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Similarity of solution branches for two-point semilinear problems *

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Abstract

For semilinear autonomous two-point problems, we show that all Neumann branches and all Dirichlet branches with odd number of interior roots have the same shape. On the other hand, Dirichlet branches with even number of roots may look differently. While this result has been proved previously by Schaaf [2], our approach appears to be simpler.

1 Introduction

We are interested in positive, negative and sign-changing solution branches of the problem

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \tag{1.1}$$

subject to either Dirichlet or Neumann boundary conditions. Here λ is a positive parameter, and we are interested in how the solutions evolve with λ. Typically the problem (1.1) will have infinitely many solution branches, each with a different number of interior roots (or monotonicity changes). It turns out that these branches tend all to have similar shapes, and there is a correspondence between the Dirichlet and Neumann branches.

We begin with a simple "warm up" example. Let us assume that $f(u)$ is an odd function, and consider the problem (1.1) together with the Dirichlet conditions (2.2) below. Then all solution curves are similar, and moreover each sign-changing solution consists of self-similar parts. This follows from two simple observations. First, any solution of (1.1) is symmetric with respect to any of its critical points. Secondly, if u is a solution of (1.1) and (2.2), then so is $-u$. Hence, for any sign-changing solution all positive humps are identical up to translation, and they are identical to the negative humps via reflection across the x-axis. So that after rescaling, we identify any sign-changing solution with a positive one, at a different λ, and hence all solution curves have the same form. For example, consider $f(u) = u - u^3$. Then a familiar picture of parabola-like curves bifurcating to the right from the zero solution at $\lambda_k = k^2\pi^2$ (see e.g. [1]) follows immediately, since the same is true for positive solutions bifurcating

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at λ_1 (as can be easily seen using super- and subsolutions). Exactly the same picture, with the same justification, holds for the pendulum equation, when $f(u) = \sin u$.

It turns out that self-similarity holds for general $f(u)$. In particular, it always holds for Neumann problems. For the Dirichlet problems, we distinguish between the cases of odd and even number of interior roots. It turns out that all Neumann branches and all Dirichlet branches with odd number of interior roots have the same shape. On the other hand, Dirichlet branches with even number of roots may behave differently. These results were previously observed by R. Schaaf [2].

2 Similarity and self-similarity of Dirichlet and Neumann branches

We consider the problem

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \tag{2.1}$$

subject to either Dirichlet

$$u(0) = u(1) = 0, \tag{2.2}$$

or Neumann

$$u'(0) = u'(1) = 0 \tag{2.3}$$

boundary conditions. We shall consider the solutions as the positive parameter λ varies, and refer to the solution curves as either Dirichlet or Neumann branches, depending on the boundary conditions used. Observe that any solution of the equation (2.1) is symmetric with respect to any of its critical points. This implies, in particular, that either minimum or maximum occurs at any critical point. It follows that any solution of Neumann problem is determined by its values on any subinterval $I \subset (0,1)$, whose end-points are two consecutive critical points of $u(x)$. We can then obtain the solution on the entire interval $(0,1)$ through reflections and translations. We refer to I, and any other interval uniquely determining the solution through reflections and translations, as a *determining interval*. The interval I, joining two consecutive critical points of $u(x)$, is also a determining interval for the Dirichlet problem. Another determining interval for Dirichlet problem is (ξ, η), where $0 \le \xi < \theta < \eta \le 1$ are three consecutive roots of $u(x)$. This interval contains both positive and negative humps (and all positive (negative) humps are translations of one another).

The natural way to distinguish the Dirichlet branches is by the number of interior roots, and the Neumann branches can be identified by the number of changes of monotonicity (both properties clearly remain constant on the solution curves). Any solution of the Dirichlet problem with at least one interior root contains a solution of the Neumann problem on a subinterval of $(0,1)$. Indeed, just consider the solution between two consecutive critical points. In order for solutions of the Neumann problem to contain in turn a solution of the Dirichlet problem, we need to impose some conditions on $f(u)$. Namely, we assume that

$$f(0) = 0, \tag{2.4}$$

and there exist two constants $-\infty \le m < 0 < M \le \infty$ so that

(f1) $f(u) > 0$ for $u \in (0, M)$,

(f2) $f(u) < 0$ for $u \in (m, 0)$.

Lemma 2.1 *Under the conditions (2.4), (f1) and (f2) any solution of the Neumann problem for (2.1), satisfying*

$$m < u(x) < M \quad \textit{for all } x \tag{2.5}$$

has a root between any two critical points.

Proof: Follows immediately, by multiplying the equation (2.1) by u', and integrating between any two consecutive critical points. ♢

Definition. We call two solution branches of (2.1) to be *similar* if for any solution on the either branch there is a determining interval so that by stretching of x, or by reflection $x \to 2a - x$, for some $a \in (0, 1)$, we obtain a solution from the other branch on a (different) determining interval.

Clearly, if solution branches are similar then the corresponding solution curves in (λ, u) "plane" have the same shape.

Theorem 2.2 *All Neumann branches of (2.1) are similar, and if $f(u)$ satisfies the conditions (2.4), (f1) and (f2), while all solutions satisfy (2.5), then the Neumann branches are similar to the Dirichlet ones with an odd number of interior roots (and these Dirichlet branches are also all similar).*

Proof: We begin with Neumann branches. If a Neumann solution changes monotonicity twice, then its increasing part is a reflection of its decreasing part with respect to $x = \frac{1}{2}$. If a Neumann solution changes monotonicity n times, then all critical points occur at i/n, $i = 1, \dots, n-1$, and the graphs of solution on all intervals where it is increasing (decreasing) are translations of one another. Since an interval connecting any two critical points is a determining interval, the equivalence of the Neumann branches follows (via rescaling).

If a Dirichlet solution has $2k - 1$ interior roots, it has k identical positive humps and k identical negative humps. Assume for definiteness that solution starts with a negative hump, followed by a positive one, and so on. If ξ is the first point of (negative) minimum of $u(x)$, then the first interior root occurs at 2ξ. If $2\xi + \eta$ is the point of the first (positive) maximum, then the second interior root occurs at $2\xi + 2\eta$. The last critical point, a positive maximum, occurs at $1 - \eta$. Observe that $k(2\xi + 2\eta) = 1$, i.e. $\xi + \eta = \frac{1}{2k}$. So while both ξ

and η vary with λ, $u(x)$ solves the Neumann problem on the interval $(\xi, 1-\eta)$, and this interval has a *fixed* length of

$$1-\eta-\xi=\frac{2k-1}{2k}.$$

So that any Dirichlet solution curve "carries" inside it a solution of a Neumann problem on a fixed interval (which can be made to be $(0,1)$ by rescaling), and hence the Dirichlet branch cannot have any more complexity (like extra turns) than any Neumann branch.

Conversely, consider the Neumann problem with $2k+1$ changes of monotonicity. Assume for definiteness that $u(0)<0$. Then $u(1)>0$. Assume that $\xi=\xi(\lambda)$ is the smallest interior root, and $1-\eta$ is the largest one, $\eta=\eta(\lambda)$. On the interval $(0,1)$ we then have $2k+1$ negative half-humps, each of width ξ, and $2k+1$ positive ones, each of width η. So that $\xi+\eta=\frac{1}{2k+1}$. On the interval $(\xi, 1-\eta)$ we have a solution of the Dirichlet problem with $2k-1$ interior roots, and the length of this interval is

$$1-\eta-\xi=\frac{2k}{2k+1},$$

which does not vary with λ. So that any Neumann branch "carries" inside it a solution of a Dirichlet problem on a fixed interval, and hence the Neumann branch cannot have any more complexity than the corresponding Dirichlet branch with an odd number of interior zeroes.

Finally, the Dirichlet branches with odd number of interior zeroes are all similar, since any two such branches are similar to a pair of Neumann branches, but Neumann branches are all similar. $\diamondsuit$

The Dirichlet branches with even number of interior zeroes may behave differently, as the following example due to Schaaf [2] shows.

Example ([2]) For the problem

$$u''+\lambda(e^u-1)=0 \quad \text{on } (0,1), \quad u(0)=u(1)=0$$

the branch bifurcating from the principal eigenvalue does not turn, while all other branches have exactly one turn.

References

[1] S.-N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag (1982).

[2] R. Schaaf, Global Solution Branches of Two Point Boundary Value Problems, Lecture Notes in Mathemathics, no. **1458**, Springer-Verlag (1990).

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