequality

$$\|u\|_4 \le \|u\|_2^{1/2}\|u\|_{H^1}^{1/2} = \|u\|_2^{1/2}(\|\nabla u\|_2 + \|u\|_2)^{1/2}. \tag{3.9}$$

Therefore, by Young inequality, we have

$$\begin{aligned}\int_\Omega u^2|\nabla v|^2 dx &\le \|u\|_4^2\|\nabla v\|_4^2 \\ &\le \|u\|_2(\|\nabla u\|_2 + \|u\|_2)\|\nabla v\|_4^2 \\ &\le \varepsilon\|\nabla u\|_2^2 + C_\varepsilon\|u\|_2^2(\|\nabla v\|_4^4 + 1).\end{aligned}$$

Hence, (3.7) and Poincaré inequality imply

$$\frac{d}{dt}\int_\Omega u^2 dx + \frac{d}{4}\int_\Omega u^2 dx \le C\|u\|_2^2(\|\nabla v\|_4^4 + 1) + \omega(t). \tag{3.10}$$

We will show that

$$\int_t^{t+1}\int_\Omega |\nabla v(x,s)|^4 dx\, ds \in \mathcal{P}. \tag{3.11}$$

With (3.11) and (3.3), we can use the uniform Gronwall inequality (see [25, Lemma 1.1, Chap.3]) to assert from (3.10) that $\|u(\cdot,t)\|_2 \in \mathcal{P}$ and conclude our proof. □

To prove (3.11) we need to estimate the norms of ∇v and v_t.

Lemma 3.4 *We assert that*

$$\|\nabla v(\cdot,t)\|_2 \in \mathcal{P}, \tag{3.12}$$

$$\int_t^{t+1}\int_\Omega v_t^2(x,s) dx\, ds \in \mathcal{P}. \tag{3.13}$$

Proof First of all, using the boundary condition for v, we notice that

$$\begin{aligned}\int_\Omega \nabla(Q\nabla v)Qv_t\, dx &= -\int_\Omega Q\nabla v(Q_v\nabla v v_t + Q\nabla(v_t))dx + \int_{\partial\Omega} Q\frac{\partial v}{\partial n}Qv_t\, d\sigma \\ &= -\frac{1}{2}\int_\Omega \frac{d}{dt}(Q^2|\nabla v|^2)dx - \int_{\partial\Omega} r_0(x)Qvv_t\, d\sigma.\end{aligned}$$

Therefore, by multiplying the equation for v by Qv_t, we get

$$\int_\Omega Qv_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega Q^2|\nabla v|^2 dx \le \int_\Omega f(u,v)Qv_t\, dx - \frac{d}{dt}\int_{\partial\Omega} r_0(x)\hat{Q}(v)\, d\sigma,$$

where $\hat{Q}(v) = \int_0^v Q(s)s\, ds$. The above then gives

$$\int_\Omega Qv_t^2 dx + \frac{d}{dt}\int_\Omega Q^2|\nabla v|^2 dx \le \int_\Omega f^2(u,v)Q\, dx - \frac{d}{dt}\int_{\partial\Omega} r_0(x)\hat{Q}(v)\, d\sigma. \quad (3.14)$$

On the other hand, let $\bar{Q}(v) = \int_0^v Q(s)ds$ and multiply the equation for v by $\bar{Q}(v)$ to obtain

$$\int_\Omega \bar{Q}v_t dx = -\int_\Omega Q^2|\nabla v|^2 dx - \int_{\partial\Omega} r_0 v\bar{Q}\, d\sigma + \int_\Omega f(u,v)\bar{Q}(v)\, dx.$$

But

$$\int_\Omega Qv_t^2 dx \ge -2\int_\Omega \bar{Q}v_t\, dx - \int_\Omega \frac{\bar{Q}^2}{Q}\, dx$$

by Young inequality. We now set

$$y(t) = \int_\Omega Q^2|\nabla v|^2 dx + \int_{\partial\Omega} r_0(x)\hat{Q}(v)\, d\sigma$$

and add $2\int_{\partial\Omega} r_0\hat{Q}\, d\sigma$ to both sides of (3.14). Using the above inequalities, we easily obtain

$$y'(t) + 2y(t) \le \int_\Omega [f^2Q + \frac{\bar{Q}^2}{Q} + 2f\bar{Q}]dx - 2\int_{\partial\Omega} r_0 v\bar{Q}\, d\sigma + 2\int_{\partial\Omega} r_0\hat{Q}\, d\sigma.$$

From the assumption $f(u,v) \le C(v)(1+u)$ and (3.3) we see that the above implies $y(t) \in \mathcal{P}$. But v, and therefore $\int_{\partial\Omega} r_0\hat{Q}\, d\sigma$ and $\int_{\partial\Omega} r_0 v\bar{Q}\, d\sigma$, belongs to $\mathcal{P}$. We conclude that $\int_\Omega Q^2|\nabla v|^2 dx \in \mathcal{P}$. This and (2.3) give (3.12).

Finally, we can integrate (3.14) and use (3.12), (2.3) to obtain (3.13). $\square$

Let us go back to (3.11). Using (3.9), we note that

$$\|\nabla v\|_4^4 \le \|\nabla v\|_2^2\|\nabla v\|_{H^1}^2 = \|\nabla v\|_2^2(\|\Delta v\|_2^2 + \|\nabla v\|_2^2).$$

Taking into account (3.12), in order to prove (3.11) and conclude our proof , we need only to estimate $\int_t^{t+1} \|\Delta v\|_2^2$. From the equation for v, we have

$$\|\nabla(Q\nabla v)\|_2^2 \le \|v_t\|_2^2 + \|u\|_2^2 + \omega(t). \quad (3.15)$$

By (3.13) and (3.3) we conclude that

$$\int_t^{t+1} \|\nabla(Q\nabla v)\|_2^2\, dt \in \mathcal{P}. \quad (3.16)$$

Since $\nabla(Q\nabla v) = Q\Delta v + Q_v|\nabla v|^2$ and (2.3) we have

$$|\Delta v|^2 \le C(|\nabla(Q\nabla v)|^2 + |\nabla v|^4).$$

Thus, $\|\Delta v\|_2 \le C(\|\nabla(Q\nabla v\|_2 + \|\nabla v\|_4^2)$. But

$$\|\nabla v\|_4^2 \le C\|Q\nabla v\|_4^2 \le C\|Q\nabla v\|_2(\|\nabla(Q\nabla v)\|_2 + \|Q\nabla v\|_2).$$

Hence, by (3.12) we have

$$\|\Delta v\|_2 \le C\|\nabla(Q\nabla v\|_2(1+\|\nabla v\|_2) + C\|\nabla v\|_2^2 \le C\omega(t)(\|\nabla(Q\nabla v)\|_2 + 1).$$

Integrating the above from t to $t+1$ we obtain

$$\int_t^{t+1} \|\Delta v\|_2^2 dt \le \omega(t)\Big(\int_t^{t+1} \|\nabla(Q\nabla v\|_2^2 dt + 1\Big).$$

This and (3.16) give (3.11). Our proof is then complete.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic press, New York, 1975.

[2] H. Amann. Dynamic theory of quasilinear parabolic equations I. abstract evolution equations. *Nonlinear Ananlysis T.M.A.*, pages 895–919, 12(1988).

[3] H. Amann. Dynamic theory of quasilinear parabolic equations-II. reaction-diffusion systems. *Diff. Int. Equations*, pages 13–75, 3(1990).

[4] H.Amann. Dynamic theory of quasilinear parabolic systems-III. global existence. *Math. Z.*, pages 219–250, 202(1989).

[5] N. Boudiba and M. Pierre. Global existence for coupled reaction diffusion systems. *J. Math. Anal. Appl.*, vol.250, pp. 1–12, 2000.

[6] D. Le. Coexistence with chemotaxis.. *SIAM J. Math. Analysis.*, Vol. 32, No. 3, pp. 504–521, (2000).

[7] D. Le. On a time dependent bio-reactor model with chemotaxis. *Applied Math. and Computation.* Vol. 131, pp. 531–558 (2002).

[8] D. Le. Global attractors and steady state solutions for a class of reaction diffusion systems. *J. Diff. Eqns.*, Vol. 147, pp.1–29 (1998).

[9] D. Le. Remark on Hölder continuity for parabolic equations and the convergence to global attractors. *Nonlinear Analysis T.M.A.*, Vol. 41, pp. 921–941 (2000).

[10] D. Le. Hölder regularity for certain strongly coupled parabolic systems. *J. Diff. Eqn.*, Vol.151, pp. 313-344 (1999).

[11] D. Le. Ultimately uniform boundedness of solutions and gradients for degenerate parabolic systems. *Nonlinear Analysis T.M.A.*, Vol. 39, pp. 157–171 (2000).

[12] D. Le. Dissipativity and global attractor for a class of quasilinear parabolic systems. *Communications in P.D.Es*, pages 413–433, 22(3&4) (1997).

[13] D. Le and H. Smith. Steady states of models of microbial growth and competition with chemotaxis. *J. Math. Anal. App.*, Vol.229, pp. 295–318 (1999).

[14] D. Le and H. L. Smith. On a parabolic system modelling microbial competition in an unmixed bio-reactor. *J. of Diff. Eqns*, pages 59–91, 130 No.1 (1996).

[15] A. Friedman. *Partial Differential Equations.* New York, 1969.

[16] J. Hale. *Asymptotic Behavior of Dissipative Systems.* American Math. Soc. Math. Surveys and Monographs, 25(1988).

[17] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag Lecture Notes in Math., 840, 1981.

[18] K. H. W. Küfner. Global existence for a certain strongly coupled quasilinear parabolic system in population dynamics. *Analysis*, pages 343–357, 15 (1995).

[19] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and Quasilinear Equations of Parabolic Type.* AMS Transl. Monographs, 23(1968).

[20] Y. Lou and W. Ni. Diffusion, Self-Diffusion and Cross-Diffusion. *J. Diff. Eqn.*, vol.131, pp. 79-131, 1996.

[21] Y. Lou, W. Ni and Y. Wu. On the global existence of a Cross-Diffusion system. *Discrete and Continuous Dyn. Sys.*, vol.4, No.2 pp. 193-203, 1998.

[22] R. Redlinger. Existence of the global attractor for a strongly coupled parabolic system arising in population dynamics. *J. Diff. Eqns.*, 118:219–252, 1995.

[23] R. Schaff. Stationary solutions of chemotaxis systems. *Trans. Amer. Math. Soc.*, 292:531–556, 1985.

[24] N. Shigesada, K. Kawasaki and E. Teramoto. Spatial segregation of interacting species. *J. Theoretical Biology*, 79:83–99, 1979.

[25] R. Temam. *Infinite Dimensional Dynamical Systems in Mechanics and Physics* Springer-Verlag New York, 2nd edition (1998).

[26] A. Yagi. Global solution to some quasilinear parabolic system in population dynamics. *Nonlinear Analysis T.M.A.*, vol.21, no. 8, pp. 531–556, 1993.

[27] A. Yagi. A priori estimates for some quasilinear parabolic system in population dynamics. *Kobe J. Math.*, vol.14, no. 2, pp. 91–108, 1997.

[28] X. Wang. Qualitative behavior of solutions of a chemotactic diffusion system: Effects of motility and chemotaxis and dynamics. preprint, 1998.

Dung Le
Department of Applied Mathematics
University of Texas at San Antonio
6900 North Loop 1604 West
San Antonio, TX 78249, USA
Email: `dle@math.utsa.edu`