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A selfadjoint hyperbolic boundary-value problem *

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Abstract

We consider the eigenvalue wave equation

$$u_{tt} - u_{ss} = \lambda p u,$$

subject to $u(s,0) = 0$, where $u \in \mathbb{R}$, is a function of $(s,t) \in \mathbb{R}^2$, with $t \geq 0$. In the characteristic triangle $T = \{(s,t) : 0 \leq t \leq 1, t \leq s \leq 2-t\}$ we impose a boundary condition along characteristics so that

$$\alpha u(t,t) - \beta \frac{\partial u}{\partial n_1}(t,t) = \alpha u(1+t, 1-t) + \beta \frac{\partial u}{\partial n_2}(1+t,1-t), \quad 0 \leq t \leq 1.$$

The parameters α and β are arbitrary except for the condition that they are not both zero. The two vectors n_1 and n_2 are the exterior unit normals to the characteristic boundaries and $\frac{\partial u}{\partial n_1}$, $\frac{\partial u}{\partial n_2}$ are the normal derivatives in those directions. When $p \equiv 1$ we will show that the above characteristic boundary value problem has real, discrete eigenvalues and corresponding eigenfunctions that are complete and orthogonal in $L_2(T)$. We will also investigate the case where $p \geq 0$ is an arbitrary continuous function in T.

1 Introduction

Consider the wave equation

$$u_{tt} - u_{ss} = \lambda p u, \quad (s,t) \in T := \{(s,t) : 0 \leq t \leq 1, t \leq s \leq 2-t\}, \tag{1.1}$$

where λ is a parameter, and $p \geq 0$ is a continuous function in T. We impose the boundary conditions,

$$u(s,0) = 0, \quad 0 \leq s \leq 2, \tag{1.2}$$

$$\alpha u(t,t) - \beta \frac{\partial u}{\partial n_1}(t,t) = \alpha u(1+t,1-t) + \beta \frac{\partial u}{\partial n_2}(1+t,1-t), \;\; 0 \leq t \leq 1, \tag{1.3}$$

where the parameters α and β are arbitrary with $\alpha^2 + \beta^2 \neq 0$. The two vectors n_1 and n_2 are the exterior unit normals to the characteristic boundaries

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and $\frac{\partial u}{\partial n_1}$, $\frac{\partial u}{\partial n_2}$ are the normal derivatives in those directions. Our problem is a generalization of the problems studied by Kalmenov [6], and Kreith [7], where they consider the boundary conditions,

$$\begin{aligned} u(s,0) &= 0 = u(1,1), \quad 0 \le s \le 2 \\ u(t,t) &= u(1+t,1-t), \quad 0 \le t \le 1. \end{aligned}$$

We note two facts. First, if we only prescribe the values of u along the characteristics, say, $u(t,t) = f_1(t)$ and $u(1+t,1-t) = f_2(t)$ then we have a classical characteristic initial value problem,(see, e.g., Garabedian [1]) and equation (1.1) will have a solution for all values of λ. However, the conditions (1.2), (1.3) provide a boundary value problem with spectral properties. Second, if we set $\beta = 0$ in condition (1.3) we have the case of Kreith [7] . If in addition, we set $p \equiv 1$ we have the case of Kalmenov [6]. As a start we will let $p \equiv 1$ and use the technique of Kalmenov [6] to show that the equation, $u_{tt} - u_{ss} = \lambda u, \quad (s,t) \in T$ subject to conditions (1.2), (1.3), is selfadjoint. Then we will study equation (1.1) subject to conditions (1.2), (1.3) by converting the problem into a nonhomogeneous eigenvalue integral equation and using the method of the Fredholm alternative. In this case we will assume that u along the characteristics is given. This will not weaken the problem because we still require that condition (1.2) be satisfied. As a result we will see that not all values of λ will produce a solution.

We would like to add here that the investigation into the spectral properties of the characteristic initial value problems for the wave equation has been conducted in different directions by several authors. In this context, beside the above mentioned references, the works of Haws [2], Kreith [8, 9] and the author [3, 4, 5] are the ones most closely related to the present work.

2 The selfadjoint problem

Consider the equation

$$u_{tt} - u_{ss} = \lambda u, \quad (s,t) \in T, \tag{2.1}$$

subject to the conditions (1.2), (1.3). Extend u as an odd function of t and define,

$$\tilde{u}(s,t) = \begin{cases} u(s,t) & \text{if } t > 0, \quad (s,t) \in T \\ -u(s,-t) & \text{if } t < 0, \quad (s,-t) \in T \end{cases}$$

and reflect T in $t = 0$ axis to obtain the rectangle

$$R = \{(s,t) : |t| \le 1,\ |t| \le s \le 2 - |t|\}. \tag{2.2}$$

Now $\tilde{u}$ must satisfy the following conditions:

$$\tilde{u}_{tt} - \tilde{u}_{ss} = \lambda \tilde{u}, \quad (s,t) \in R, \tag{2.3}$$

$$\alpha \tilde{u}(t,t) - \beta \frac{\partial \tilde{u}}{\partial n_1}(t,t) = \alpha \tilde{u}(1+t,1-t) + \beta \frac{\partial \tilde{u}}{\partial n_2}(1+t,1-t), \quad |t| \le 1, \tag{2.4}$$

$$\tilde{u}(s,t) = -\tilde{u}(s,-t), \quad (s,t) \in R. \tag{2.5}$$

Using the change of variables, $x = s - t$, $y = s + t$ and denoting $\tilde{u}(s,t) = \tilde{u}(\frac{x+y}{2}, \frac{y-x}{2})$ by $\tilde{U}(x,y)$, the rectangle R maps into $S = \{(x,y) : 0 \le x \le 2, 0 \le y \le 2\}$, and we have,

$$-4\tilde{U}_{x,y} = \lambda \tilde{U}, \quad (x,y) \in S, \tag{2.6}$$
$$\alpha\tilde{U}(0,y) + \beta\tilde{U}_x(0,y) = \alpha\tilde{U}(y,2) + \beta\tilde{U}_y(y,2), \tag{2.7}$$
$$\tilde{U}(x,y) = -\tilde{U}(y,x). \tag{2.8}$$

From this equation, we have $\tilde{U}_x(x,y) = -\tilde{U}_y(y,x)$; therefore,

$$\tilde{U}(0,y) = -\tilde{U}(y,0) \tag{2.9}$$
$$\tilde{U}_x(0,y) = -\tilde{U}_y(y,0). \tag{2.10}$$

These two identities change the boundary condition (2.7) to,

$$-\alpha\tilde{U}(y,0) - \beta\tilde{U}_y(y,0) = \alpha\tilde{U}(y,2) + \beta\tilde{U}_y(y,2). \tag{2.11}$$

Now let $\tilde{U}(x,y) = \phi(x)\psi(y)$, and plug it into equation (2.6), we have,

$$[2i\phi'(x)][2i\psi'(y)] = \lambda\phi(x)\psi(y) \tag{2.12}$$

which leads to

$$\frac{2i\phi'(x)}{\phi(x)} = \lambda\frac{\psi(y)}{2i\psi'(y)} = \mu,$$

which in turn leads to two ODE equations:

$$\psi'(y) = -\frac{i\lambda}{2\mu}\psi(y) = -\frac{i\delta}{2}\psi(y), \quad \delta = \frac{\lambda}{\mu}, \quad \mu \ne 0, \tag{2.13}$$
$$\phi'(x) = -\frac{i\mu}{2}\phi(x). \tag{2.14}$$

The effect of separation of variables on the boundary condition (2.11) will be

$$-\alpha\phi(y)\psi(0) - \beta\phi(y)\psi'(0) = \alpha\phi(y)\psi(2) + \beta\phi(y)\psi'(2). \tag{2.15}$$

Upon cancelling $\phi(y)$ here, we have the boundary-value problem

$$\psi'(y) = -\frac{i\delta}{2}\psi(y), \quad 0 \le y \le 2, \tag{2.16}$$
$$-\alpha\psi(0) - \beta\psi'(0) = \alpha\psi(2) + \beta\psi'(2). \tag{2.17}$$

We note that from (2.16) we have

$$\psi'(0) = -\frac{i\delta}{2}\psi(0), \tag{2.18}$$
$$\psi'(2) = -\frac{i\delta}{2}\psi(2). \tag{2.19}$$

These relations simplify the condition (2.17) to

$$\psi'(0) = -\psi(2), \tag{2.20}$$

The selfadjoint boundary-value problem (2.16), (2.20) is the same as the one in [6]. The eigenvalues and eigenfunctions of problem (2.16), (2.20) are

$$\delta = (-2n-1)\pi, \quad \psi_n(y) = \exp(\frac{(2n+1)\pi i y}{2}), \quad n = 0, \pm1, \pm2, \dots \tag{2.21}$$

Using (2.8) differently to write $\tilde{U}_y(x,y) = -\tilde{U}_x(y,x)$,

$$\tilde{U}(y,2) = -\tilde{U}(2,y) \tag{2.22}$$

$$\tilde{U}_y(y,2) = -\tilde{U}_x(2,y). \tag{2.23}$$

The substitution of (2.22), (2.23) into the boundary condition (2.7) yields

$$\alpha\tilde{U}(0,y) + \beta\tilde{U}_x(0,y) = -\alpha\tilde{U}(2,y) - \beta\tilde{U}_x(2,y). \tag{2.24}$$

If we again use the separation of variables $\tilde{U}(x,y) = \phi(x)\psi(y)$ in the equation (2.24) and proceed as for the case of the function $\psi(y)$, we will have

$$\phi'(x) = -\frac{i\mu}{2}\phi(x), \quad 0 \le x \le 2, \tag{2.25}$$

$$-\alpha\phi(0) - \beta\phi'(0) = \alpha\phi(2) + \beta\phi'(2). \tag{2.26}$$

The condition (2.26) again can be replaced with

$$\phi'(0) = -\phi(2), \tag{2.27}$$

The eigenvalues and eigenfunctions of problem (2.25), (2.27) are

$$\mu = (-2m-1)\pi, \quad \phi_m(y) = \exp(\frac{(2m+1)\pi i x}{2}), \quad m = 0, \pm1, \pm2, \dots \tag{2.28}$$

Both sets of eigenfunctions $\{\psi_n(y)\}$ and $\{\phi_m(x)\}$ are complete and orthogonal in $L_2(0,2)$. The eigenfunctions and the eigenvalues of (2.6)–(2.8) will be

$$\begin{gathered} \lambda_{mn} = (2m+1)(2n+2)\pi^2, \\ \tilde{U}_{mn}(x,y) = \exp(\frac{(2m+1)x}{2} + \frac{(2n+1)y}{2})\pi i, \quad m,n = 0, \pm1, \pm2, \dots \end{gathered} \tag{2.29}$$

To find the eigenfunctions of the problem (1.1)–(1.3), we introduce the functions

$$\begin{aligned} u_{mn}(x,y) &= \tilde{U}_{mn}(x,y) - \tilde{U}_{mn}(y,x) \\ &= \exp(\frac{(2m+1)x}{2} + \frac{(2n+1)y}{2})\pi i \\ &\quad - \exp(\frac{(2m+1)y}{2} + \frac{(2n+1)x}{2})\pi i, \end{aligned} \tag{2.30}$$

for $m,n = 0, \pm1, \pm2, \dots$. The set $\{u_{mn}(x,y)\}$ is complete and orthogonal in $L_2(T')$ where $T' = \{(x,y) : 0 \le y \le 2, 0 \le x \le y\}$.

Theorem 2.1 *Problem* (1.1)–(1.3) *when* $p \equiv 1$ *is selfadjoint. It has eigenvalues* (2.29) *and eigenfunctions* (2.30). *The eigenfunctions are complete and orthogonal in* $L_2(T)$.

3 The non-selfadjoint problem

Now consider the problem (1.1)–(1.3) and make the change of variables $x = s - t$ and $y = s + t$, to obtain

$$U_{xy} = \gamma P(x,y)U(x,y), \quad (x,y) \in T', \tag{3.1}$$

$$T' = \{(x,y) : 0 \le y \le 2, 0 \le x \le y\} \tag{3.2}$$

$$\alpha U(0,y) + \beta U_x(0,y) = \alpha U(y,2) + \beta U_y(y,2), \quad 0 \le y \le 2 \tag{3.3}$$

$$U(x,x) = 0, \quad 0 \le x \le 2, \tag{3.4}$$

where $U(x,y) = u(\frac{x+y}{2}, \frac{y-x}{2})$, $P(x,y) = p(\frac{x+y}{2}, \frac{y-x}{2})$, and $\gamma = -\lambda/4$. Integrating (3.1) in T' from 0 to ξ, we have

$$U_y(\xi,y) - U_y(0,y) = \gamma \int_0^{\xi} PU\,dx. \tag{3.5}$$

The above equation when $\xi = y$ and $y = 2$ will be

$$U_y(y,2) - U_y(0,2) = \gamma \int_0^{y} P(x,2)U(x,2)\,dx. \tag{3.6}$$

Integrating equation (3.1) in T' from y to η, we have

$$U_x(x,\eta) - U_x(x,y) = \gamma \int_y^{\eta} PU\,dy.$$

When $x = 0$ and $\eta = 2$ this equation becomes

$$U_x(0,2) - U_x(0,y) = \gamma \int_y^{2} P(0,y)U(0,y)\,dy. \tag{3.7}$$

In (3.5) let ξ lie on the line $y = x$ and integrate the equation from ξ to η in T',

$$U(\xi,\eta) - U(\xi,\xi) - U(0,\eta) + U(0,\xi) = \gamma \int_{\xi}^{\eta} \int_0^{\xi} PU dxdy. \tag{3.8}$$

Since $U(\xi,\xi) = 0$ by the boundary condition (3.4), we have

$$U(\xi,\eta) = U(0,\eta) - U(0,\xi) + \gamma \int_{\xi}^{\eta} \int_0^{\xi} PU\,dx\,dy\,. \tag{3.9}$$

From (3.6) when y is replaced with η we have

$$U_y(\eta,2) = U_y(0,2) + \gamma \int_0^{\eta} P(x,2)U(x,2)\,dx. \tag{3.10}$$

From (3.7), when y is replaced with η we have

$$U_x(0,\eta) = U_x(0,2) - \gamma \int_{\eta}^{2} P(0,y)U(0,y)\,dy. \tag{3.11}$$

Finally from equation (3.9) we have,

$$U(\eta,2)=U(0,2)-U(0,\eta)+\gamma\int_\eta^2\int_0^\eta PU\,dx\,dy\,. \tag{3.12}$$

Now, we substitute the right hand side of the equations (3.10), (3.11), (3.12) into the boundary condition (3.3) with y replaced with η,

$$\begin{aligned}&\alpha U(0,\eta)+\beta(U_x(0,2)-\gamma\int_\eta^2 P(0,y)U(0,y)dy)\\&=\alpha(U(0,2)-U(0,\eta)+\gamma\int_\eta^2\int_0^\eta PU\,dx\,dy)+\beta(U_y(0,2)\\&\quad+\gamma\int_0^\eta P(x,2)U(x,2)dx).\end{aligned}$$

Placing $\alpha U(0,\eta)$ in right-hand side and combining,

$$\begin{aligned}&2\alpha U(0,\eta)+\beta(U_x(0,2)+\gamma\int_\eta^2 P(0,y)U(0,y)dy)\\&=\alpha(U(0,2)+\gamma\int_\eta^2\int_0^\eta PU\,dx\,dy)+\beta(U_y(0,2)\\&\quad+\gamma\int_0^\eta P(x,2)U(x,2)dx).\end{aligned} \tag{3.13}$$

Rewrite this equation with η replaced with ξ,

$$\begin{aligned}&2\alpha U(0,\xi)+\beta(U_x(0,2)+\gamma\int_\xi^2 P(0,y)U(0,y)dy)\\&=\alpha(U(0,2)+\gamma\int_\xi^2\int_0^\xi PU\,dx\,dy)+\beta(U_y(0,2)\\&\quad+\gamma\int_0^\xi P(x,2)U(x,2)dx).\end{aligned} \tag{3.14}$$

Subtract equation (3.14) from (3.13),

$$\begin{aligned}&2\alpha(U(0,\eta)-U(0,\xi))+\beta\gamma(\int_\eta^2 P(0,y)U(0,y)dy-\int_\xi^2 P(0,y)U(0,y)dy)\\&=\alpha\gamma(\int_\eta^2\int_0^\eta PU dxdy-\int_\xi^2\int_0^\xi PU\,dx\,dy)\\&\quad+\beta\gamma(\int_0^\eta P(x,2)U(x,2)dx-\int_0^\xi P(x,2)U(x,2)dx).\end{aligned} \tag{3.15}$$

Solve for $U(0,\eta)-U(0,\xi)$ in (3.15), assuming $\alpha \neq 0$, and substitute in (3.9),

$$\begin{aligned}
U(\xi,\eta) = &-\frac{\beta}{2\alpha}\gamma\Big(\int_{\xi}^{2} P(0,y)U(0,y)dy - \int_{\eta}^{2} P(0,y)U(0,y)dy\Big)\\
&+\frac{\gamma}{2}\Big(\int_{\eta}^{2}\int_{0}^{\eta} PU\,dx\,dy - \int_{\xi}^{2}\int_{0}^{\xi} PU\,dx\,dy\Big)\\
&+\frac{\beta}{2\alpha}\gamma\Big(\int_{0}^{\eta} P(x,2)U(x,2)dx - \int_{0}^{\xi} P(x,2)U(x,2)dx\Big)\\
&+\gamma\int_{\xi}^{\eta}\int_{0}^{\xi} PU\,dx\,dy\,.
\end{aligned} \tag{3.16}$$

Rewrite (3.16) in a compact form using the Green's function, $G(\xi,\eta;x,y)$ described in Figure 1,

$$\begin{aligned}
U(\xi,\eta) = &\gamma\Big(\int\int_{T'} G(\xi,\eta;x,y)PU(x,y)\,dx\,dy\\
&-\int_{0}^{2} g(\xi,\eta,x)(P(0,x)U(0,x)-P(x,2)U(x,2))dx\Big),
\end{aligned} \tag{3.17}$$

where

$$g(\xi,\eta,x) = \begin{cases} 0 & \text{if } 0 \le x \le \xi\\ \beta/(2\alpha) & \text{if } \xi \le x \le \eta\\ 0 & \text{if } \eta \le x \le 2 \end{cases}$$

Now, let G be the operator,

$$G[U] = \int\int_{T'} G(\xi,\eta;x,y)PU(x,y)\,dx\,dy \tag{3.18}$$

defined on the Hilbert space of weighted square integrable functions $H = L_2^P(T')$. Assume the function U along the characteristics $x=0$ and $y=0$ in T' is given, and denote,

$$f(\xi,\eta) = \int_{0}^{2} g(\xi,\eta,x)(P(0,x)U(0,x)-P(x,2)U(x,2))dx. \tag{3.19}$$

Also, let $\Gamma = 1/\gamma$ when $\gamma \neq 0$. Then, (3.13) can be written as

$$G[U] = \Gamma U + f. \tag{3.20}$$

The operator G in (3.18), is the same as the one in [7], where it is shown to be selfadjoint in H. Denote the normalized eigenfunctions of G by E_k, and the inner product in H by $\langle .,. \rangle$. Using the standard Fredholm alternative [10] we have the following theorem.

Theorem 3.1 *Let Γ_k be the eigenvalues of $GU = \Gamma U$, where by the selfadjointness of G, are real and satisfy $|\Gamma_k| \to 0$, as $k \to \infty$ then, we have*

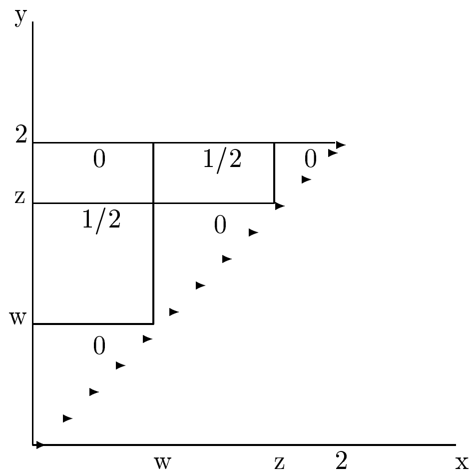

Figure 1: The Green's function $G(w, z; x, y)$

1. $\Gamma \neq \Gamma_k$ *for any integer* k. *A unique solution of* (3.20) *exists and is given in the form* $U = -\frac{f}{\Gamma} + \sum_k \frac{\Gamma_k}{\Gamma(\Gamma_k - \Gamma)} \langle f, E_k \rangle E_k$.

2. $\Gamma = \Gamma_m$, *one of the eigenvalues of* G, *and* Γ_m *is not degenerate. If* $\langle f, E_m \rangle \neq 0$ *the equation* (3.20) *has no solution. If* $\langle f, E_m \rangle = 0$ *then,* (3.20) *has infinitely many solutions*

$$U = -\frac{f}{\Gamma_m} + \sum_{k \neq m} \frac{\Gamma_k}{\Gamma_m(\Gamma_k - \Gamma_m)} \langle f, E_k \rangle E_k + cE_m,$$

with c *an arbitrary constant.*

3. Γ_m *is degenerate,* $\Gamma_{m_1} = \Gamma_{m_2} = \cdots = \Gamma_{m_j}$, *for successive indices* m_1, $m_2, \ldots m_j$, *with some* $m_i = m$, $i = 1, 2, \ldots j$, *where* j *is the multiplicity of* Γ_m. *Then, unless* $\langle f, E_i \rangle = 0$, $i = 1, 2, \ldots, j$, *the equation* (3.20) *has no solution. If however, these* j *solvability conditions are satisfied, the solution can be represented by,*

$$U = -\frac{f}{\Gamma_m} + \sum_{k \neq m_i} \frac{\Gamma_k}{\Gamma_m(\Gamma_k - \Gamma_m)} \langle f, E_k \rangle E_k + \sum_{i=1}^{j} c_i E_{m_i},$$

where c_i*'s are arbitrary constants.*

Remark When $p \equiv 1$ in problem (1.1)–(1.3), the problem is selfadjoint if $\alpha^2 + \beta^2 \neq 0$. When p is not necessarily 1, the case $\alpha \neq 0, \beta = 0$, reduces the problem to the selfadjoint problem of [7]. When $\alpha \neq 0, \beta \neq 0$, Theorem 2 holds. When $\alpha = 0$, no conclusion can be drawn.

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