Fifth Mississippi State Conference on Differential Equations and Computational Simulations,
Electronic Journal of Differential Equations, Conference 10, 2003, pp 101–107.
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A sign-changing solution for a superlinear Dirichlet problem, II *

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Abstract

In previous work by Castro, Cossio, and Neuberger [2], it was shown that a superlinear Dirichlet problem has at least three nontrivial solutions when the derivative of the nonlinearity at zero is less than the first eigenvalue of $-\Delta$ with zero Dirichlet boundry condition. One of these solutions changes sign exactly-once and the other two are of one sign. In this paper we show that when this derivative is between the k-th and $k+1$-st eigenvalues there still exists a solution which changes sign at most k times. In particular, when $k=1$ the sign-changing *exactly-once* solution persists although one-sign solutions no longer exist.

1 Introduction

Let Ω be a smooth bounded region in $\mathbb{R}^N$, Δ the Laplacian operator, and $f \in C^1(\mathbb{R},\mathbb{R})$ such that $f(0)=0$. In this paper we study the boundary-value problem

$$\begin{aligned} \Delta u + f(u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{1.1}$$

We assume that there exist constants $A>0$ and $p \in (1, \frac{N+2}{N-2})$ such that $|f'(u)| \le A(|u|^{p-1}+1)$ for all $u \in \mathbb{R}$. Hence f is *subcritical*, i.e., there exists $B>0$ such that $|f(u)| \le B(|u|^p+1)$. Also, we assume that there exists $m \in (0,1)$ and $\eta > 0$ such that

$$muf(u) \ge 2F(u) \tag{1.2}$$

for $|u|>\eta$, where $F(u)=\int_0^u f(s)\,ds$. Finally, we make the assumption that f satisfies

$$f'(u) > \frac{f(u)}{u} \quad \text{for } u \neq 0, \quad \text{and} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = \infty \quad (f \quad \text{is superlinear}). \tag{1.3}$$

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* *Mathematics Subject Classifications:* 35J20, 35J25, 35J60.

Key words: Dirichlet problem, superlinear, subcritical, sign-changing solution, deformation lemma.

Published February 28, 2003.

P. Drabek was partially supported by Ministry of Education of the Czech Republic, MSM 235200001.

J. Neuberger was partially supported by the National Science Foundation DMS-0074326.

Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, d\zeta$ (see [1] or [8]). Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. We let $\{\phi_1, \phi_2, \dots\}$ denote a complete orthonormal set in $L^2(\Omega)$ of eigenfunctions corresponding to the latter eigenvalues.

Our main result is as follows.

Theorem 1 *If $f'(0) \in [\lambda_k, \lambda_{k+1})$ then (1.1) has a solution w which changes sign at most k times, i.e., $\Omega - w^{-1}\{0\}$ consists of at most $k+1$ non-empty connected sets.*

Corollary 2 *If $f'(0) \in [\lambda_1, \lambda_2)$ then (1.1) has a solution w which changes sign exactly once.*

Our proofs here combine Lyapunov-Schmidt reduction arguments [5], the mountain pass lemma [11], Sard's Lemma [12], and the index of critical points of mountain pass type [6].

To the best of our knowledge, [2] was the first to establish the existence of a *sign-changing* solution to (1.1) for a general region in the superlinear case where $f'(0) < \lambda_1$. The proofs in [2] are based on the study of the Nehari manifold

$$S = \{u \in H : u \neq 0, \int_\Omega (\|\nabla u\|^2 - u f(u)) \, d\zeta = 0\}.$$

Unlike the work in [2], where S is a differentible manifold homeomorphic to the unit sphere and bounded away from 0, here 0 is a limit point of S. Also $S \cup 0$ has a singularity at 0. The intersection of S with planes spanned by $\{\phi_1, \phi_k\}$, $k = 2, \dots$ is a figure eight. The semipostione result in [10] is another example where a more complicated variational structure is successfully analyzed via our techniques.

For historical remarks concerning the existence of sign changing solutions to semilinear elliptic boundary value problems we refer the reader to [4]. See also [13].

Remark 1 One can easily see that when $f'(0) \geq \lambda_1$ there can be no one signed solutions. In fact suppose to the contrary that $f'(0) \geq \lambda_1$ and that u is (for example) a positive solution. Let ϕ_1 be a positive eigenfunction corresponding to λ_1. Then, by multiplying (1.1) by ϕ_1 and integrating we obtain

$$\begin{aligned} \int_\Omega \{\Delta u + f(u)\}\phi_1 \, d\zeta &= \int_\Omega \{u\Delta\phi_1 + f(u)\phi_1\} \, d\zeta \\ &= \int_\Omega \{\frac{f(u)}{u} - \lambda_1\} u\phi_1 \, d\zeta \\ &= \int_\Omega \{f'(v) - \lambda_1\} u\phi_1 \, d\zeta \\ &> \int_\Omega \{f'(0) - \lambda_1\} u\phi_1 \, d\zeta \geq 0, \end{aligned} \tag{1.4}$$

where we have used the mean value theorem to find $v \in (0, u)$.

Remark 2 Theorem 1 and Corollary 2 are also valid when the Dirichlet boundary condition in (1.1) is replaced by a homogeneous boundary condition for which the spectrum of the Laplacian operator consists of isolated eigenvalues of finite multiplicity converging to ∞. This is the case, for example, of the Neumann boundary condition $(\partial u/\partial\eta)(x) = 0$ for region with Lipschitzian boundary.

2 Preliminary Lemmas

Let k be a positive integer and $f'(0) \in (\lambda_k, \lambda_{k+1})$. We define $J : H \to \mathbb{R}$ by

$$J(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 - F(u)\}\, d\zeta.$$

By regularity theory for elliptic boundary value problems [8], u is a solution to (1.1) if and only if u is a critical point of J. Because f is subcritical, $J \in C^2(H, \mathbb{R})$ (see [11]). The gradient and Hessian of J are given by

$$J'(u)(v) = \langle \nabla J(u), v\rangle = \int_\Omega \{\nabla u \cdot \nabla v - f(u)v\}\, d\zeta, \quad \text{for all } v \in H, \tag{2.1}$$

and

$$\langle D^2 J(u)v, w\rangle = \int_\Omega \{\nabla v \cdot \nabla w - f'(u)vw\}\, d\zeta, \quad \text{for all } u, v, w \in H. \tag{2.2}$$

Let X be the linear subspace generated by $\{\phi_1, \dots, \phi_k\}$ and Y the subspace of H generated by $\{\phi_{k+1}, \dots\}$. By orthogonality properties of eigenfunctions $H = X \oplus Y$. Since (1.3) implies that $f'(t) \geq f'(0) > \lambda_k$, one sees that there exists $m_1 > 0$ such that

$$\langle D^2 J(u)x, x\rangle \leq -m_1\|x\|^2 \quad \text{for all } u \in H,\ x \in X. \tag{2.3}$$

Arguing as in Theorem 4 of [5], one sees that there exists a function $\psi \in C^1(Y, X)$ such that

$$\hat{J}(y) \equiv J(y + \psi(y)) = \max_{x\in X} J(x + y) \quad \text{for all } y \in Y, \tag{2.4}$$

and

$$\langle \nabla \hat{J}(y), v\rangle = \langle \nabla J(y + \psi(y)), v\rangle = \int_\Omega \{\nabla y \cdot \nabla v - f(y + \psi(y))v\}\, d\zeta, \tag{2.5}$$

for all $y, v \in Y$. In addition,

$$\psi(y) \text{ is the only critical point of } x \to J(x + y) \quad \text{for all} \quad y \in Y. \tag{2.6}$$

Thus

$$y \text{ is a critical point of } \hat{J} \text{ if and only if } y + \psi(y) \text{ is a critical point of } J. \tag{2.7}$$

Although not obvious (see [5]), $\hat{J}$ is of class C^2 in spite of only $\psi \in C^1(Y,X)$. From (2.5) we see that $\nabla\hat{J}(y) = y + K(y)$ where K is a compact function of y. Also since $\nabla\hat{J}$ is a variational vector field of class C^1 we have $\dim\ker(\nabla\hat{J})'(y) = \dim\ker(D^2\hat{J}(y)) = \operatorname{codim}(D^2(\hat{J}(y))(Y)) = \operatorname{codim}(\nabla J)'(y)(Y)$ for all $y \in Y$. Thus we may apply Sard's lemma [12] to conlcude the following lemma.

Lemma 1 *There exists $\{q_n\} \subset Y$ with $q_n \downarrow 0$ as $n \to \infty$ such that if $\nabla\hat{J}(u) = q_n$ then the Hessian $D^2\hat{J}(u)$ is invertible.*

Proof This proof follows immediately from the fact that the regular values of $\nabla\hat{J}$ is the complement of a set of first category. In partiuclar it is dense. $\diamondsuit$

Let q_n be as in the previous lemma. We define $J_n : Y \to \mathbb{R}$ by $J_n(y) = \hat{J}(y) - \langle q_n, y\rangle$. We note that $D^2J_n(y) = D^2\hat{J}(y)$ for each $y \in Y$. Also from (12) of [5]

$$\langle D^2\hat{J}(y)h, h\rangle = \langle D^2J(y+\psi(y))(h+\psi'(y)h), h+\psi'(y)h\rangle. \tag{2.8}$$

Lemma 2 *The functional J_n has a critical point y_n such that the Morse index of $D^2J(y_n+\psi(y_n))$ is less than or equal to $k+1$.*

proof In order to establish the existence of the critical point u_n we prove that J_n satisifies the hypotheses of the Mountain Pass Lemma [11]. Since $f'(0) < \lambda_{k+1}$, we have $\langle D^2J(0)y, y\rangle = J''(0)(y,y) \geq (1 - f'(0)/\lambda_{k+1})\int_\Omega |\nabla y|^2\, d\zeta > 0$ for $y \in Y$. Thus 0 is a strict local minimum of J restricted to Y. Thus there exist $\delta, \eta > 0$ such that $J(y) \geq \eta$ for all $||y|| = \delta$. This and (2.4) imply that for $||y|| = \delta$ we have

$$\hat{J}(y) \geq J(y) \geq \eta > 0\,. \tag{2.9}$$

Now taking n sufficiently large so that $||q_n|| \leq \eta/(2\delta)$ we have

$$J_n(y) \geq \eta - ||q_n||\delta > \eta/2 > 0\,, \tag{2.10}$$

for $||y|| = \delta$.

Next we note that, since 0 is a critical point of J, $\psi(0) = 0$. Hence $J_n(0) = 0$. Additionally, since we are assuming f to be superlinear, there exit numbers $m_1 > \lambda_{k+1}$ and m_2 such that

$$2F(t) \geq m_1t^2 + m_2 \tag{2.11}$$

for all $t \in \mathbb{R}$. Hence

$$\begin{aligned} J_n(t\phi_{k+1}) &= \hat{J}(t\phi_{k+1}) - \langle q_n, t\phi_{k+1}\rangle \leq J(t\phi_{k+1}) - \langle q_n, t\phi_{k+1}\rangle \\ &\leq (t||\phi_{k+1}||^2 - m_1\int_\Omega (t\phi_{k+1})^2\, d\zeta - m_2|\Omega|)/2 + t||q_n||||\phi_{k+1}|| \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned} \tag{2.12}$$

where we have used that $||\phi_{k+1}||^2 = \lambda_{k+1}\int_\Omega \phi_{k+1}^2\, d\zeta$. For future reference we note that, without loss of generality, we may assume that $||q_n|| \leq 1$ for all n.

Thus (2.12) implies that

$$J_n(t\phi_{k+1}) \le \frac{(\lambda_{k+1}-m_1)(-m_2|\Omega|)-\lambda_{k+1}\|\phi_{k+1}\|}{2(\lambda_{k+1}-m_1)} \equiv \tilde{K} \quad \text{for all } n,t \ge 0. \tag{2.13}$$

From (2.12), for each n, there exists a real number $t_n > 0$ with $\|t_n\phi_{k+1}\| \ge 2\delta$ such that $\hat{J}_n(t_n\phi_{k+1}) < 0$.

Next we show that J_n satisfies the Palais-Smale condition. Suppose that $\{y_j\}$ is a sequence so that $\{J_n(y_j)\}$ is bounded, say $|J_n(y_j)| \le M$ for all j and $\nabla J_n(y_j) \to 0$ as $j \to \infty$. For ease of notation, let $u = y_j + \psi(y_j)$ and $T = \frac{m}{2}uf(u) - F(u)$. Then

$$\begin{aligned} M + \frac{m}{2}||u|| &\ge J_n(y_j) - \frac{m}{2}[\langle \nabla J_n(y_j), y_j\rangle + \langle \nabla J(y_j+\psi(y_j)), \psi(y_j)\rangle] \\ &= (\frac{1}{2}-\frac{m}{2})||u||^2 + \int_\Omega T\, d\zeta - (1-\frac{m}{2})\langle q_n, u\rangle \\ &\ge (\frac{1}{4}-\frac{m}{4})(||u||^2 - \|q_n\|^2) + M_1|\Omega|, \end{aligned} \tag{2.14}$$

where $M_1 \in \mathbb{R}$ is a lower bound for T (see (1.2)). The latter inequality implies that $\{y_j + \psi(y_j)\}$ is bounded. Hence without loss of generality we may assume that $\{y_j\}$ converges weakly to $\bar{y} \in Y$ and that $\psi(y_j)$ converges to $\bar{x} \in X$. Since the imbedding of H in $L^{p+1}(\Omega)$ is compact, we may assume that $f(y_j + \psi(y_j))$ converges strongly in $L^1(\Omega)$. Since $\nabla J_n(y_j) = y_j + K(y_j) + q_n \to 0$ with K compact, we see that $\{y_j\}$ has a convergent subsequence. This proves that J_n satisfies the Palais-Smale condition.

Now by the Mountain Pass Lemma there exists $y_n \in Y$ such that $\nabla J_n(y_n) = 0$ and

$$J_n(y_n) = \inf_{\sigma\in\Sigma} \big[\max_{t\in[0,1]} J_n(\sigma(t))\big], \tag{2.15}$$

where $\Sigma = \{\sigma : [0,1] \to Y; \sigma \text{ is continuous}, \sigma(0) = 0 \text{ and } \sigma(1) = t_n\phi_{k+1}\}$. Thus from (2.10), (2.13) and (2.15) we have

$$\eta/2 \le J_n(y_n) \le \tilde{K}. \tag{2.16}$$

In addition, since $D^2J_n(y_n)$ is invertible (see Lemma 1), by Theorem 2 of [6], we may assume that the Morse index of $D^2J_n(y_n)$ is 1. Since $D^2J(y_n + \psi(y_n))$ is negative definite in a subspace of dimension k, namely X, then we conclude that the Morse index of $D^2J(y_n + \psi(y_n))$ is at most $k+1$, and this concludes the proof. ♢

3 Proof of Main Theorem

First we consider the case $f'(0) > \lambda_k$. Let $\{y_n\}$ be as in Lemma 2. By (2.16) one sees that $\{y_n + \psi(y_n)\}$ is bounded. Thus, without loss of generality, we may assume that $\{y_n\}$ converges weakly to $\bar{y} \in Y$ and $\{\psi(y_n)\}$ converges to $\bar{x} \in X$.

From (2.4) we have $0 = \nabla J_n(y_n) = y_n + K(y_n) - q_n$ where K is a compact operator. Since, in addition $\{q_n\}$ converges to 0, actually $\{y_n\}$ converges strongly to $\bar{y}$. Also since $J_n(y_n) \geq \eta/2 > 0$, we see that $\hat{J}(\bar{y}) \geq \eta/2 > 0$ and $\bar{y} \neq 0$. Now for $v \in X$ one has

$$\begin{aligned} &\langle \bar{x}, v\rangle - \int_\Omega \{f(\bar{y}+\bar{x})v\}\,d\zeta \\ &= \lim_{n\to\infty} \Big[\langle \psi(y_n), v\rangle - \int_\Omega \{vf(y_n+\psi(y_n))\}\,d\zeta - \langle q_n, \psi(y_n)\rangle\Big] \\ &= 0. \end{aligned} \tag{3.1}$$

Hence $\bar{x} = \psi(\bar{y})$ and $\nabla J(\bar{x}+\bar{y}) = 0$. Thus $\bar{x}+\bar{y} \neq 0$ is a solution to (1.1). Let us see that $\bar{x}+\bar{y}$ has at most $k+1$ nodal regions. If not, by defining v_j, $j = 1, \dots, k+2$, as $\bar{x}+\bar{y}$ on W_j and as zero on $\bar{\Omega} - W_j$, then from (1.3), (2.1) and (2.2), we see that $\langle D^2J(\bar{x}+\bar{y})v_j, v_j\rangle < 0$. Since the $v_j's$ are mutually orthogonal then we have that $D^2J(\bar{x}+\bar{y})$ is negative definite on a $k+2$-dimensional subspace. By continuity then $D^2J(y_n+\psi(y_n))$ is negative definite on the same $k+2$-dimensional subspace. This contradicts that the Morse index of $D^2J(y_n+\psi(y_n))$ is less than or equal to $k+1$. This contradiction proves that $\bar{x}+\bar{y}$ is a solution to (1.1) having at most $k+1$ nodal regions.

Finally we consider the case $f'(0) = \lambda_k$. Let $\{\epsilon_j\}$ be a sequence of positive numbers converging to 0. Without loss of generality we may assume that $\epsilon_j < (\lambda_{k+1} - \lambda_k)/2$ for all positive integers j. By our previous arguments, there exists a sequence $\{u_j = \bar{x}_j + \bar{y}_j\}$ of functions in H that satisfy

$$\begin{gathered} \Delta u_j + \epsilon_j u_j + f(u_j) = 0 \quad \text{in } \Omega \\ u_j = 0 \quad \text{in } \partial\Omega. \end{gathered} \tag{3.2}$$

In addition, each $x_j + y_j$ is the limit a of a sequence $\{x_{n,j} + y_{n,j}\}$ with each $x_{n,j} + y_{n,j}$ satisfying (2.16). Hence, by continuity $\eta/2 \leq J(\bar{x}_j + \bar{y}_j) \leq \tilde{K}$ and

$$\int_\Omega \{\nabla v \cdot \nabla v - (f'(u_j) + \epsilon_j)v^2\}\,d\zeta$$

defines a quadratic form of Morse index at most $k+1$. Arguing as above one sees that $\bar{x}_j + \bar{y}_j$ has a convergent subsequence with limit $\bar{x}+\bar{y}$. The function $\bar{x}+\bar{y}$ is a solution to (1.1) and, as in the case $f'(0) > \lambda_k$, it has at most $k+1$ nodal regions. This concludes the proof of our main theorem; the corollary is obvious.

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