Fifth Mississippi State Conference on Differential Equations and Computational Simulations,
Electronic Journal of Differential Equations, Conference 10, 2003, pp 143–152.
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A non-resonant multi-point boundary-value problem for a p-Laplacian type operator *

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Abstract

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ with $\phi(0) = 0$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and $e(t) \in L^1[0,1]$. Let $\xi_i \in (0,1)$, $a_i \in \mathbb{R}$, $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2} a_i \neq 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ be given. This paper is concerned with the problem of existence of a solution for the multi-point boundary-value problem

$$
\begin{gathered}
(\phi(x'(t)))' = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\
x(0) = 0, \quad \phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i)).
\end{gathered}
$$

This paper gives conditions for the existence of a solution for the above boundary-value problem using some new Poincaré type a priori estimates. In the case $\phi(t) \equiv t$ for $t \in \mathbb{R}$, this problem was studied earlier by Gupta, Trofimchuk in [2] and by Gupta, Ntouyas and Tsamatos in [1]. We give a priori estimates needed for this problem that are similar to a priori estimates obtained by Gupta, Trofimchuk in [2]. We then obtain existence theorems for the above multi-point boundary-value problem using the a priori estimates and Leray-Schauder continuation theorem.

1 Introduction

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ with $\phi(0) = 0$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions, $e : [0,1] \mapsto \mathbb{R}$ be a function in $L^1[0,1]$, $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2} a_i \neq 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ be given. We study the problem of existence of solutions for the m-point boundary-value problem

$$
\begin{gathered}
(\phi(x'(t)))' = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\
x(0) = 0, \quad \phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i)).
\end{gathered}
\tag{1.1}
$$

* *Mathematics Subject Classifications:* 34B10, 34B15, 34G20.
Key words: Multi-point boundary-value problem, three-point boundary-value problem, p-Laplacian, Leray Schauder Continuation theorem, Caratheodory's conditions.
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Published February 28, 2003.

This problem was studied earlier in the case $\phi(t) \equiv t$ for $t \in \mathbb{R}$, by Gupta, Trofimchuk in [2] and by Gupta, Ntouyas and Tsamatos in [1]. Gupta, Ntouyas and Tsamatos had studied the problem (1.1) when all of the $a_i \in \mathbb{R}$, $i = 1, 2, \dots, m-2$, had the same sign by first studying the three-point boundary-value problem, for a given $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $\eta \in (0,1)$,

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = \alpha x'(\eta), \end{gathered} \tag{1.2}$$

while Gupta, Trofimchuk in [2] studied the problem (1.1) when the $a_i \in \mathbb{R}$, $i = 1, 2, \dots, m-2$, do not necessarily have the same sign.

We also study the three-point boundary-value problem analogue of (1.1), for a given $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $\eta \in (0,1)$,

$$\begin{gathered} (\phi(x'(t)))' = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \quad \phi(x'(1)) = \alpha\phi(x'(\eta)). \end{gathered} \tag{1.3}$$

The purpose of this paper is to obtain conditions for the existence of a solution for the boundary-value problem (1.1), using new estimates and inequalities involving a function $x(t)$, its derivative $x'(t)$, the functions $\phi(x'(t))$ and its derivative $(\phi(x'(t)))'$. These results are motivated by the so called *nonlocal* boundary-value problem studied by Il'in and Moiseev in [5]. We may mention that the reason for studying the three-point boundary-value problem (1.3) is that in this case we obtain a better existence theorem using a priori estimates involving L_2 norm.

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev spaces $W^{2,k}_\phi(0,1)$, $k = 1, 2$ defined by

$$W^{2,k}_\phi(0,1) = \{x : [0,1] \to R : x, x' \text{ abs. cont. on } [0,1], (\phi(x'(t)))' \in L^k[0,1]\}$$

with its usual norm. We denote the norm in $L^k[0,1]$ by $\|\cdot\|_k$, and the norm in $L^\infty[0,1]$ by $\|\cdot\|_\infty$.

2 A Priori Estimates

Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$ be given. Let $x(t) \in W^{2,1}_\phi(0,1)$ be such that $x(0) = 0$, $\phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i))$ be given. We are interested in obtaining a priori estimates of the form $\|\phi(x'(t))\|_\infty \leq C\|(\phi(x'(t)))'\|_1$. The following theorem gives such an estimate. We recall that for $a \in \mathbb{R}$, $a_+ = \max\{a, 0\}$, $a_- = \max\{-a, 0\}$ so that $a = a_+ - a_-$ and $|a| = a_+ + a_-$.

Theorem 2.1 *Let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$ be given. Then for $x(t) \in W_\phi^{2,1}(0,1)$ with $x(0) = 0$, $\phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i))$ we have*

$$\|\phi(x'(t))\|_\infty \leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_1 \tag{2.1}$$

where either $\tau = 0$ or

$$\tau = \min\Big\{\frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_- + 1}, \frac{\sum_{i=1}^{m-2}(a_i)_- + 1}{\sum_{i=1}^{m-2}(a_i)_+}\Big\}.$$

Proof We see that the assumption $\phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i))$ implies

$$\phi(x'(1)) + \sum_{i=1}^{m-2}(a_i)_-\phi(x'(\xi_i)) = \sum_{i=1}^{m-2}(a_i)_+\phi(x'(\xi_i))$$

and thus there exist λ_1, $\lambda_2 \in [0,1]$ such that

$$\big(1 + \sum_{i=1}^{m-2}(a_i)_-\big)\phi(x'(\lambda_1)) = \sum_{i=1}^{m-2}(a_i)_+\phi(x'(\lambda_2)). \tag{2.2}$$

If, now, either $x'(\lambda_1) = 0$ or $x'(\lambda_2) = 0$, so that either $\phi(x'(\lambda_1)) = 0$ or $\phi(x'(\lambda_2)) = 0$, then we clearly have

$$\|\phi(x'(t))\|_\infty \leq \|(\phi(x'(t)))'\|_1. \tag{2.3}$$

Suppose, now, that $x'(\lambda_1) \neq 0$ and $x'(\lambda_2) \neq 0$, so that $\phi(x'(\lambda_1)) \neq 0$ and $\phi(x'(\lambda_2)) \neq 0$. It then follows easily from equation (2.2) that $\phi(x'(\lambda_1)) \neq \phi(x'(\lambda_2))$, in view of the assumption $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$. It then follows from equation (2.2), the estimate (2.3) and the equations

$$\begin{aligned}\phi(x'(t)) &= \phi(x'(\lambda_1)) + \int_{\lambda_1}^t (\phi(x'(s)))'ds,\\ \phi(x'(t)) &= \phi(x'(\lambda_2)) + \int_{\lambda_2}^t (\phi(x'(s)))'ds,\end{aligned}$$

that

$$\|\phi(x'(t))\|_\infty \leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_1$$
$$\text{with } \tau = \min\{\frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_- + 1}, \frac{\sum_{i=1}^{m-2}(a_i)_- + 1}{\sum_{i=1}^{m-2}(a_i)_+}\}.$$

This completes the proof of the theorem. □

Remark We note that if $a_i \leq 0$ for every $i = 1, 2, \ldots, m-2$, then $\tau = 0$. Also, if $a_i \geq 0$ for every $i = 1, 2, \ldots, m-2$ so that $\alpha = \sum_{i=1}^{m-2} a_i = \sum_{i=1}^{m-2} (a_i)_+ \geq 0$ then $\tau = \min\{\alpha, 1/\alpha\} \in [0, 1)$ since $\alpha \neq 1$, by assumption.

The following theorem gives a better estimate for the three-point boundary-value problem in the case of the L^2 norm.

Theorem 2.2 *Let $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $\eta \in (0,1)$ be given. Let $x(t) \in W_{\phi}^{2,2}(0,1)$ be such that $\phi(x'(1)) = \alpha\phi(x'(\eta))$. Then*

$$\|\phi(x'(t))\|_2 \leq C(\alpha,\eta)\|(\phi(x'(t)))'\|_2, \tag{2.4}$$

where

$$C(\alpha,\eta) = \begin{cases} \min\{\sqrt{F(\alpha,\eta)}, 2/\pi\} & \text{if } \alpha \leq 0, \\ \sqrt{F(\alpha,\eta)}, & \text{if } \alpha > 0, \end{cases}$$
$$F(\alpha,\eta) = \frac{1}{2(\alpha-1)^2}[\alpha^2(1-\eta)^2 + (\alpha^2 - 2\alpha)\eta^2 + 1].$$

Proof If $\alpha \leq 0$, we note from $\phi(x'(1)) = \alpha\phi(x'(\eta))$ that there exists an $\xi \in (\eta, 1)$ such that $\phi(x'(\xi)) = 0$. It follows from the Wirtinger's inequality [3, Theorem 256] that

$$\|\phi(x'(t))\|_2 \leq \frac{2}{\pi}\|(\phi(x'(t)))'\|_2. \tag{2.5}$$

Next, we note, again, from $\phi(x'(1)) = \alpha\phi(x'(\eta))$ that for $0 < t < 1$,

$$\phi(x'(t)) = \int_0^t (\phi(x'(s)))'ds + \frac{\alpha}{1-\alpha}\int_0^\eta (\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_0^1 (\phi(x'(s)))'ds. \tag{2.6}$$

Accordingly, we have for $t \in [0, \eta]$

$$\begin{aligned} \phi(x'(t)) &= \int_0^t (\phi(x'(s)))'ds + \frac{\alpha}{1-\alpha}\int_0^\eta (\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_0^1 (\phi(x'(s)))'ds \\ &= \int_0^t (1 + \frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha})(\phi(x'(s)))'ds \\ &\quad + \int_t^\eta (\frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha})(\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_\eta^1 (\phi(x'(s)))'ds \\ &= -\int_t^\eta (\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_\eta^1 (\phi(x'(s)))'ds, \end{aligned} \tag{2.7}$$

and, for $t \in [\eta, 1]$

$$\begin{aligned}\phi(x'(t)) &= \int_0^t (\phi(x'(s)))'ds + \frac{\alpha}{1-\alpha}\int_0^\eta (\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_0^1 (\phi(x'(s)))'ds \\ &= \int_0^\eta (1 + \frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha})(\phi(x'(s)))'ds + \int_\eta^t (1 - \frac{1}{1-\alpha})(\phi(x'(s)))'ds \\ &\quad -\frac{1}{1-\alpha}\int_t^1 (\phi(x'(s)))'ds \\ &= -\int_\eta^t \frac{\alpha}{1-\alpha}(\phi(x'(s)))'ds - \frac{1}{1-\alpha}\int_t^1 (\phi(x'(s)))'ds. \end{aligned} \tag{2.8}$$

Let us, now, define a function $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(t,s) = \begin{cases} -\chi_{[t,\eta]}(s) - \frac{1}{1-\alpha}\chi_{[\eta,1]}(s), & \text{for } t \in [0,\eta], s \in [0,1], \\ -\frac{\alpha}{1-\alpha}\chi_{[\eta,t]}(s) - \frac{1}{1-\alpha}\chi_{[t,1]}(s), & \text{for } t \in [\eta,1], s \in [0,1]. \end{cases} \tag{2.9}$$

Now, we see from equations (2.7), (2.8) that

$$\phi(x'(t)) = \int_0^1 K(t,s)(\phi(x'(s)))'ds, \quad \text{for } t \in [0,1], \tag{2.10}$$

and

$$\|\phi(x'(t))\|_2^2 \le (\int_0^1 \int_0^1 (K(t,s))^2\, ds\, dt)\|(\phi(x'(s)))'\|_2^2. \tag{2.11}$$

Now, it is easy to see that

$$\int_0^1 \int_0^1 (K(t,s))^2\, ds\, dt = \frac{1}{2(\alpha-1)^2}[\alpha^2(1-\eta)^2 + (\alpha^2 - 2\alpha)\eta^2 + 1]. \tag{2.12}$$

For $\alpha \le 0$ the estimate (2.4) is now immediate from (2.5), (2.11), (2.12) and for $\alpha > 0$, $\alpha \ne 1$, by (2.11), (2.12). This completes the proof of the theorem. $\square$

Remark It is easy to see that $C(-0.1,\eta) = 2/\pi$, for all $\eta \in (0,1)$. Indeed, $\sqrt{F(-0.1,\eta)} \ge 0.648986183$ and $2/\pi \approx 0.6366197724$. Also $C(-2,1/3) = \sqrt{11/54}$ and $C(-2,15/16) = 2/\pi$, since $\sqrt{F(-2,15/16)} = \sqrt{1030}/48 > 2/\pi$.

3 Existence Theorems

Definition A function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Caratheodory's conditions if

(i) For each $(x,y) \in \mathbb{R}^2$, the function $t \in [0,1] \to f(t,x,y) \in \mathbb{R}$ is measurable on $[0,1]$

(ii) for a.e. $t \in [0,1]$, the function $(x,y) \in \mathbb{R}^2 \to f(t,x,y) \in \mathbb{R}$ is continuous on $\mathbb{R}^2$

(iii) for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $|f(t,x,y)| \le \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x,y) \in \mathbb{R}^2$ with $\sqrt{x^2+y^2} \le r$.

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,x_1,x_2)| \le p(t)\phi(|x_1|) + q(t)\phi(|x_2|) + r(t) \tag{3.1}$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in \mathbb{R}^2$. Also let $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \ne 1$ be given. Then the boundary-value problem (1.1) has at least one solution in $C^1[0,1]$ provided

$$\|p(t)\|_1 + \|q(t)\|_1 + \tau < 1. \tag{3.2}$$

where τ is as defined in Theorem 2.1.

Proof It is easy to see that the boundary-value problem (1.1) is equivalent to the fixed point problem

$$x(t) = \int_0^t \phi^{-1}\Big(\int_0^s [f(\tau,x(\tau),x'(\tau)) + e(\tau)]d\tau + A\Big)ds, \tag{3.3}$$

where

$$\begin{aligned} A \;&=\; \sum_{i=1}^{m-2}\Big(\frac{a_i}{1-\sum_{i=1}^{m-2} a_i}\Big)\int_0^{\xi_i} [f(\tau,x(\tau),x'(\tau)) + e(\tau)]d\tau \\ &\quad -\frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 [f(\tau,x(\tau),x'(\tau)) + e(\tau)]d\tau. \end{aligned}$$

It is standard to check that the mapping

$$x(t) \in C^1[0,1] \mapsto \int_0^t \phi^{-1}\Big(\int_0^s [f(\tau,x(\tau),x'(\tau)) + e(\tau)]d\tau + A\Big)ds \in C^1[0,1],$$

is a compact mapping. We apply the Leray-Schauder Continuation theorem (see, e.g. [4]) to obtain the existence of a solution for (3.3) or equivalently to the boundary-value problem (1.1).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$\begin{gathered} (\phi(x'(t)))' = \lambda f(t,x(t),x'(t)) + \lambda e(t), \quad 0 < t < 1, \\ x(0) = 0, \quad \phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i)), \end{gathered} \tag{3.4}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. We observe that if $x \in W^{2,1}_{\phi}(0,1)$, with $x(0) = 0$, $\phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i))$ then $x(t) = \int_0^t x'(s)ds$. Hence, $|x(t)| \le \|x'\|_\infty$ for $t \in [0,1]$ and

$$\|\phi(x'(t))\|_\infty \le \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_1,$$

where τ is as defined in Theorem 2.1. Also, it is easy to see that $\phi(\|x'\|_\infty) \leq \|\phi(x'(t))\|_\infty$.

Let, now, $x(t)$ be a solution of (3.4) for some $\lambda \in [0,1]$, so that $x \in W_\phi^{2,1}(0,1)$ with $x(0) = 0, \phi(x'(1)) = \sum_{i=1}^{m-2} a_i \phi(x'(\xi_i))$. We then get from the equation in (3.4) and Theorem 2.1 t hat

$$\begin{aligned}
\|\phi(x'(t))\|_\infty &\leq \frac{\lambda}{1-\tau}\|f(t,x(t),x'(t)) + e(t)\|_1 \\
&\leq \frac{1}{1-\tau}(\|p(t)\phi(|x(t)|) + q(t)\phi(|x'(t)|) + r(t)\|_1 + \|e(t)\|_1) \\
&\leq \frac{1}{1-\tau}\big(\{\|p(t)\|_1 + \|q(t)\|_1\}\|\phi(|x'(t)|)\|_\infty + \|r(t)\|_1 + \|e(t)\|_1\big) \\
&\leq \frac{1}{1-\tau}(\|p(t)\|_1 + \|q(t)\|_1)\|\phi(x'(t))\|_\infty + \frac{1}{1-\tau}(\|r(t)\|_1 + \|e(t)\|_1).
\end{aligned}$$

It follows from the assumption (3.2) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x\|_\infty \leq \|x'\|_\infty \leq c.$$

It is now immediate that the set of solutions of the family of equations (3.4) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$. This completes the proof of the theorem. ☐

Remark Suppose that the the odd increasing homeomorphism ϕ in Theorem 3.1 is k-homogeneous, in the sense that $\phi(tx) = t^k\phi(x)$ for $t \geq 0$ and $x \in \mathbb{R}$. Then the existence condition 3.2 in Theorem 3.1 becomes

$$\|t^k p(t)\|_1 + \|q(t)\|_1 + \tau < 1.$$

Theorem 3.2 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^2(0,1)$ such that*

$$|f(t,x_1,x_2)| \leq p(t)\phi(|x_1|) + q(t)\phi(|x_2|) + r(t) \tag{3.5}$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in \mathbb{R}^2$. Also let $\alpha \neq 1$, and $\eta \in (0,1)$ be given. Then for any given $e(t)$ in $L^2(0,1)$ the boundary-value problem (1.3) has at least one solution in $C^1[0,1]$ provided

$$\frac{1}{1-\tau}\|p\|_2 + C(\alpha,\eta)\|q\|_2 < 1, \tag{3.6}$$

where $C(\alpha,\eta)$ is as in Theorem 2.2.

Proof As in the proof of Theorem 3.1 it suffices to prove that the set of all possible solutions of the family of equations

$$\begin{gathered}
(\phi(x'(t)))' = \lambda f(t,x(t),x'(t)) + \lambda e(t), \quad 0 < t < 1, \\
x(0) = 0, \quad \phi(x'(1)) = \alpha\phi(x'(\eta)),
\end{gathered} \tag{3.7}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. For an $x \in W_\phi^{2,2}(0,1)$, with $x(0) = 0$, and $\phi(x'(1)) = \alpha\phi(x'(\eta))$ we use Theorem 2.1 and Theorem 2.2, above, to note that

$$\begin{aligned} \|\phi(|x(t)|)\|_2 &\leq \phi(\|x\|_\infty) \leq \phi(\|x'\|_\infty) \leq \|\phi(x'(t))\|_\infty \\ &\leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_1 \leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_2 \end{aligned}$$

and

$$\|\phi(x'(t))\|_2 \leq C(\alpha,\eta)\|(\phi(x'(t)))'\|_2. \tag{3.8}$$

Now, for a solution x of the family of equations (3.7) for some $\lambda \in [0,1]$ we have

$$\begin{aligned} \|(\phi(x'(t)))'\|_2 &\leq \lambda\|f(t,x(t),x'(t)) + e(t)\|_2 \\ &\leq \|p(t)\phi(|x(t)|) + q(t)\phi(|x'(t)|) + r(t)\|_2 + \|e\|_2 \\ &\leq \|p\|_2\|\phi(|x(t)|)\|_2 + \|q\|_2\|\phi(|x'(t)|)\|_2 + \|r\|_2 + \|e\|_2 \\ &\leq (\frac{1}{1-\tau}\|p\|_2 + C(\alpha,\eta)\|q\|_2)\|(\phi(x'(t)))'\|_2 + \|r(t)\|_2 + \|e\|_2, \end{aligned}$$

in view of estimate (3.8), for a solution x of the family of equations (3.7) for some $\lambda \in [0,1]$. It then follows from (3.6) that there is a constant c independent of $\lambda \in [0,1]$ such that

$$\|(\phi(x'(t)))'\|_2 \leq c,$$

for a solution x of the family of equations (3.7) for some $\lambda \in [0,1]$. Finally, we see, using Theorem 2.1 that

$$\phi(\|x\|_\infty) \leq \phi(\|x'\|_\infty) \leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_1 \leq \frac{1}{1-\tau}\|(\phi(x'(t)))'\|_2$$

and accordingly, the set of solutions of the family of equations (3.7) is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. This completes the proof of Theorem 3.2. $\square$

Example 3.3 Let $\alpha \leq 0$ and $\eta \in (0,1)$ be given and $A \in \mathbb{R}$. For $e(t) \in L^1(0,1)$, we consider the three point boundary-value problem

$$\begin{gathered} (\phi(x'(t)))' = t^{\frac{1}{2}}\phi(|x(t)|) + At\phi(|x'(t)|) + e(t), 0 < t < 1, \\ x(0) = 0, \phi(x'(1)) = \alpha\phi(x'(\eta)). \end{gathered} \tag{3.9}$$

We apply Theorem 3.1 to obtain a condition for the existence of a solution for the three-point boundary-value problem (3.9). Here $p(t) = t^{1/2}$, $q(t) = At$ and $\tau = 0$. Now, $\|p(t)\|_1 = 2/3$ and $\|q(t)\|_1 = \frac{1}{2}|A|$. Now, if

$$\frac{2}{3} + \frac{1}{2}|A| < 1,$$

or, equivalently $|A| < 2/3$, then Theorem 3.1 implies the existence of a solution for the three-point boundary-value problem (3.9).

Example 3.4 *Let $\alpha = -2$, $\eta = \frac{1}{3}$ and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three point boundary-value problem*

$$\begin{gathered} (\phi(x'(t)))' = t^{\frac{1}{4}}\phi(|x(t)|) + At^{-\frac{1}{4}}\phi(|x'(t)|) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \quad \phi(x'(1)) = \alpha\phi(x'(\eta)). \end{gathered} \tag{3.10}$$

We apply Theorem 3.2 to obtain a condition for the existence of a solution for the three-point boundary-value problem (3.10). Here $p(t) = t^{1/4}$, $q(t) = At^{-1/4}$. Now, $\|p(t)\|_2 = \sqrt{2/3}$ and $\|q(t)\|_2 = \sqrt{2}|A|$. Now the existence condition required to apply Theorem 3.2

$$\sqrt{\frac{2}{3}} + \sqrt{2}C(\alpha,\eta)|A| < 1. \tag{3.11}$$

Since we have $C(-2,\frac{1}{3}) = \sqrt{\frac{11}{54}}$, we get from (3.11)

$$\sqrt{\frac{2}{3}} + \sqrt{\frac{22}{54}}|A| < 1.$$

Accordingly, we see from Theorem 3.2 that a solution for the three-point boundary-value problem (3.10) exists if $|A| < \sqrt{54/22}(1 - \sqrt{2/3}) = .287\,49$.

Example 3.5 Let $\alpha = -2$, $\eta = 1/3$ and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three point boundary-value problem

$$\begin{gathered} (\phi(x'(t)))' = t^{\frac{15}{32}}\phi(|x(t)|) + At\phi(|x'(t)|) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \quad \phi(x'(1)) = \alpha\phi(x'(\eta)). \end{gathered} \tag{3.12}$$

We apply Theorem 3.2 to obtain a condition for the existence of a solution for the three-point boundary-value problem (3.12). Here $p(t) = t^{15/32}$, $q(t) = At$. Now, $\|p(t)\|_2 = 4/\sqrt{31}$ and $\|q(t)\|_2 = |A|/\sqrt{3}$. Now the existence condition required to apply Theorem 3.2

$$\frac{4}{\sqrt{31}} + \frac{1}{\sqrt{3}}C(\alpha,\eta)|A|) < 1. \tag{3.13}$$

Since, $C(-2,1/3) = \sqrt{11/54}$ and we get from (3.13)

$$\frac{4}{\sqrt{31}} + \sqrt{\frac{11}{162}}|A| < 1,$$

which implies

$$|A| < \sqrt{\frac{162}{11}}(1 - \frac{4}{\sqrt{31}}) = 1.080\,6.$$

Now, to apply Theorem 3.1 we see that

$$\|p(t)\|_1 = \int_0^1 t^{\frac{15}{32}}dt = \frac{32}{47}$$

and $\|q(t)\|_1 = \frac{1}{2}|A|$. Accordingly, we see using Theorem 3.1 a solution for the three-point boundary-value problem (3.12) exists if

$$\frac{32}{47} + \frac{1}{2}|A| < 1,$$

or, equivalently, if

$$|A| < 2(1 - \frac{32}{47}) = \frac{30}{47} = 0.6383.$$

We thus see that Theorem 3.2 gives a better result than Theorem 3.1.

Remark Note that if we take for $p > 1$, the odd increasing homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = |t|^{p-2}t \quad \text{for } t \in \mathbb{R},$$

then Theorems 3.1, 3.2 give existence theorems for the analogous three-point boundary-value problems for the one-dimensional analogue of the p-Laplacian. However, Theorems 3.1, 3.2 apply to more general differential operators than a p-Laplacian, since Theorems 3.1, 3.2 do not require the homeomorphism ϕ to be homogeneous as happens to be the case for the p-Laplacian.

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